MaGIC 2005 • Ustaoset, Norway • February 2005

Exterior algebra, ODEs and numerics

Thomas J. Bridges
University of Surrey

Joint work with

Leanne Allen (Surrey), Andre Afendikov (Moscow),
Frédéric Chardard (Paris), Gianne Derks (Surrey),
Frédéric Dias (Paris), Georg Gottwald (Sydney)

Given a vector space V of dimension n, there are a number of other vector spaces that can be built on it: the dual space V^*, the spaces of $k-$vectors $\bigwedge^k(V)$, and $k-$forms $\bigwedge^k(V^*)$, for $k = 0, \ldots, n$. Given a linear ordinary differential equation (ODE) on V it is often of interest to numerically integrate the induced systems on $\bigwedge^k(V)$ or $\bigwedge^k(V^*)$. Such systems arise in the linearization of nonlinear ODEs about trajectories where V is a model for the tangent space of the phase space. These lectures will discuss the theory behind such equations and the implementation of numerical algorithms for their integration. Applications of the theory to the stability of solitary waves, solution of boundary value problems, and hydrodynamic stability will be presented.

$$
\begin{tikzcd}
V \times V \arrow[r, "\phi"] \arrow[dr, "\Phi"'] & \bigwedge^2(V) \arrow[d, "L"] \\
 & \bigwedge^2(V)
\end{tikzcd}
$$

A model problem

Consider the linear boundary value problem

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u}\,, \quad \mathbf{u} \in V\,, \quad -1 < x < +1\,, \tag{1}$$

where $V \cong \mathbb{C}^2$, $\lambda \in \Lambda \subset \mathbb{C}$, with boundary conditions

$$\langle \mathbf{a}, \mathbf{u}(-1,\lambda)\rangle_1 = 0 \quad \text{and} \quad \langle \mathbf{b}, \mathbf{u}(+1,\lambda)\rangle_1 = 0\,, \tag{2}$$

where $\mathbf{a} \in V^*$ and $\mathbf{b} \in V^*$ are given. The bracket

$$\langle \cdot,\cdot\rangle_1 \;:\; V^* \times V \to \mathbb{C}\,,$$

is the natural pairing between V^* and V.

A numerical strategy: for any fixed $\lambda \in \Lambda$, integrate

$$\begin{aligned} \mathbf{u}_x^+ &= \mathbf{A}(x,\lambda)\mathbf{u}^+\,, \quad x \geq -1\,, \quad \langle \mathbf{a}, \mathbf{u}^+(-1,\lambda)\rangle_1 = 0 \\ \mathbf{u}_x^- &= \mathbf{A}(x,\lambda)\mathbf{u}^-\,, \quad x \leq +1\,, \quad \langle \mathbf{b}, \mathbf{u}^-(+1,\lambda)\rangle_1 = 0 \end{aligned}$$

λ is an eigenvalue if these two solutions are linearly dependent,

$$\mathbf{u}^-(x,\lambda) \wedge \mathbf{u}^+(x,\lambda) = 0 \quad \forall\, x\,, \quad i.e. \quad \det[\mathbf{u}^- \,|\, \mathbf{u}^+] = 0\,,$$

i.e. vanishing Wronskian. Let $\tau(x,\lambda) = \mathrm{Tr}(\mathbf{A}(x,\lambda))$, then $\lambda \in \mathbb{C}$ is an eigenvalue if

$$\mathscr{D}(\lambda) = 0 \quad \text{where} \quad \mathscr{D}(\lambda) = \mathrm{e}^{-\int_0^x \tau(s,\lambda)\mathrm{d}s}\, \mathbf{u}^-(x,\lambda) \wedge \mathbf{u}^+(x,\lambda)\,,$$

since

$$(\mathbf{u}^- \wedge \mathbf{u}^+)_x = \mathbf{A}\mathbf{u}^- \wedge \mathbf{u}^+ + \mathbf{u}^- \wedge \mathbf{A}\mathbf{u}^+ = \tau(\mathbf{u}^- \wedge \mathbf{u}^+)\,.$$

The model problem on $\bigwedge(V)$

Let's look at some aspects of the exterior algebra of V for this problem. The total exterior algebra on V is

$$\begin{aligned} &\textstyle\bigwedge^0(V) = \mathbb{C} && \textstyle\bigwedge^0(V^*) = \mathbb{C} \\ &\textstyle\bigwedge^1(V) = V = \mathrm{span}\{\xi_1, \xi_2\} && \textstyle\bigwedge^1(V^*) = V^* = \mathrm{span}\{\eta^1, \eta^2\} \\ &\textstyle\bigwedge^2(V) = \mathrm{span}\{\xi_1 \wedge \xi_2\} && \textstyle\bigwedge^2(V^*) = \mathrm{span}\{\eta^1 \wedge \eta^2\} \end{aligned}$$

The wedge product can be viewed in this case in term of bases. For example, if

$$\mathbf{a} = a_1\xi_1 + a_2\xi_2 \quad \text{and} \quad \mathbf{b} = b_1\xi_1 + b_2\xi_2\,,$$

then

$$\mathbf{a} \wedge \mathbf{b} = (a_1b_2 - a_2b_1)\xi_1 \wedge \xi_2\,.$$

Fix a volume form $\Omega \in \bigwedge^2(V^*)$; for example, $\Omega = \eta^1 \wedge \eta^2$. Define

$$\mathbf{v} = \mathbf{u} \lrcorner\, \Omega\,, \quad \mathbf{u} \in \textstyle\bigwedge^1(V)\,, \quad \mathbf{v} \in \textstyle\bigwedge^1(V^*)\,.$$

The mapping $\mathbf{u} \mapsto \mathbf{u} \lrcorner\, \Omega$ is the *interior product* It is defined by

$$\langle \mathbf{u} \lrcorner\, \Omega, \mathbf{a}\rangle_1 = \langle \Omega, \mathbf{u} \wedge \mathbf{a}\rangle_2\,, \quad \forall\, \mathbf{a} \in \textstyle\bigwedge^1(V)\,,$$

where $\langle \cdot, \cdot \rangle_k$ is the induced pairing on $\bigwedge^k$. In coordinates

$$\langle \Omega, \mathbf{u} \wedge \mathbf{a}\rangle_2 = \det \begin{bmatrix} \langle \eta^1, \mathbf{u}\rangle_1 & \langle \eta^1, \mathbf{a}\rangle_1 \\ \langle \eta^2, \mathbf{u}\rangle_1 & \langle \eta^2, \mathbf{a}\rangle_1 \end{bmatrix}\,,$$

Hence

$$\mathbf{v} = \mathbf{u} \lrcorner\, \Omega = \langle \eta^1, \mathbf{u}\rangle_1 \eta^2 - \langle \eta^2, \mathbf{u}\rangle_1 \eta^1$$

What equation does $\mathbf{v} = \mathbf{u} \lrcorner\, \Omega$ satisfy when $\mathbf{u}$ satisfies $\mathbf{u}_x = \mathbf{A}(x, \lambda)\mathbf{u}$, $\mathbf{u} \in \bigwedge^1(V)$?

The differential equation for $\mathbf{u} \lrcorner \Omega$

What equation does $\mathbf{v} = \mathbf{u} \lrcorner \Omega$ satisfy?

$$\mathbf{v}_x = \mathbf{u}_x \lrcorner \Omega = \mathbf{A}\mathbf{u} \lrcorner \Omega \,.$$

but

$$\mathbf{A}\mathbf{u} \lrcorner \Omega = (\tau \mathbf{I} - \mathbf{A}^T)\mathbf{u} \lrcorner \Omega \,. \tag{3}$$

Hence,

$$\mathbf{v}_x = \tau \mathbf{v} - \mathbf{A}^T \mathbf{v} \,, \quad \mathbf{v} \in \bigwedge^1(V^*) \,, \quad \tau = \mathrm{Tr}(\mathbf{A}) \,.$$

A proof of (3) is as follows. For any $\mathbf{a} \in \bigwedge^1(V)$,

$$\begin{aligned}
\langle (\mathbf{A}\mathbf{u}) \lrcorner \Omega, \mathbf{a} \rangle_1 &= \langle \Omega, \mathbf{A}\mathbf{u} \wedge \mathbf{a} \rangle_2 \\
&= \langle \Omega, \mathbf{A}\mathbf{u} \wedge \mathbf{a} \rangle_2 + \langle \Omega, \mathbf{u} \wedge \mathbf{A}\mathbf{a} \rangle_2 - \langle \Omega, \mathbf{u} \wedge \mathbf{A}\mathbf{a} \rangle_2 \\
&= \tau \, \langle \Omega, \mathbf{u} \wedge \mathbf{a} \rangle_2 - \langle \mathbf{u} \lrcorner \Omega, \mathbf{A}\mathbf{a} \rangle_1 \\
&= \tau \, \langle \mathbf{u} \lrcorner \Omega, \mathbf{a} \rangle_1 - \langle \mathbf{A}^T (\mathbf{u} \lrcorner \Omega), \mathbf{a} \rangle_1 \\
&= \langle (\tau \mathbf{I} - \mathbf{A}^T) \mathbf{u} \lrcorner \Omega, \mathbf{a} \rangle_1
\end{aligned}$$

where we have used $\mathbf{A}\mathbf{u} \wedge \mathbf{a} + \mathbf{u} \wedge \mathbf{A}\mathbf{a} = \tau \mathbf{u} \wedge \mathbf{a}$.

Let

$$\widetilde{\mathbf{v}} = \mathrm{e}^{-\int_0^x \tau(s,\lambda)\mathrm{d}s} \, \mathbf{v} = \mathrm{e}^{-\int_0^x \tau(s,\lambda)\mathrm{d}s} \, \mathbf{u} \lrcorner \Omega \,,$$

then clearly

$$\widetilde{\mathbf{v}}_x = -\mathbf{A}(x,\lambda)^T \widetilde{\mathbf{v}} \,, \quad \widetilde{\mathbf{v}} \in \bigwedge^1(V^*) \,.$$

Revised model problem using $\mathbf{v} = \mathbf{u} \lrcorner \Omega$

How does $\widetilde{\mathbf{v}}^- = \mathrm{e}^{-\int_0^x \tau(s,\lambda)\mathrm{d}s}\, \mathbf{u}^- \lrcorner \Omega$ appear in the model problem?

$$
\begin{aligned}
\mathscr{D}(\lambda) &= \mathrm{e}^{-\int_0^x \tau(s,\lambda)\mathrm{d}s}\, \mathbf{u}^-(x,\lambda) \wedge \mathbf{u}^+(x,\lambda) \\
&= \mathrm{e}^{-\int_0^x \tau(s,\lambda)\mathrm{d}s}\, \langle \Omega, \mathbf{u}^-(x,\lambda) \wedge \mathbf{u}^+(x,\lambda)\rangle_2\, \xi_1 \wedge \xi_2 \\
&= \mathrm{e}^{-\int_0^x \tau(s,\lambda)\mathrm{d}s}\, \langle \mathbf{u}^- \lrcorner \Omega, \mathbf{u}^+\rangle_1\, \xi_1 \wedge \xi_2 \\
&= \langle \widetilde{\mathbf{v}}^-, \mathbf{u}^+\rangle_1\, \xi_1 \wedge \xi_2
\end{aligned}
$$

Hence eigenvalues satisfy $D(\lambda) = 0$ where $D(\lambda) = \langle \widetilde{\mathbf{v}}^-, \mathbf{u}^+\rangle_1$.

Numerical strategy: for any fixed $\lambda \in \Lambda$ integrate

$$\mathbf{u}_x^+ = \mathbf{A}(x,\lambda)\mathbf{u}^+ , \quad x \geq -1 , \quad \langle \mathbf{a}, \mathbf{u}^+(-1,\lambda)\rangle_1 = 0$$

for $-1 \leq x$, and for $x \leq 1$ integrate

$$\widetilde{\mathbf{v}}_x^- = -\mathbf{A}(x,\lambda)^T \widetilde{\mathbf{v}}_x^- , \quad x \leq +1 , \quad \widetilde{\mathbf{v}}^-(+1,\lambda) = \widetilde{\mathbf{b}} \lrcorner \Omega ,$$

where

$$\widetilde{\mathbf{b}} \in \textstyle\bigwedge^1(V) \quad \text{is any vector satisfying} \quad \langle \mathbf{b}, \widetilde{\mathbf{b}}\rangle_1 = 0 .$$

Then, at some convenient value of x evaluate $D(\lambda) = \langle \widetilde{\mathbf{v}}^-, \mathbf{u}^+\rangle_1$.
Eigenvalues satisfy $D(\lambda) = 0$.

Introduction to the interior product on $\bigwedge(V)$

Let V be an $n-$dimensional vector space, and let Ω be a fixed volume form, i.e. a fixed element of $\bigwedge^n(V^*)$. The mapping

$$\mathbf{u} \mapsto \mathbf{u} \lrcorner \Omega \qquad \mathbf{u} \in \bigwedge^k(V)$$

which takes $\mathbf{u} \in \bigwedge^k(V)$ to $\mathbf{u} \lrcorner \Omega \in \bigwedge^{n-k}(V^*)$ is a duality mapping. It is a special case of the *interior product* which is defined in general by

$$\langle \mathbf{u} \lrcorner \mathbf{U}, \mathbf{v}\rangle_{\ell-k} = \langle \mathbf{U}, \mathbf{u}\wedge\mathbf{v}\rangle_{\ell}\,, \quad \mathbf{U} \in \bigwedge^{\ell}(V^*)\,,\ \mathbf{v} \in \bigwedge^{\ell-k}(V)\,,\ \mathbf{u} \in \bigwedge^k(V)\,,$$

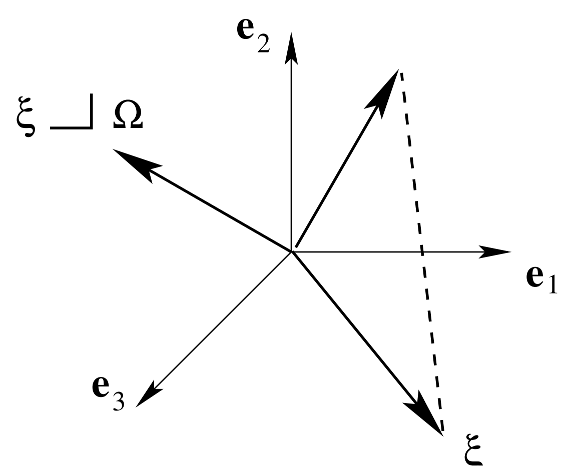
where $\langle\cdot,\cdot\rangle_k$ is the induced pairing on $\bigwedge^k$. The interior product is also denoted $\mathbf{i}_{\mathbf{u}}\mathbf{U}$ or $i(\mathbf{u})\mathbf{U}$.

Example of the interior product in $\mathbb{R}^3$ ($V = V^* = \mathbb{R}^3$). Let

$$\Omega = \mathbf{e}_1 \wedge \mathbf{e}_2\,, \quad \xi = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2 + \xi_3\mathbf{e}_3\,,$$

then

$$\xi \lrcorner \Omega = -\xi_2\mathbf{e}_1 + \xi_1\mathbf{e}_2$$

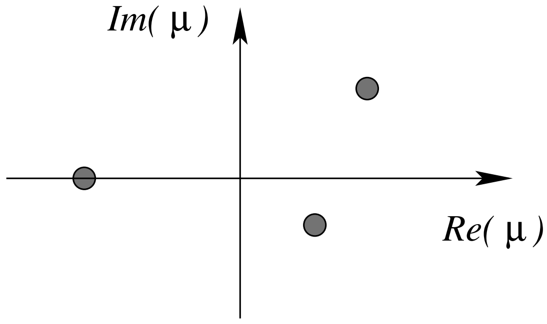


2D subspaces of 3D vector spaces

Let $\mathbf{A}(\lambda) : V \to V$ be a linear transformation depending analytically on λ for all $\lambda \in \Lambda \subset \mathbb{C}$. Suppose that the spectrum of $\mathbf{A}(\lambda)$

$$\{\mu \in \mathbb{C} \ : \ \det[\mu(\lambda)\mathbf{I} - \mathbf{A}(\lambda)] = 0\} \,,$$

is of the qualitative form shown in the figure for all $\lambda \in \Lambda$.



Generically the two eigenvalues with positive real part (denoted $\mu_2(\lambda)$ and $\mu_3(\lambda)$) may collide for some λ, resulting in a square root singularity. But the *subspace* is analytic. If

$$[\mathbf{A}(\lambda) - \mu_j(\lambda)\mathbf{I}]\xi_j(\lambda) = 0$$

then the cross product of $\xi_1(\lambda) \times \xi_2(\lambda)$ can be analytic and

$$\mathbf{A}^{(2)}(\lambda)\xi_1 \times \xi_2 = (\mu_1 + \mu_2)\xi_1 \times \xi_2 \,,$$

where

$$\mathbf{A}^{(2)}(\lambda)\xi_1 \times \xi_2 := \mathbf{A}(\lambda)\xi_1 \times \xi_2 + \xi_1 \times \mathbf{A}(\lambda)\xi_2 \,.$$

In fact

$$\mathbf{A}^{(2)}(\lambda) = \tau\mathbf{I} - \mathbf{A}(\lambda)^T \,, \quad \tau = \mathrm{Tr}(\mathbf{A}) \,.$$

Introductory example: linear stability of pulses

Consider the parabolic PDE

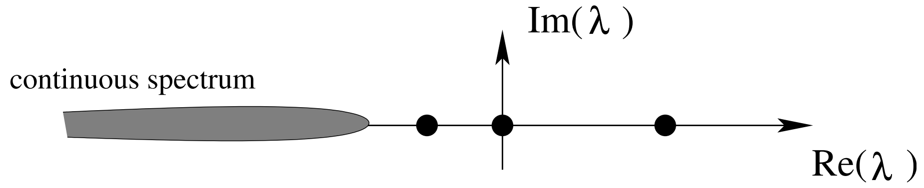
$$u_t = u_{xx} - 4u + 6u^2 \,,$$

which has the steady solution $\widehat{u}(x) = \mathrm{sech}^2(x)$. Linearising about $\widehat{u}$

$$u_t = u_{xx} - 4u + 12\widehat{u}(x)u \,,$$

with associated spectral problem on the real line

$$\mathscr{L}u = \lambda u \quad \text{with} \quad \mathscr{L} := \frac{d^2}{dx^2} - 4 + 12\widehat{u}(x)\,.$$



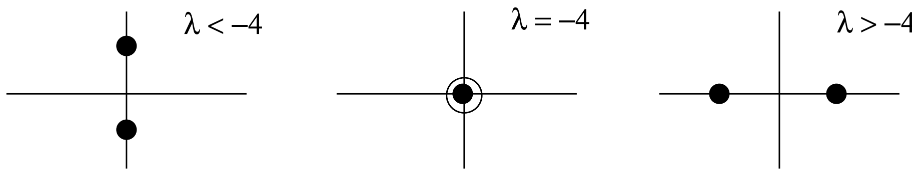
Dynamical systems view of the spectral problem

Let $\mathbf{u} = (u_1, u_2)$, with $u_1 := u$ and $u_2 = u_x$

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u}\,, \quad \mathbf{A}(x,\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + 4 - 12\widehat{u}(x) & 0 \end{pmatrix},$$

with

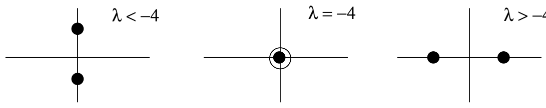
$$\lim_{x\to\pm\infty} \mathbf{A}(x,\lambda) = \mathbf{A}_{\pm}(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda + 4 & 0 \end{pmatrix}.$$



An "Evans function" formulation

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u} \quad \mathbf{u} \in \mathbb{C}^2$$

with spectrum of $\mathbf{A}_\infty(\lambda)$ of the form:



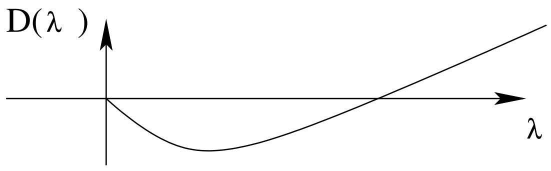
Using standard results for linear ODEs, there exist functions $\mathbf{u}^+(x,\lambda)$ and $\mathbf{u}^-(x,\lambda)$ for all $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > -4$ with

$$\begin{array}{rclcrcl} \lim_{x\to+\infty} \mathrm{e}^{-\mu_+(\lambda)x}\mathbf{u}^+(x,\lambda) & = & \zeta^+(\lambda)\,, & & (\mathbf{A}_\infty(\lambda) - \mu_+(\lambda)\mathbf{I})\zeta^+(\lambda) & = & 0 \\ \lim_{x\to-\infty} \mathrm{e}^{-\mu_-(\lambda)x}\mathbf{u}^-(x,\lambda) & = & \zeta^-(\lambda)\,, & & (\mathbf{A}_\infty(\lambda) - \mu_-(\lambda)\mathbf{I})\zeta^-(\lambda) & = & 0 \end{array}$$

$\mathbf{u}^+$ is bounded as $x \to +\infty$ and $\mathbf{u}^-$ is bounded as $x \to -\infty$. When they are linearly dependent for some λ, they are bounded (and decay exponentially) as $x \to \pm\infty$, and that value of λ is an *eigenvalue*. Define

$$D(\lambda) = \det[\,\mathbf{u}^-(x,\lambda)\,\big|\,\mathbf{u}^+(x,\lambda)\,]$$

- $D(\lambda)$ is independent of x.
- $D(\lambda)$ is an analytic function of λ for $\mathrm{Re}(\lambda) > -4$.
- $D(\lambda)$ is real if λ is real.
- $D(0) = 0$, $D'(0) < 0$ and $D(\lambda) > 0$ for λ real and large.



Numerical integration of model problem

Strategy for numerical integration on the interval $-L < x < L$

For any fixed λ, integrate

$$\frac{d\mathbf{u}^+}{dx} = \mathbf{A}(x,\lambda)\mathbf{u}^+ \quad L > x > 0 \quad \text{with} \quad \mathbf{u}^+(L,\lambda) = \zeta^+(\lambda)$$

using any standard numerical integrator. Similarly, integrate

$$\frac{d\mathbf{u}^-}{dx} = \mathbf{A}(x,\lambda)\mathbf{u}^- \quad -L < x < 0 \quad \text{with} \quad \mathbf{u}^-(-L,\lambda) = \zeta^-(\lambda)$$

At $x = 0$ construct

$$D(\lambda) = \det[\,\mathbf{u}^-(0,\lambda)\,\big|\,\mathbf{u}^+(0,\lambda)\,]$$

Now study the function $D(\lambda)$

- Along the real axis (if only real unstable eigenvalues are anticipated)
- Along a contour in the complex λ plane; then use Cauchy's theorem to count complex eigenvalues with positive real part.

Contrast with matrix methods where $u_{xx} + a(x)u = \lambda u$, $u \in \mathrm{L}^2(\mathbb{R})$ is replaced by a discretization using exact asymptotic b.c.

$$\begin{aligned} u_{xx} + a(x)u &= \lambda u \quad -L < x < L \\ u_x \pm \sqrt{\lambda}\,u &= 0 \quad \text{at} \quad x = \pm L \end{aligned}$$

Hence matrix eigenvalue routines not applicable. Can also apply articial ($\lambda-$independent) boundary conditions, but other problems may arise (see later discussion).

Stability of pulses – when $n = 3$

The Fitzhugh-Nagumo equation, which models the transmission of nerve impulses, can be written in the form

$$\phi_t = \phi_{xx} + f(\phi) - \psi \quad \text{and} \quad \psi_t = b(\phi - d\psi)$$

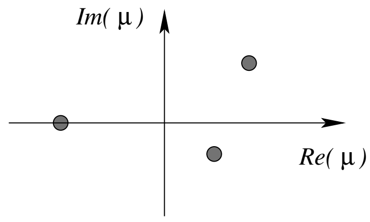
where $f(\phi)$ is some smooth cubic function. Linearizing this PDE about a pulse solution leads to a spectral problem of the form

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u} \quad \mathbf{u} \in \mathbb{C}^3$$

with

$$\lim_{x\to\pm\infty} \mathbf{A}(x,\lambda) = \mathbf{A}_\infty(\lambda)$$

where $\mathbf{A}_\infty(\lambda)$ has one eigenvalue with negative real part and two with positive real part, for all $\lambda \in \mathbb{C}$ such that $\mathbf{Re}(\lambda) > 0$.



Using standard results for linear ODEs, there exist functions $\mathbf{u}_j(x,\lambda)$, $j = 1,2,3$ for all $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) > 0$ with

$$\begin{aligned}
\lim_{x\to+\infty} \mathrm{e}^{-\mu_1(\lambda)x}\mathbf{u}_1(x,\lambda) &= \zeta_1(\lambda)\,, \quad (\mathbf{A}_\infty(\lambda) - \mu_1(\lambda)\mathbf{I})\zeta_1(\lambda) = 0 \\
\lim_{x\to-\infty} \mathrm{e}^{-\mu_j(\lambda)x}\mathbf{u}_j(x,\lambda) &= \zeta_j(\lambda)\,, \quad (\mathbf{A}_\infty(\lambda) - \mu_j(\lambda)\mathbf{I})\zeta_j(\lambda) = 0 \quad j = 2,3
\end{aligned}$$

But (a) $\mu_2(\lambda)$ and $\mu_3(\lambda)$ may not be analytic for all admissible λ (eigenvalue collisions); (b) when integrating from $x = -L$ to $x = 0$ there will be problems with computing two independent exponentially growing solutions.

Integrating the stability equation on two-planes

How do we numerically integrate

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u} \quad \mathbf{u} \in \mathbb{C}^3 \quad -L < x < 0$$

when there are two eigenvalues of $\mathbf{A}_\infty(\lambda)$ with positive real part?

One can use continuous or discrete orthonormalization, but orthonormalization converts a linear ODE to a nonlinear ODE, and orthonormalization does not preserve analyticity in general.

Integrate the ODE on two planes. Let

$$\xi_1 = \mathbf{e}_2 \times \mathbf{e}_3\,, \quad \xi_2 = \mathbf{e}_3 \times \mathbf{e}_1\,, \quad \xi_3 = \mathbf{e}_1 \times \mathbf{e}_2\,,$$

be a basis for the 2-planes in $\mathbb{C}^3$, and define

$$\mathbf{A}^{(2)}(\mathbf{a} \times \mathbf{b}) := \mathbf{A}\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{A}\mathbf{b}$$

Then, with $\mathbf{w} = w_1\boldsymbol{\xi}_1 + w_2\boldsymbol{\xi}_2 + w_3\boldsymbol{\xi}_3$,

$$\mathbf{A}^{(2)}\mathbf{w} = \tau\,\mathbf{w} - \mathbf{A}^T\mathbf{w}$$

where $\tau = \mathrm{Trace}(\mathbf{A})$. Hence

$$\mathbf{u}_x^- = -\mathbf{A}^T\mathbf{u}^- \quad \text{when} \quad \mathbf{u}^- = \mathrm{e}^{-\int_0^x \tau(s)\,\mathrm{d}s}\mathbf{w}(x)$$

Hence, the numerical strategy is: for any fixed λ, integrate

$$\frac{d\mathbf{u}^+}{dx} = \mathbf{A}(x,\lambda)\mathbf{u}^+ \quad L > x > 0 \quad \text{with} \quad \mathbf{u}^+(L,\lambda) = \zeta_1(\lambda)$$

using any standard numerical integrator. Similarly, integrate

$$\frac{d\mathbf{u}^-}{dx} = -\mathbf{A}(x,\lambda)^T\mathbf{u}^- \quad -L < x < 0 \quad \text{with} \quad \mathbf{u}^-(-L,\lambda) = \zeta_2(\lambda) \times \zeta_3(\lambda)$$

At $x = 0$ construct $D(\lambda) = \mathbf{u}^-(0,\lambda) \cdot \mathbf{u}^+(0,\lambda)$.

Exterior algebra of a vector space V

Let $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ be a basis for the $n-$dimensional vector space V; then the d distinct combinations

$$\boldsymbol{\xi}_{i_1} \wedge \cdots \wedge \boldsymbol{\xi}_{i_k}\,, \quad \text{where} \quad d = \frac{n!}{k!(n-k)!}$$

form a basis for $\bigwedge^k(V)$. Label and introduce an ordering for this basis, $\omega_1, \ldots, \omega_d$. Then any element $\mathbf{U} \in \bigwedge^k(V)$ can be represented as

$$\mathbf{U} = \sum_{j=1}^{d} U_j\, \omega_j\,.$$

Similarly, any element $\mathbf{V} \in \bigwedge^{n-k}(V)$ can be represented as

$$\mathbf{V} = \sum_{j=1}^{d} V_j\, \alpha_j\,,$$

where $\alpha_1, \ldots, \alpha_d$ is a basis for $\bigwedge^{n-k}(V)$.

Now, when studying the linear stability of solitary waves, one would like to integrate the ODE,

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\,\mathbf{u}\,, \quad \mathbf{u} \in V \cong \bigwedge^1(V)\,. \qquad (*)$$

on the various spaces $\bigwedge^k(V)$, $\bigwedge^{n-k}(V)$, as well as $\bigwedge^k(V^*)$, when

$$\lim_{x\to\pm\infty} \mathbf{A}(x,\lambda) = \mathbf{A}_\infty(\lambda)\,,$$

and the spectrum of $\mathbf{A}_\infty(\lambda)$ has a $k \times (n-k)$ splitting.

The wedge product on $\bigwedge^2(V)$ and $\bigwedge^k(V)$

Let V be an $n-$dimensional vector space. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \bigwedge^1(V)$, the *wedge product* satisfies the four rules

- $(a\mathbf{u} + \mathbf{v}) \wedge \mathbf{w} = a\,\mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}$ for all scalars a
- $\mathbf{u} \wedge (b\mathbf{v} + \mathbf{w}) = b\,\mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{w}$ for all scalars b
- $\mathbf{u} \wedge \mathbf{u} = 0$
- If $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis for V, then the nonzero elements of $\{\mathbf{u}_i \wedge \mathbf{u}_j\ 1 \leq i, j \leq n\}$ form a basis for $\bigwedge^2(V)$.

Note that $(\mathbf{u} + \mathbf{v}) \wedge (\mathbf{u} + \mathbf{v}) = 0$ implies

$$\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u} \quad \text{for any} \quad \mathbf{u}, \mathbf{v} \in \bigwedge^1(V)\,.$$

These properties have natural generalizations to any $\bigwedge^k(V)$ for $1 \leq k \leq n$ with properties

- associativity: $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$
- $\mathbf{U} \wedge \mathbf{V} = (-1)^{k\ell}\mathbf{V} \wedge \mathbf{U}$ when $\mathbf{U} \in \bigwedge^k(V)$ and $\mathbf{V} \in \bigwedge^\ell(V)$.
- $\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k = 0$ if $\mathbf{u}_i = \mathbf{u}_j$ for some $i \neq j$
- $\dim \bigwedge^k(V) = \frac{n!}{k!(n-k)!}$
- $\dim \bigwedge(V) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} = 2^n$

Fundamentals of $\bigwedge(V)$

There are various starting points for an axiomatic construction of the exterior algebra of a vector space. One can start with the tensor product of vector spaces, and project onto the anti-symmetric tensors. More directly, one can start with *alternating multilinear mappings*

Let V, W be vector spaces. A mapping $h : V \times \cdots \times V \to W$ is called

- *multilinear* if $h(\mathbf{u}_1, \ldots, \mathbf{u}_k)$ is linear in each argument
- *alternating* (or anti-symmetric) if

  $$h(\mathbf{u}_{\pi(1)}, \ldots, \mathbf{u}_{\pi(k)}) = \mathrm{sgn}(\pi) h(\mathbf{u}_1, \ldots, \mathbf{u}_k)$$

  here π is a permutation on $k-$symbols.

Denote the space of $k-$linear alternating maps by $\mathscr{A}_k(V, W)$. An important property of multilinear alternating mappings is the *universal factorization property*:

Consider a $k-$linear mapping $g(\mathbf{u}_1, \ldots, \mathbf{u}_k)$. There exists a unique linear mapping $\widehat{g} : \bigwedge^k(V) \to W$ such that

$$g(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \widehat{g}(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k)$$

such that the following diagram commutes

$$\begin{tikzcd}
V \times \ldots \times V \arrow[rr] \arrow[dr, "g"'] & & \bigwedge^k(V) \arrow[dl, "\hat{g}"] \\
& W &
\end{tikzcd}$$

Integrating ODEs on $\bigwedge^k(V)$

The system

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\,\mathbf{u}\,,\quad \mathbf{u} \in \textstyle\bigwedge^1(V) \cong V\,.$$

with a $k \times (n-k)$ splitting of $\mathbf{A}_\infty(\lambda)$ where

$$\lim_{x\to\pm\infty} \mathbf{A}(x,\lambda) = \mathbf{A}_\infty(\lambda)\,,$$

generates induced systems

$$\mathbf{U}_x = \mathbf{A}^{(k)}(x,\lambda)\,\mathbf{U}\,,\quad \mathbf{V}_x = \mathbf{A}^{(n-k)}(x,\lambda)\,\mathbf{V}\,,$$

where $\mathbf{A}^{(k)}(x,\lambda)$ and $\mathbf{A}^{(n-k)}(x,\lambda)$ are $d \times d$ matrices.

- Integrate on $\bigwedge^{n-k}(\mathbb{C}^n)$ from $x = -L_\infty$ to $x = 0$
- Integrate on $\bigwedge^{k}(\mathbb{C}^n)$ from $x = L_\infty$ to $x = 0$
- Starting vector: at $x = -L_\infty$ choose $\xi^-(\lambda)$ as starting vector: it is the eigenvector corresponding to the *simple* eigenvalue of $\mathbf{A}_\infty^{(n-k)}(\lambda)$ of largest real part. (With a similar construction at $x = +L_\infty$.)
- How to match at $x = x_0$? One uses the "Evans function"

$$\widetilde{D}(\lambda) = \mathrm{e}^{-\int_0^{x_0} \tau(s,\lambda)\,\mathrm{d}s}\,\mathbf{U}(x_0,\lambda) \wedge \mathbf{V}(x_0,\lambda)\,.$$

- One can use metric-free duality $\bigwedge^{n-k}(V^*) \cong \bigwedge^k(V)$ or Hodge duality: $\bigwedge^k(\mathbb{C}^n) \cong \bigwedge^{n-k}(\mathbb{C}^n)$ to simplify the Evans function. For example, let $\bigstar : \bigwedge^{n-k}(\mathbb{C}^n) \to \bigwedge^k(\mathbb{C}^n)$ be the Hodge star operator; then

$$\widetilde{D}(\lambda) = D(\lambda)\,\mathsf{Vol}\quad D(\lambda) = [\![\bigstar\mathbf{V},\mathbf{U}]\!]_d\,.$$

Total exterior algebra when V is three-dimensional

Let V be a $3-$dimensional complex vector space. The total exterior algebra is

$$\bigwedge^0(V) = \mathbb{C} \qquad \bigwedge^0(V^*) = \mathbb{C}$$
$$\bigwedge^1(V) = V = \mathrm{span}\{\xi_1, \xi_2, \xi_3\} \qquad \bigwedge^1(V^*) = V^* = \mathrm{span}\{\eta^1, \ldots\}$$
$$\bigwedge^2(V) = \mathrm{span}\{\xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \xi_2 \wedge \xi_3\} \qquad \bigwedge^2(V^*) = \mathrm{span}\{\eta^1 \wedge \eta^2, \ldots\}$$
$$\bigwedge^3(V) = \mathrm{span}\{\xi_1 \wedge \xi_2 \wedge \xi_3\} \qquad \bigwedge^3(V^*) = \mathrm{span}\{\eta^1 \wedge \eta^2 \wedge \eta^3\}$$

Given a linear ODE on $\bigwedge^1(V)$, we are interested in the induced ODEs on $\bigwedge^k(V)$ for $k = 2, 3$. They are determined as follows. Consider

$$\mathbf{V}_x = \mathbf{A}^{(2)}\mathbf{V}\,, \quad \mathbf{V} \in \bigwedge^2(V)\,.$$

The linear operator $\mathbf{A}^{(2)} : \bigwedge^2(V) \to \bigwedge^2(V)$ is defined by

$$\mathbf{A}^{(2)}\mathbf{u} \wedge \mathbf{v} = \mathbf{A}\mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{A}\mathbf{v}\,, \quad \text{for any} \quad \mathbf{u}, \mathbf{v} \in \bigwedge^1(V)\,,$$

and then extended by linearity to any element in $\bigwedge^2(V)$. By the universal factorization theorem, there is an induced *linear* operator on $\bigwedge^2(V)$.

A matrix representation for $\mathbf{A}^{(2)}$ acting on $\bigwedge^2(V)$ is obtained by computing $\mathbf{A}^{(2)}$ on a basis for $\bigwedge^2(V)$; i.e. let

$$\begin{aligned} \omega_1 &= \xi_1 \wedge \xi_2\,, \quad \omega_2 = \xi_1 \wedge \xi_3\,, \quad \omega_3 = \xi_2 \wedge \xi_3 \\ \theta_1 &= \eta^1 \wedge \eta^2\,, \quad \theta^2 = \eta^1 \wedge \eta^3\,, \quad \theta_3 = \eta^2 \wedge \eta^3 \end{aligned}$$

then

$$(\mathbf{A}^{(2)})_{i,j} = \langle \theta^i \mathbf{A}^{(2)} \omega_j \rangle_2\,.$$

Induced matrix $\mathbf{A}^{(2)}$ on $\bigwedge^2(V)$

Compute

$$(\mathbf{A}^{(2)})_{i,j} = \langle \theta^i \mathbf{A}^{(2)} \omega_j \rangle_2 \,.$$

For example

$$\begin{aligned}
(\mathbf{A}^{(2)})_{1,1} &= \langle \theta^1, \mathbf{A}^{(2)} \omega_1 \rangle_2 \\
&= \langle \eta^1 \wedge \eta^2, \mathbf{A}\xi_1 \wedge \xi_2 + \xi_1 \wedge \mathbf{A}\xi_2 \rangle_2 \\
&= \det \begin{bmatrix} \langle \eta^1, \mathbf{A}\xi_1 \rangle_1 & \langle \eta^1, \xi_2 \rangle_1 \\ \langle \eta^2, \mathbf{A}\xi_1 \rangle_1 & \langle \eta^2, \xi_2 \rangle_1 \end{bmatrix} + \det \begin{bmatrix} \langle \eta^1, \xi_1 \rangle_1 & \langle \eta^1, \mathbf{A}\xi_2 \rangle_1 \\ \langle \eta^2, \xi_1 \rangle_1 & \langle \eta^2, \mathbf{A}\xi_2 \rangle_1 \end{bmatrix} \\
&= \langle \eta^1, \mathbf{A}\xi_1 \rangle_1 + \langle \eta^2, \mathbf{A}\xi_2 \rangle_1 \\
&= a_{11} + a_{22} \,.
\end{aligned}$$

Given $\mathbf{A}^{(k)}$ on $\bigwedge^k(V)$, the linear mappings on $\bigwedge^k(V^*)$ can be obtained directly, in a coordinate free way, using the duality mapping,

$$\mathbf{U} \mapsto \mathbf{U} \lrcorner \Omega \in \bigwedge^{n-k}(V^*) \,, \quad \mathbf{U} \in \bigwedge^k(V) \,.$$

The duality map on $\bigwedge^k(V)$

Let V be an $n-$dimensional vector space, and consider the ODE

$$\mathbf{u}_x = \mathbf{A}\mathbf{u}\,, \quad \mathbf{u} \in \textstyle\bigwedge^1(V)\,.$$

On any $\bigwedge^k(V)$ there is an induced ODE,

$$\mathbf{U}_x = \mathbf{A}^{(k)}\mathbf{U}\,, \quad \mathbf{U} \in \textstyle\bigwedge^k(V)\,.$$

Fix the volume form $\Omega \in \bigwedge^n(V^*)$. The duality map

$$\mathbf{U} \mapsto \mathbf{U} \lrcorner\, \Omega \in \textstyle\bigwedge^{n-k}(V^*)$$

takes elements from $\bigwedge^k(V)$ to elements in $\bigwedge^{n-k}(V^*)$.

To deduce the induced equation on $\bigwedge^{n-k}(V^*)$, let

$$\mathbf{V} = \mathbf{U} \lrcorner\, \Omega\,,$$

then

$$\mathbf{V}_x = \mathbf{U}_x \lrcorner\, \Omega = \mathbf{A}^{(k)}\mathbf{U} \lrcorner\, \Omega\,.$$

But, for any $\mathbf{W} \in \bigwedge^{n-k}(V)$,

$$\begin{aligned}
\langle \mathbf{A}^{(k)}\mathbf{U} \lrcorner\, \Omega, \mathbf{W}\rangle_{n-k} &= \langle \Omega, \mathbf{A}^{(k)}\mathbf{U} \wedge \mathbf{W}\rangle_n \\
&= \langle \Omega, \mathbf{A}^{(k)}\mathbf{U} \wedge \mathbf{W}\rangle_n + \langle \Omega, \mathbf{U} \wedge \mathbf{A}^{(k)}\mathbf{W}\rangle_n - \langle \Omega, \mathbf{U} \wedge \mathbf{A}^{(k)}\mathbf{W}\rangle_n \\
&= \tau\langle \Omega, \mathbf{U} \wedge \mathbf{W}\rangle_n - \langle \mathbf{U} \lrcorner\, \Omega, \mathbf{A}^{(k)}\mathbf{W}\rangle_n \\
&= \tau\langle \mathbf{U} \lrcorner\, \Omega, \mathbf{W}\rangle_n - \langle (\mathbf{A}^{(k)})^T\mathbf{U} \lrcorner\, \Omega, \mathbf{W}\rangle_n
\end{aligned}$$

Hence

$$\mathbf{V}_x = -(\mathbf{A}^{(k)})^T\mathbf{V} + \tau\mathbf{V}\,, \quad \mathbf{V} \in \textstyle\bigwedge^{n-k}(V^*)\,.$$

Additonal properties of the interior product

Main formula: for $\mathbf{U} \in \bigwedge^{\ell}(V^*)$, $\mathbf{v} \in \bigwedge^{\ell-k}(V)$ and $\mathbf{u} \in \bigwedge^{k}(V)$,

$$\langle \mathbf{u} \lrcorner \mathbf{U}, \mathbf{v} \rangle_{\ell-k} = \langle \mathbf{U}, \mathbf{u} \wedge \mathbf{v} \rangle_{\ell} \,.$$

Further properties

$$\mathbf{u} \lrcorner \mathbf{v} = \langle \mathbf{v}, \mathbf{u} \rangle_1 \quad \mathbf{u} \in \bigwedge^1(V)\,, \quad \mathbf{v} \in \bigwedge^1(V^*)$$

For any scalars c_1 and c_2,

$$\mathbf{u} \lrcorner (c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) = c_1\, \mathbf{u} \lrcorner \mathbf{v}_1 + c_2 \mathbf{u} \lrcorner \mathbf{v}_2)\,.$$

For $\mathbf{U} \in \bigwedge^{k}(V^*)$ and $\mathbf{V} \in \bigwedge^{\ell}(V^*)$,

$$\mathbf{u} \lrcorner (\mathbf{U} \wedge \mathbf{V}) = (\mathbf{u} \lrcorner \mathbf{U}) \wedge \mathbf{V} + (-1)^k \mathbf{U} \wedge (\mathbf{u} \lrcorner \mathbf{V})\,.$$

Remark. One can also talk about right interior products versus left interior products, and interior products taking $\bigwedge(V^*)$ to $\bigwedge(V)$. They are all defined using the dual pairing.

$\bigwedge(V)$ for four-dimensional vector spaces V

Let V be a $4-$dimensional complex vector space. The total exterior algebra is

$$\begin{array}{rcll} \bigwedge^0(V) & = & \mathbb{C} & 1-\text{dimensional} \\ \bigwedge^1(V) & = & V = \mathrm{span}\{\xi_1, \xi_2, \xi_3, \xi_4\} & 4-\text{dimensional} \\ \bigwedge^2(V) & = & \mathrm{span}\{\xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \ldots\} & 6-\text{dimensional} \\ \bigwedge^3(V) & = & \mathrm{span}\{\xi_1 \wedge \xi_2 \wedge \xi_3, \ldots\} & 4-\text{dimensional} \\ \bigwedge^4(V) & = & \mathrm{span}\{\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4\} & 1-\text{dimensional} \end{array}$$

and $\bigwedge^k(V^*)$, $k = 0, \ldots, 4$, with $\bigwedge^1(V^*) = V^* = \mathrm{span}\{\eta^1, \ldots, \eta^4\}$.

Example: consider the boundary-value problem

$$\mathbf{u}_x = \mathbf{A}(x, \lambda)\mathbf{u}\,, \quad \mathbf{u} \in V\,, \quad -1 < x < 1\,,$$

with homogeneous boundary conditions

$$\langle \mathbf{a}^j, \mathbf{u}(-1, \lambda)\rangle_1 = 0\,, \quad \langle \mathbf{b}^j, \mathbf{u}(+1, \lambda)\rangle_1 = 0\,, \quad j = 1, 2\,.$$

This system has natural two-dimensional subspaces, defined by the boundary conditions. Integrate the induced system on $\bigwedge^2(V)$

$$\mathbf{U}_x = \mathbf{A}^{(k)}(x, \lambda)\mathbf{U}\,, \quad \mathbf{U} \in \bigwedge^2(V)\,.$$

When integrating from $x = -1$ to any value of x, what is the initial condition? Let

$$W = \{\, \mathbf{u} \in \bigwedge^1(V) \;:\; \langle \mathbf{a}^1, \mathbf{u}\rangle_1 = 0 \,\}\,.$$

W is a 2D subspace of V. The starting vector is

$$\mathbf{U}(-1, \lambda) = \Sigma$$

where Σ is any *Grassmann representative* for the subspace W.

A Grassmann representative of a subspace

Let V be an $n-$dimensional complex vector space with basis $\{\xi_1, \ldots, \xi_n\}$, and suppose W is a $k-$dimensional subspace with basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$. The subspace is an equivalence class of bases.

$\{\widetilde{\mathbf{w}}_1, \ldots, \widetilde{\mathbf{w}}_k\}$ is also a basis for W if

$$[\widetilde{\mathbf{w}}_1 \mid \cdots \mid \widetilde{\mathbf{w}}_k] = [\mathbf{w}_1 \mid \cdots \mid \mathbf{w}_k]\mathbf{m}$$

where $\mathbf{m}$ is any $k \times k$ invertible matrix, i.e. $\mathbf{m} \in G\ell(k, \mathbb{C})$.

A *Grassmann representative* of W is

$$\Sigma = \{\, C\,\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k\,, \quad C \in \mathbb{C}\,\}$$

since a change of basis for W induces a scalar multiple of the Grassmann representative. For example

$$\widetilde{\mathbf{w}}_1 \wedge \cdots \wedge \widetilde{\mathbf{w}}_k = \det(\mathbf{m})\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_k$$

In other words $k-$dimensional subspaces of V are associated with lines in $\bigwedge^k(V)$. It is sometimes said that Σ is the *Plücker line* representing the subspace W.

Decomposability and the Grassmannian $G_{2,4}$

An arbitrary element $\mathbf{U} \in \bigwedge^2(V)$ does not necessarily represent a two-dimensional subspace of V. If $\mathbf{U}$ represents a $2-$dimensional subspace of V it is called *decomposable* and can be expressed in the form

$$\mathbf{U} = \mathbf{u} \wedge \mathbf{v} \quad \text{for some} \quad \mathbf{u}, \mathbf{v} \in \textstyle\bigwedge^1(V)\,.$$

It is a result from algebraic geometry that $\mathbf{U} \in \bigwedge^2(V)$ for any vector space V is decomposable if and only if

$$\mathbf{U} \wedge \mathbf{U} = 0$$

Now, $\mathbf{U} \wedge \mathbf{U} \in \bigwedge^4(V)$ and so

$$\begin{aligned} \mathbf{U} \wedge \mathbf{U} &= \langle \Omega, \mathbf{U} \wedge \mathbf{U} \rangle_4 \mathsf{Vol} \\ &= \langle \mathbf{U} \lrcorner \Omega, \mathbf{U} \rangle_2 \mathsf{Vol} \\ &:= I(\mathbf{U}) \mathsf{Vol}\,. \end{aligned}$$

The manifold $I^{-1}(0)$ is the *Grassmannian*, $G_{2,4}$. The manifold of two dimensional subspaces of V. It is an invariant manifold of the induced ODE on $\bigwedge^2(V)$ since

$$\begin{aligned} I_x &= \langle \mathbf{U}_x \lrcorner \Omega, \mathbf{U} \rangle_2 + \langle \mathbf{U} \lrcorner \Omega, \mathbf{U}_x \rangle_2 \\ &= \langle \mathbf{A}^{(2)}\mathbf{U} \lrcorner \Omega, \mathbf{U} \rangle_2 + \langle \mathbf{U} \lrcorner \Omega, \mathbf{A}^{(2)}\mathbf{U} \rangle_2 \\ &= \langle \Omega, \mathbf{A}^{(2)}\mathbf{U} \wedge \mathbf{U} \rangle_4 + \langle \Omega, \mathbf{U} \wedge \mathbf{A}^{(2)}\mathbf{U} \rangle_4 \\ &= \mathrm{Tr}(\mathbf{A}) \langle \mathbf{U} \lrcorner \Omega, \mathbf{U} \rangle_2 \\ &= \tau I\,. \end{aligned}$$

Hence $I_x = 0$ for all x if $I = 0$ for some value of x.

The Grassmannian $G_{2,4}$ and the manifold $S^2 \times S^2$

In standard coordinates,

$$\begin{aligned} \omega_1 &= \xi_1 \wedge \xi_2\,, \quad \omega_2 = \xi_1 \wedge \xi_3\,, \quad \omega_3 = \xi_1 \wedge \xi_4\,, \\ \omega_4 &= \xi_2 \wedge \xi_3\,, \quad \omega_5 = \xi_2 \wedge \xi_4\,, \quad \omega_6 = \xi_3 \wedge \xi_4\,. \end{aligned}$$

and any $\mathbf{U} \in \bigwedge^2(V)$ can be expressed in the form

$$\mathbf{U} = \sum_{j=1}^{6} U_j \omega_j\,.$$

In these coordinates the Grassmannian is associated with the quadric

$$I = U_1 U_6 - U_2 U_5 + U_3 U_4\,.$$

Consider the following alternative basis for $\bigwedge^2(V)$

$$\begin{aligned} \omega_1 &= \xi_1 \wedge \xi_2 - \xi_3 \wedge \xi_4\,, \quad \omega_2 = \xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_4\,, \\ \omega_3 &= \xi_1 \wedge \xi_4 - \xi_2 \wedge \xi_3\,, \quad \omega_4 = \xi_1 \wedge \xi_4 + \xi_2 \wedge \xi_3\,, \\ \omega_5 &= -\xi_1 \wedge \xi_3 + \xi_2 \wedge \xi_4\,, \quad \omega_6 = \xi_1 \wedge \xi_2 + \xi_3 \wedge \xi_4\,. \end{aligned} \tag{4}$$

The quadric I is transformed to

$$I = U_1^2 + U_2^2 + U_3^2 - U_4^2 - U_5^2 - U_6^2 = 0\,,$$

and so $I = 0$ corresponds to $U_1^2 + U_2^2 + U_3^2 = U_4^2 + U_5^2 + U_6^2$. By a suitable scaling of the magnitude, these coordinates illustrate the result in algebraic geometry that when V is a *real* vector space, $G_{2,4} = S^2 \times S^2$.

Finding a basis for a decomposable $2-$vector

For illustration, take $V = \mathbb{R}^4$ with the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. Let $\mathbf{U}$ be a *decomposable* element of $\bigwedge^2(\mathbb{R}^4)$. If $\mathbf{U} = \sum U_j\omega_j$ with the standard basis for $\bigwedge^2(\mathbb{R}^4)$, then $U_1U_6 - U_2U_5 + U_3U_4 = 0$. Another way to characterize decomposability is the given by

Lemma. *A nonzero* $\mathbf{U} \in \bigwedge^2(\mathbb{R}^4)$ *is decomposable if and only if there exists linearly independent vectors* $\mathbf{a}, \mathbf{b} \in \bigwedge^1(\mathbb{R}^4)$ *such that* $\mathbf{U} \wedge \mathbf{a} = \mathbf{U} \wedge \mathbf{b} = 0$.

Let $\mathbf{a} \in \bigwedge^1(V)$, then $\mathbf{a} \wedge \mathbf{U} \in \bigwedge^3(V)$. Taking the standard basis for $\bigwedge^3(V)$,

$$\mathbf{a} \wedge \mathbf{U} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} U_4 & -U_2 & U_1 & 0 \\ U_5 & -U_3 & 0 & U_1 \\ U_6 & 0 & -U_3 & U_2 \\ 0 & U_6 & -U_5 & U_4 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

For non-triviality, $\mathbf{a} \neq 0$, and hence the determinant of the coefficient matrix is required to vanish,

$$\det(\mathbf{M}) = (U_1U_6 - U_2U_5 + U_3U_4)^2 , \quad \mathbf{M} := \begin{bmatrix} U_4 & -U_2 & U_1 & 0 \\ U_5 & -U_3 & 0 & U_1 \\ U_6 & 0 & -U_3 & U_2 \\ 0 & U_6 & -U_5 & U_4 \end{bmatrix} .$$

$\mathbf{U}$ is decomposable if and only if $\det(\mathbf{M}) = 0$, recovering the usual condition for decomposability.

How to find a basis for the decomposable form ?

Finding a basis for a decomposable $2-$vector

How to find a basis for the decomposable form $\mathbf{U} \in \bigwedge^2(V)$, with V four dimensional ?

The subspace is given by $\mathrm{Ker}(\mathbf{M})$. The kernel of $\mathbf{M}$ is spanned by $\{\phi_1, \phi_2, \phi_3, \phi_4\}$, where

$$\phi_1 = \begin{pmatrix} -U_3 \\ -U_5 \\ -U_6 \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} U_2 \\ U_4 \\ 0 \\ -U_6 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} -U_1 \\ 0 \\ U_4 \\ U_5 \end{pmatrix}, \quad \phi_4 = \begin{pmatrix} 0 \\ -U_1 \\ -U_2 \\ -U_3 \end{pmatrix}.$$

These vectors satisfy

$$\begin{aligned} \phi_1 \wedge \phi_2 &= U_6\,\mathbf{U}\,, \quad \phi_1 \wedge \phi_3 = -U_5\,\mathbf{U}\,, \quad \phi_1 \wedge \phi_4 = U_3\,\mathbf{U}\,, \\ \phi_2 \wedge \phi_3 &= U_4\,\mathbf{U}\,, \quad \phi_2 \wedge \phi_4 = -U_2\,\mathbf{U}\,, \quad \phi_3 \wedge \phi_4 = U_1\,\mathbf{U}\,. \end{aligned}$$

For a *fixed* nonzero $\mathbf{U} \in \bigwedge^2(\mathbb{R}^4)$ it is straightforward to find a basis for the two-dimensional subspace. For example, if $U_6 \neq 0$ then take

$$\mathbf{U} = \mathbf{u} \wedge \mathbf{v} \quad \text{with} \quad \mathbf{u} = \frac{1}{U_6}\phi_1\,, \quad \mathbf{v} = \phi_2\,.$$

Now suppose $\mathbf{U}$ depends on x or λ. How can one construct a smoothly varying basis ? Theoretically the existence is assured under suitable hypotheses, but how to construct this smoothly varying basis numerically?

Making the Grassmannian $G_{2,4}$ attracting

For ODEs on the real line of the form arising in spectral problems for pulses, the Grassmannian can be made attractive in a natural way. When the Grassmannian is always attractive, one has greater freedom in choosing the numerical integration scheme.

Consider the integration of $\mathbf{U}^+(x,\lambda)$ with $V = \mathbb{C}^4$

$$\frac{d}{dx}\mathbf{U}^+ = \mathbf{A}^{(2)}(x,\lambda)\mathbf{U}^+ \quad \mathbf{U}^+ \in \bigwedge^2(V) \quad L > x > 0$$

Introduce the transformation

$$\mathbf{U}^+(x,\lambda) = \mathrm{e}^{\sigma_+(\lambda)x}\,\widehat{\mathbf{U}}^+(x,\lambda)$$

where $\sigma_+(\lambda)$ is the sum of the eigenvalues of $\mathbf{A}_\infty(\lambda)$ with negative real part. Then $\widehat{\mathbf{U}}^+$ satisfies

$$\frac{d}{dx}\widehat{\mathbf{U}}^+ = [\mathbf{A}^{(2)}(x,\lambda) - \sigma_+(\lambda)\mathbf{I}]\widehat{\mathbf{U}}^+ \qquad L > x > 0$$

The Grassmanian is still an invariant manifold of this equation: *when τ is constant, the Grassmannian is attracting.* Let

$$\widehat{I} = \langle \widehat{\mathbf{U}}^+ \lrcorner\, \Omega, \widehat{\mathbf{U}}^+\rangle_2\,.$$

Then

$$\widehat{I}_x = \tau\widehat{I} - 2\sigma_+\widehat{I}\,.$$

If τ is constant, then $\tau = \sigma_+ + \sigma_-$ with $\mathrm{Re}(\sigma_+) < 0$ and $\mathrm{Re}(\sigma_-) > 0$. Hence

$$\widehat{I}_x = (\sigma_- - \sigma_+)\widehat{I} = (|\sigma_-| + |\sigma_+|)\widehat{I}$$

and so $\widehat{I}$ is strictly decreasing when integrating from $x = L$ to $x = 0$. A similar argument can be constructed for integration on the interval $-L < x < 0$. In this latter case

$$\widehat{I}_x = -(|\sigma_+| + |\sigma_-|)\widehat{I}\,.$$

Forcing the Grassmannian $G_{2,4}$ to be attracting?

Consider the pair of equations

$$\begin{aligned} \mathbf{U}_x &= \mathbf{A}^{(2)}(x,\lambda)\mathbf{U}\,, \quad \mathbf{U}\in\textstyle\bigwedge^2(V) \\ I_x &= \tau(x)\,I\,, \quad I = \langle \mathbf{U} \lrcorner\, \Omega, \mathbf{U}\rangle_2\,. \end{aligned}$$

Using an idea of ASCHER & REICH we can try to *force* the Grassmannian to be attracting. Replace the above equation by

$$\mathbf{U}_x = \mathbf{A}^{(2)}(x,\lambda)\mathbf{U} + \gamma\, I\,\mathbf{U}\,, \quad \mathbf{U}\in\textstyle\bigwedge^2(V)$$

with γ some constant to be determined. The modified equation for I is

$$I_x = \tau(x)\,I + 2\gamma\,I^2\,.$$

If $\tau(x)=0$ then the idea fails since $I_x = 2\gamma I^2$ blows up.

If $\tau(x)\neq 0$ is there a choice of γ such that $I=0$ is attracting?

Suppose the exponential part is also subtracted off as well:

$$\mathbf{U}^+(x,\lambda) = \mathrm{e}^{\sigma_+(\lambda)x}\,\widehat{\mathbf{U}}^+(x,\lambda)$$

where $\sigma_+(\lambda)$ is the sum of the eigenvalues of $\mathbf{A}_\infty(\lambda)$ with negative real part. Then $\widehat{\mathbf{U}}^+$ satisfies

$$\frac{d}{dx}\widehat{\mathbf{U}}^+ = [\mathbf{A}^{(2)}(x,\lambda) - \sigma_+(\lambda)\mathbf{I}]\widehat{\mathbf{U}}^+ + \gamma\,I\widehat{\mathbf{U}}^+ \qquad L > x > 0$$

Then

$$\widehat{I}_x = \tau(x)\widehat{I} - 2\sigma_+\widehat{I} + 2\gamma\widehat{I}^2\,.$$

or

$$\widehat{I}_x = (\tau(x)-\tau_\infty)\widehat{I} + (\sigma_- - \sigma_+)\widehat{I} + 2\gamma\widehat{I}^2\,.$$

Can γ be chosen so that $I=0$ is attracting when τ is nonconstant?

The Hodge star operator

The Hodge star operator can be introduced when the vector space has both an *orientation* and an *inner product*, denoted by $[\![\cdot,\cdot]\!]$ (or $[\![\cdot,\cdot]\!]_1$). It enables one to relate the spaces $\bigwedge^k(V)$ and $\bigwedge^{n-k}(V)$ and $\bigstar : \bigwedge^{n-k}(V) \to \bigwedge^k(V)$ can be defined by

$$\mathbf{V} \wedge \mathbf{U} = [\![\bigstar\mathbf{V}, \mathbf{U}]\!]_k \mathsf{Vol}\,,$$

where $[\![\cdot,\cdot]\!]_k$ is the induced inner product on $\bigwedge^k(V)$.

Properties of Hodge star

- $\mathbf{U} \wedge \bigstar\mathbf{V} = \mathbf{V} \wedge \bigstar\mathbf{U}$ for $\mathbf{U}, \mathbf{V} \in \bigwedge^k(V)$
- $\bigstar\bigstar\mathbf{U} = (-1)^{k(n-k)}$ when $\mathbf{U} \in \bigwedge^k(V)$
- $[\![\mathbf{U}, \mathbf{V}]\!]_{n-k} = [\![\bigstar\mathbf{U}, \bigstar\mathbf{V}]\!]_k$

Another way to define the Hodge star operator, which shows its connection with the duality operator is

$$[\![\bigstar\mathbf{U}, \mathbf{V}]\!]_k = [\![\Omega, \mathbf{U} \wedge \mathbf{V}]\!]_n \quad \text{where} \quad \mathbf{U} \in \textstyle\bigwedge^{n-k}(V)\,, \quad \mathbf{V} \in \textstyle\bigwedge^k(V)\,,$$

and $\Omega \in \bigwedge^n(V)$ is a volume form.

Another identity which is useful for constructing induced systems is

$$\bigstar\mathbf{A}^{(k)} + (\mathbf{A}^{(n-k)})^T\bigstar = \mathrm{Tr}(\mathbf{A})\bigstar$$

With the Hodge star operator the Grassmannian $G_{2,4}$ is defined by the zero set of the quadric

$$I(\mathbf{U}) = [\![\bigstar\mathbf{U}, \mathbf{U}]\!]_2\,.$$

Stability of pulse solutions of the cGL equation

Consider the complex Ginzburg-Landau equation

$$\mathcal{A}_t = b_1\,\mathcal{A}_{xx} + b_2\,\mathcal{A} + b_3\,|\mathcal{A}|^2\mathcal{A}\,, \quad x \in \mathbb{R}\,.$$

Suppose there exists a pulse solution. An example is the Hocking-Stewartson pulse, which exists for a subset of values of b_1, b_2 and b_3 and has the form:

$$\mathcal{A}(x,t) = a_0 L\,\mathrm{e}^{i\nu t}(\mathrm{sech}\,\alpha x)^{1+i\omega}$$

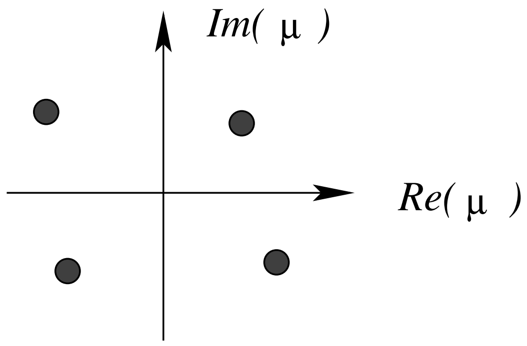
Are such solutions stable? The LSE is

$$\mathcal{B}_t = b_1\,\mathcal{B}_{xx} + b_2\,\mathcal{B} + 2b_3\,|\mathcal{A}|^2\mathcal{B} + b_3\mathcal{A}^2\overline{\mathcal{B}}\,,$$

Introduce real coordinates $\mathcal{B} = \hat{u}_1 + i\,\hat{u}_2$ and a spectral ansatz $\hat{u}(x,t) = \mathrm{Re}(\mathrm{e}^{\lambda t}u(x))$, the result is a system of ODEs which can be written as a first-order system

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\,\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{C}^4\,.$$

where $\lim_{x\to\pm\infty}\mathbf{A}(x,\lambda) = \mathbf{A}_\infty(\lambda)$ and the spectrum of the matrix $\mathbf{A}_\infty(\lambda)$ (for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$) is of the form



Integrating the linearized CGL on $\bigwedge^2(\mathbb{C}^4)$

The system

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\,\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{C}^4\,.$$

with a 2×2 splitting of $\mathbf{A}_\infty(\lambda)$ generates induced systems

$$\mathbf{U}_x = \mathbf{A}^{(2)}(x,\lambda)\,\mathbf{U}\,, \quad \mathbf{V}_x = \mathbf{A}^{(2)}(x,\lambda)\,\mathbf{V}\,,$$

where $\mathbf{A}^{(2)}(x,\lambda)$ and $\mathbf{A}^{(4-2)}(x,\lambda)$ are 6×6 matrices.

- Integrate $\mathbf{U}$ equation on $\bigwedge^2(\mathbb{C}^4)$ from $x = -L_\infty$ to $x = 0$.
- Integrate $\mathbf{V}$ equation on $\bigwedge^2(\mathbb{C}^4)$ from $x = L_\infty$ to $x = 0$.
- **Starting vectors.** At $x = -L_\infty$ choose $\xi^-(\lambda)$ as starting vector. It is the eigenvector corresponding to the *simple* eigenvalue of $\mathbf{A}_\infty^{(2)}(\lambda)$ of largest real part. (With a similar construction at $x = +L_\infty$.)
- At $x = 0$

  $$\widetilde{D}(\lambda) = D(\lambda)\,\mathsf{Vol}\,, \quad D(\lambda) = [\![\bigstar\mathbf{V}, \mathbf{U}]\!]_6\,,$$

  where the Hodge star operator in standard coordinates is

  $$\bigstar\mathbf{V} = \mathbf{S}\overline{\mathbf{V}} \quad \text{with} \quad \mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

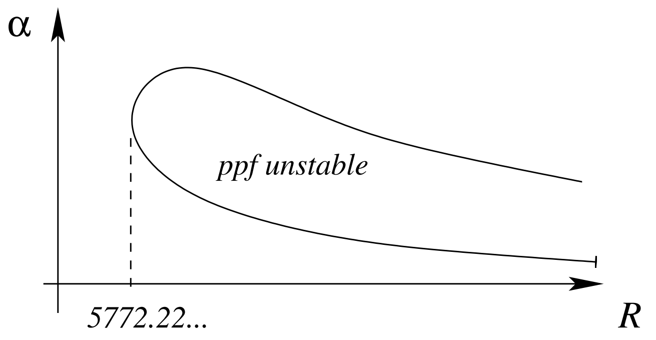
Instability of the Hocking-Stewartson pulse

The cGL equation associated with *spanwise* modulation of PPF (Afendikov & Mielke, 1999) in scaled form

$$\rho\, \mathrm{e}^{i\psi}\, \mathcal{A}_t = \mathcal{A}_{zz} - (1+i\omega)^2\, \mathcal{A} + (1+i\omega)(2+i\omega)\, |\mathcal{A}|^2 \mathcal{A}\,,$$

has the exact Hocking-Stewartson solution

$$\mathcal{A}(z,t) = \widehat{\mathcal{A}}(z) = (\cosh z)^{-1-i\omega}\,.$$

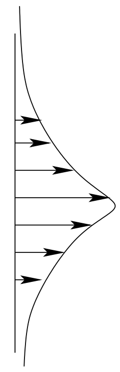


Computed unstable exponent along the neutral curve

α	ρ	ψ	ω	λ
1.0973	.069720	-1.219908	-3.399210	0.650261
1.09	.072627	-1.306955	-4.228913	0.896550
1.08	.074454	-1.356560	-4.946956	1.120561
1.06	.077080	-1.422892	-6.462923	1.628036
1.05	.078173	-1.449108	-7.377601	1.957210
1.03	.080122	-1.494230	-9.803213	2.914473
$\boxed{1.02055}$	.080965	-1.513175	-11.39539	$\boxed{3.609562}$
1.00	.082672	-1.550681	-16.86346	6.400421
0.988	.083603	-1.567483	-21.51673	9.270247

Orr-Sommerfeld equation and Bickley jet

The stability of fluid flows in unbounded domains, such as jets, wakes and mixing layers, is often studied using the Orr-Sommerfeld equation. An example is the *Bickley jet*. Mathematically, the stability problem for the Bickley jet is identical to the stability problem for a solitary wave such as the HS pulse.



In scaled variables, the horizontal velocity field for the Bickley jet takes the form

$$U(x) = \mathrm{sech}^2 x\,, \quad -\infty < x < \infty\,. \tag{5}$$

The *Orr-Sommerfeld equation* can be expressed in the form,

$$\mathbf{U}_x = \mathbf{A}^{(2)}(x,\lambda)\,\mathbf{U}\,, \tag{6}$$

where $\lambda = -\mathrm{i}\alpha c$ is the stability exponent. For interesting parameter values, $\mathbf{A}_\infty(\lambda)$ has two eigenvalues with positive real part.

Computing the neutral curve for the Bickley jet

The eigenvectors of $\mathbf{A}_{\infty}^{(2)}(\lambda)$ associated with $\sigma^{\pm}(\lambda)$ can be explicitly calculated.

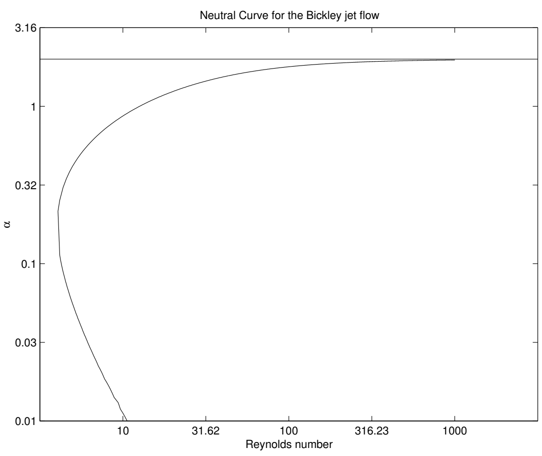
The system (6) is integrated from $x = L_\infty$ to $x = 0$ with starting vector $\xi^+(\lambda)$, and L_∞ is taken to be $L_\infty = 10$ in the results reported here. Call this solution $\mathbf{U}^+(x, \lambda)$. The system (6) is then integrated from $x = -L_\infty$ to $x = 0$ with starting vector $\xi^-(\lambda)$. Call this solution $\mathbf{U}^-(x, \lambda)$.

A value of $\lambda \in \mathbb{C}$ is an eigenvalue if $D(\lambda) = 0$ where

$$D(\lambda) = [\![\mathbf{U}^+, \mathbf{S}\,\mathbf{U}^-]\!]_2\,, \tag{7}$$

where $\mathbf{S}$ is the Hodge star operator in standard coordinates.

Numerical calculation of the neutral curve for the Bickley jet using the above algorithm is shown in the figure, and the curve agrees to graphical accuracy with known results.



Newton's method and continuation were used to compute the points on the neutral curve. The calculations were done using the implicit midpoint rule, which is only second-order accurate, but is clearly adequate for graphical accuracy.

Other examples on $\bigwedge^2(V)$ when V is $4D$

ADROVER ET AL. have computed Lyapunov exponents on wedge spaces. Exterior algebra is useful for computing LEs for low dimensional systems, but will be severely limited for large systems since the dimension of $\bigwedge^k(V)$ increases rapidly with dimension.

For systems of large dimension, orthonormalization and Stiefel manifold integrators are much more practical for computing LEs in general.

- A. ADROVER, S. CERBELLI & M. GIONA. *Exterior algebra-based algorithms to estimate liapunov spectra and stretching statistics in high-dimensional and distributed systems*, Int. J. Bifur. Chaos **12** 353–368 (2002).

Two-dimensional subspaces of 5D vector spaces

Going from $2D$ subspaces of $4D$ vector spaces to $2D$ or $3D$ subspaces of $5D$ vector spaces is predominantly straightforward.

Given an ODE on V, a $5D$ vector space, constructing the induced system on $\bigwedge^2(V)$,

$$\mathbf{U}_x = \mathbf{A}^{(2)}(x,\lambda)\mathbf{U}\,, \quad \mathbf{U} \in \bigwedge^2(V)\,,$$

follows the usual algorithm. The space $\bigwedge^2(V)$ has dimension 10, so the linear operator $\mathbf{A}^{(2)}(x,\lambda)$ is represented by a 10×10 matrix.

A representation for the Hodge star operator is constructed as follows. Starting with an orthonormal basis for $V \cong \mathbb{C}^5$, fix a volume form, for example, the standard volume form

$$\mathsf{Vol} = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_5\,,$$

and let $\mathbf{a}_1,\ldots,\mathbf{a}_{10}$ be the induced orthonormal basis on $\bigwedge^2(\mathbb{C}^5)$. Using a standard lexical ordering, this basis can be taken to be

$$\begin{aligned}
\mathbf{a}_1 &= \mathbf{e}_1 \wedge \mathbf{e}_2\,, \quad \mathbf{a}_2 = \mathbf{e}_1 \wedge \mathbf{e}_3\,, \quad \mathbf{a}_3 = \mathbf{e}_1 \wedge \mathbf{e}_4\\
\mathbf{a}_4 &= \mathbf{e}_1 \wedge \mathbf{e}_5\,, \quad \mathbf{a}_5 = \mathbf{e}_2 \wedge \mathbf{e}_3\,, \quad \mathbf{a}_6 = \mathbf{e}_2 \wedge \mathbf{e}_4\,,\\
\mathbf{a}_7 &= \mathbf{e}_2 \wedge \mathbf{e}_5\,, \quad \mathbf{a}_8 = \mathbf{e}_3 \wedge \mathbf{e}_4\,, \quad \mathbf{a}_9 = \mathbf{e}_3 \wedge \mathbf{e}_5\,,\\
\mathbf{a}_{10} &= \mathbf{e}_4 \wedge \mathbf{e}_5\,.
\end{aligned}$$

Any $\mathbf{U} \in \bigwedge^2(V)$ can be expressed as $\mathbf{U} = \sum_{j=1}^{10} U_j \mathbf{a}_j$.

Now, let $\{\mathbf{b}_1,\ldots,\mathbf{b}_{10}\}$ be an orthogonal basis for $\bigwedge^3(V)$. The action of Hodge star is defined by its action on basis vectors

$$[\![\overline{\star\mathbf{b}_j}, \mathbf{a}_i]\!]_k\, \mathsf{Vol} = \mathbf{b}_j \wedge \mathbf{a}_i\,, \quad \text{for} \quad i,j = 1,\ldots d\,.$$

where the conjugation on the left nullifies the conjugation in the Hermitian inner product.

Using a standard lexical ordering for the basis of $\bigwedge^3(V)$, a matrix representation for the star operator, denoted by $\mathbf{S} \in \mathbb{R}^{10\times 10}$, is then defined by

$$\star \mathbf{b}_j = \sum_{l=1}^{10} S_{jl}\mathbf{a}_l\,, \quad j = 1, \ldots, 10\,, \tag{8}$$

with

$$\mathbf{S} = \begin{bmatrix} \mathbf{0} & \mathbf{S}_1 \\ \mathbf{S}_1 & \mathbf{0} \end{bmatrix}, \quad \mathbf{S}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

Remarks.

- Note that $\mathbf{S}$ is symmetric, isometric, and an involution.
- In contrast to the case of $G_{2,4}$ there does not appear to be any relation between Hodge star and the Grassmannian $G_{2,5}$.
- How to characterize the Grassmannian $G_{2,5}$?

The Grassmannian in $\bigwedge^2(\mathbb{C}^5)$

The set of all decomposable 2-forms is a quadratic submanifold of the projectified ambient space $\bigwedge^2(V) \cong \mathbb{C}^{10}$. This manifold is the Plücker embedding of the Grassmannian $G_{2,5}$. This submanifold has dimension 6. Explicit expressions for the quadrics which define $G_{2,5}$ are obtained as follows.

An element $\mathbf{U} \in \bigwedge^2(V)$ is decomposable if and only if $\mathbf{U} \wedge \mathbf{U} = 0$ (note that this simple characterization of decomposability does not generalize to $k > 2$). Now, $\mathbf{U} \wedge \mathbf{U} \in \bigwedge^4(V)$, hence this form will have 5 components. To obtain explicit expressions, introduce bases for all the spaces involved.

Take the standard basis for $\mathbb{C}^5$, the standard lexicographically ordered bases for $\bigwedge^2(\mathbb{C}^5)$ and $\bigwedge^3(\mathbb{C}^5)$, and the following orthonormal basis for $\bigwedge^4(\mathbb{C}^5)$,

$$\begin{aligned}
\mathbf{c}_1 &= -\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5\,, \quad \mathbf{c}_2 = -\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5\,,\\
\mathbf{c}_3 &= \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 \quad \mathbf{c}_4 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_5\,,\\
\mathbf{c}_5 &= -\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4\,.
\end{aligned}$$

Then for $\mathbf{U} = \sum_{j=1}^{10} U_j \omega_j$,

$$\mathbf{U} \wedge \mathbf{U} = 2 \sum_{j=1}^{5} I_j \mathbf{c}_j\,,$$

where $I_1,\ldots,I_5$ are defined by

$$
\begin{aligned}
I_1 &= -U_1U_9+U_2U_7-U_4U_5\\
I_2 &= -U_5U_{10}+U_6U_9-U_7U_8\\
I_3 &= U_2U_{10}-U_3U_9+U_4U_8\\
I_4 &= U_1U_8-U_2U_6+U_3U_5\\
I_5 &= -U_1U_{10}+U_3U_7-U_4U_6\,.
\end{aligned}
\tag{10}
$$

These quadrics are not all independent, they satisfy the two relations,

$$U_2\,I_3+U_3\,I_4+U_4\,I_5=0 \quad \text{and} \quad U_5\,I_3+U_6\,I_4+U_7\,I_5=0\,.$$

The quadric surface defined by $\mathbf{I}=0$, where $\mathbf{I}=(I_1,\ldots,I_5)\in\mathbb{C}^5$, is the Grassmannian $G_{2,5}$. The 5×10 matrix $\nabla_u\mathbf{I}$ has rank 3.

When $\mathbf{I}$ is evaluated on a solution of an induced system on $\bigwedge^2(V)$ it satisfies the equation

$$\frac{d}{dx}\mathbf{I}=\tau(x,\lambda)\mathbf{I}-\mathbf{A}(x,\lambda)^T\mathbf{I}\,. \tag{11}$$

It is immediate from (11) that – mathematically – if $\mathbf{I}=0$ at the starting value, it is preserved by the differential equation on $\bigwedge^2(V)$: $G_2(\mathbb{C}^5)$ is an invariant manifold of the induced system on $\bigwedge^2(V)$.

On the other hand, numerically these invariants may not be preserved. The vectorfield $\mathbf{I}_x$ is not identically zero, but vanishes in general only when $\mathbf{I}=0$. Hence, the Grassmannian $G_{2,5}$ is a *weak* constraint.

What is the appropriate numerical integrator?

Gauss-Legendre Runge-Kutta methods appear to work for this problem, but theoretical justification is lacking.

Magnus integrators also appear to have natural properties for integration on $\bigwedge^k(V)$.

Stability of solitary waves – fifth order KdV

The fifth-order KdV equation or *Kawahara equation* is a model equation appearing in water waves, plasma physics, etc,

$$u_t + u u_x + \alpha\, u_{xxx} + \beta\, u_{xxxxx} = 0\,,$$

or more generally

$$u_t + \partial_x f(u, u_x, u_{xx}) + \alpha\, u_{xxx} + \beta\, u_{xxxxx} = 0\,.$$

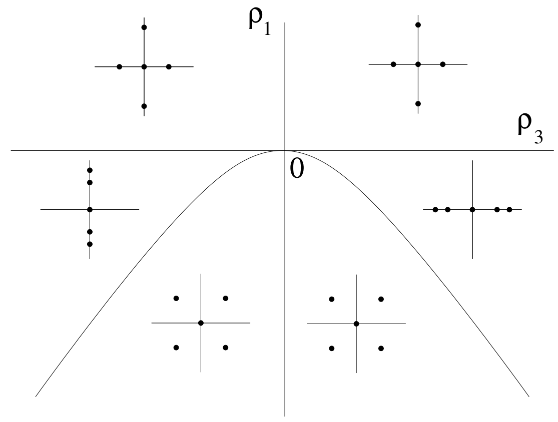
Travelling solitary wave states $u(x,t) = \hat{u}(x - ct)$ satisfy,

$$\beta\, \hat{u}_{xxxx} + \alpha \hat{u}_{xx} + f(\hat{u}, \hat{u}_x, \hat{u}_{xx}) - c\hat{u} = 0\,.$$

The linearization about solitary waves leads to a system of the form

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{C}^5\,, \text{ with } \lim_{x\to\pm\infty} \mathbf{A}(x,\lambda) = \mathbf{A}_\infty(\lambda)\,,$$

and dependent on parameter values, $k = 1$ or $k = 2$ and $n = 5$.



Most interesting case is $k = 2$. Hence, integrate

$$\mathbf{U}_x = \mathbf{A}^{(2)}(x,\lambda)\mathbf{U}\,, \quad x \geq 0 \tag{12}$$

$$\mathbf{V}_x = \mathbf{A}^{(3)}(x,\lambda)\mathbf{V}\,, \quad x \le 0$$

on $\bigwedge^2(\mathbb{C}^5)$ and $\bigwedge^3(\mathbb{C}^5)$ which have dimension 10.

One can also use Hodge duality to simplify:

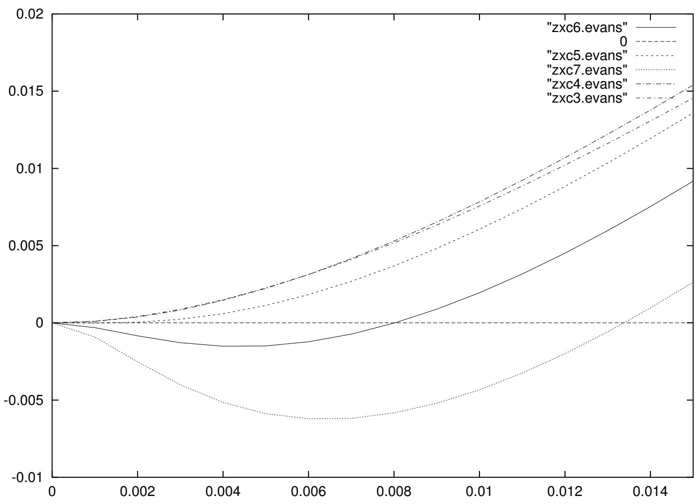
$$\mathbf{S}\mathbf{A}^{(3)} + [\mathbf{A}^{(2)}]^T\mathbf{S} = \tau\,\mathbf{S}\,,$$

and then integrate the adjoint of (12) for $x \le 0$. For the former case, the computable expression for the Evans function is

$$D(\lambda) = [\![\mathbf{S}\mathbf{V}, \mathbf{U}]\!]_2\,.$$

Results: For the classical KdV equation (nonlinear uu_x) we find that all *one-pulse* solitary waves are stable using numerical implementation of Cauchy's Theorem, and stability of multi-pulses in progress.

For the nonlinearity $u^p u_x$ there is an exact solution for one value of $c(p)$, and we find instability of this one pulse for $p > 4.80...$ which is in agreement with an analytical conjecture of Karpman, and suggests that Karpman's conjecture is sharp.



Analyticity of starting vectors

One of the issues that arises when the dimension of V is 5 or greater is that the eigenvalues of the "system at infinity" $\mathbf{A}_\infty(\lambda)$ may not be computable analytically. Hence, for each λ it may be required to obtain the eigenvalues numerically.

This computation does not present any numerical difficulties, but it creates a problem with *numerical analytic continuation*.

The problem reduces to computing a simple eigenvalue $\sigma(\lambda)$ and its eigenvector $\xi(\lambda)$ (and adjoint eigenvector $\eta(\lambda)$) along a path in the complex λ plane of a matrix $\mathbf{A}(\lambda)$ which depends analytically on λ.

Consider two points λ_1 and λ_2 and compute the eigenvalues and eigenvectors of $\mathbf{A}(\lambda_1)$ and $\mathbf{A}(\lambda_2)$. There will be a simple unique eigenvalue $\sigma(\lambda_j)$ of each, and as $\lambda_2 \to \lambda_1$, $\sigma(\lambda_2) \to \sigma(\lambda_1)$. The problem is with the eigenvectors $\xi(\lambda_2)$ and $\xi(\lambda_1)$. This computation will not in general even be continuous.

Hence an algorithm for numerical analytic continuation of eigenvectors is needed.

Analyticity of starting vectors

Pragmatic approach: compute $\eta(\lambda)$ and $\xi(\lambda)$ at isolated values and normalize: $[\![\eta(\lambda), \xi(\lambda)]\!] = 1$. Neither $\xi(\lambda)$ or $\eta(\lambda)$ are analytic – but $D(\lambda)$ is analytic. This approach is special to computing the Evans function. See TJB, DERKS & GOTTWALD (2002).

Elegant approach: integrate the following ODE (and a similar one for $\eta(\lambda)$) along paths in the complex λ plane to obtain analytic starting vectors

$$\begin{bmatrix} \mathbf{A}(\lambda) - \sigma(\lambda)\mathbf{I} & -\xi(\lambda) \\ -\eta(\lambda)^T & 0 \end{bmatrix} \begin{pmatrix} \frac{d\xi}{d\lambda} \\ \frac{d\sigma}{d\lambda} \end{pmatrix} = \begin{pmatrix} -\mathbf{A}'(\lambda)\xi(\lambda) \\ 0 \end{pmatrix}$$

Left hand side is *always* invertible when $\sigma(\lambda)$ is a simple eigenvalue. Hence this system defines a differential equation with analytic right hand side. Solution of this equation provides an analytic path.

What is the appropriate numerical integrator? The coupled system has a quadratic invariant since $[\![\eta, \xi]\!] = 1$.

Several people pointed out at Ustaoset that *Magnus integrators* might be excellent integrators for this problem.

3D subspaces 6D vector spaces

The next interesting case is when V is a six-dimensional vector space, and it is of interest to integrate the induced system for three-dimensional subspaces.

Starting with

$$\mathbf{u}_x = \mathbf{A}(x, \lambda)\mathbf{u}\,, \quad \mathbf{u} \in \bigwedge^1(V)\,,$$

and one is interested integration on

$$\mathbf{U}_x = \mathbf{A}^{(3)}(x, \lambda)\mathbf{U}\,, \quad \mathbf{U} \in \bigwedge^3(V)\,.$$

In principle this is straightforward. The major open question is how to integrate along paths of $3-$dimensional subspaces, that is, along the Grassmannian $G_{3,6}$ which is now a $9-$dimensional submanifold of the $19-$dimensional projective space $\mathbb{P}(\bigwedge^3(V))$.

Open questions.

- Algebraic geometry books show that it is the zero set of a large number of quadrics, not all of which are independent. Find a nice characterization of the Grassmannian.
- Show that $G_{3,6}$ is an invariant manifold of the induced system.
- Choose/design an algorithm to preserve $G_{3,6}$.

Geometry of three forms on $V \cong \mathbb{R}^6$

The space of three forms (or three vectors) has much more geometry than two forms. The simplest case is three forms in six dimensional spaces. For example, HITCHIN shows that there are three equivalence classes of such forms.

Let $\mathbf{U} \in \bigwedge^3(V^*)$ be any three form and let $\mathbf{u} \in V$. Then

$$\mathbf{u} \lrcorner \mathbf{U} \wedge \mathbf{U} \in \bigwedge^5(V^*) .$$

There is a natural duality between $\bigwedge^5(V^*)$ and $V \otimes \bigwedge^6(V^*)$. Define the duality mapping $\varphi : \bigwedge^5(V^*) \to V \otimes \bigwedge^6(V^*)$. Using this duality, define the linear transformation

$$\mathbf{K_U} : V \to V \otimes \bigwedge^6(V^*)$$

by

$$\mathbf{K_U}\mathbf{u} = \varphi(\mathbf{u} \lrcorner \mathbf{U} \wedge \mathbf{U})$$

Then the following function is defined by HITCHIN to discriminate between $3-$forms

$$f(\mathbf{U}) := \frac{1}{6}\mathrm{Tr}(\mathbf{K}_\mathbf{U}^2) .$$

Every three form $\mathbf{U} \in \bigwedge^3(V^*)$ satisfies either

$$f(\mathbf{U}) < 0 , \quad f(\mathbf{U}) = 0 \quad \text{or} \quad f(\mathbf{U}) > 0$$

What are the implications for numerics of this property?

- N.J. HITCHIN. *The geometry of three forms in six dimensions*, J. Diff. Geom. **55** 547–576 (2000).
- See also LYCHAGIN & RUBTSOV (1983), BANOS (2002).

Stability of the Ekman layer with a coupled compliant surface

Linearisation of the 3D Navier Stokes equations in a rotating frame about the Ekman layer coupled to a compliant surface leads to the sixth order ODE

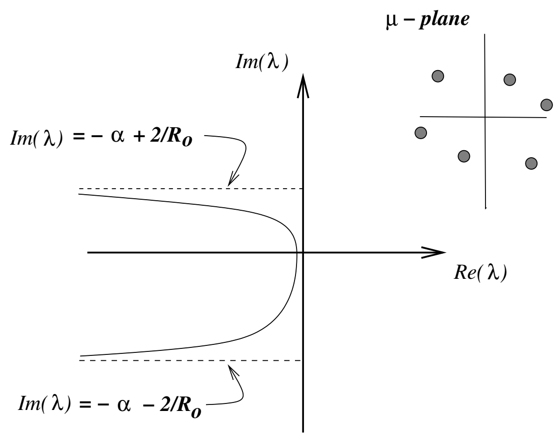
$$\phi_{xxxx} - b(x)\phi_{xx} - a(x)\phi + 2\psi_x = 0$$
$$\psi_{xx} + (\gamma^2 - b(x))\psi - \mathrm{i}\gamma \mathrm{R} V'(x)\phi - 2\phi_x = 0$$

for $0 \le x \le +\infty$ with compliant surface b.c.'s at $x = 0$.

Equivalent to the first order system

$$\mathbf{u}_x = \mathbf{A}(x, \lambda, \mathbf{p})\mathbf{u} \quad \mathbf{u} \in \mathbb{C}^6 \quad 0 < x < +\infty$$

with continuous spectrum and spectrum of $\mathbf{A}_\infty(\lambda)$:

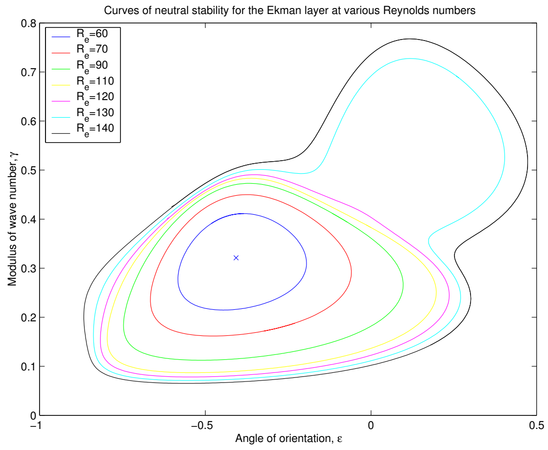


Appropriate space to integrate is $\bigwedge^3(\mathbb{C}^6)$ which has dimension 20. Integrate from $x = L$ for some large L using appropriate eigenvector for starting vector. Then at $x = 0$

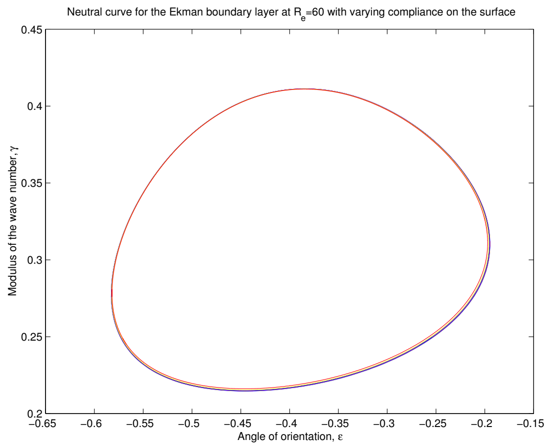
$$D(\lambda) = \langle \mathbf{b}(\lambda), \mathbf{U}(0, \lambda) \rangle_{20}$$

where $\mathbf{b}(\lambda)$ is determined by the boundary conditions at $x = 0$.

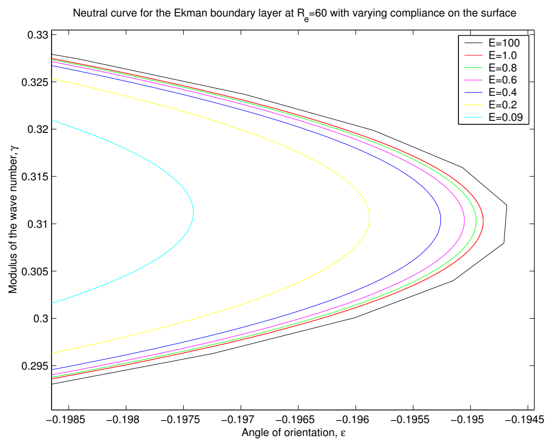
Neutral curves for rigid wall in the wavenumber(γ) – orientation angle (ϵ) plane, with R varying.



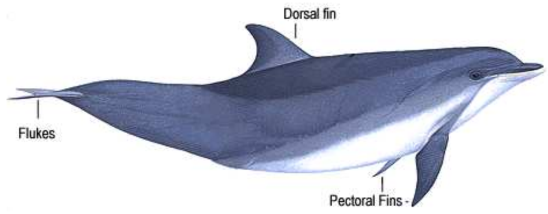
Effect of wall compliance on neutral curves.



Zooming in on the above neutral curves.



Dolphin Hydrodynamics



The idea is to test the effect of compliance on the stability of the boundary layer over the fins of a dolphin, using the attachment line boundary layer coupled to a compliant surface as a model.

Linearising the 3D Navier-Stokes equations about the exact similarity solution for the attachment line boundary layer leads to coupled pair of ODEs one fourth order and one second order, which can be formulated as

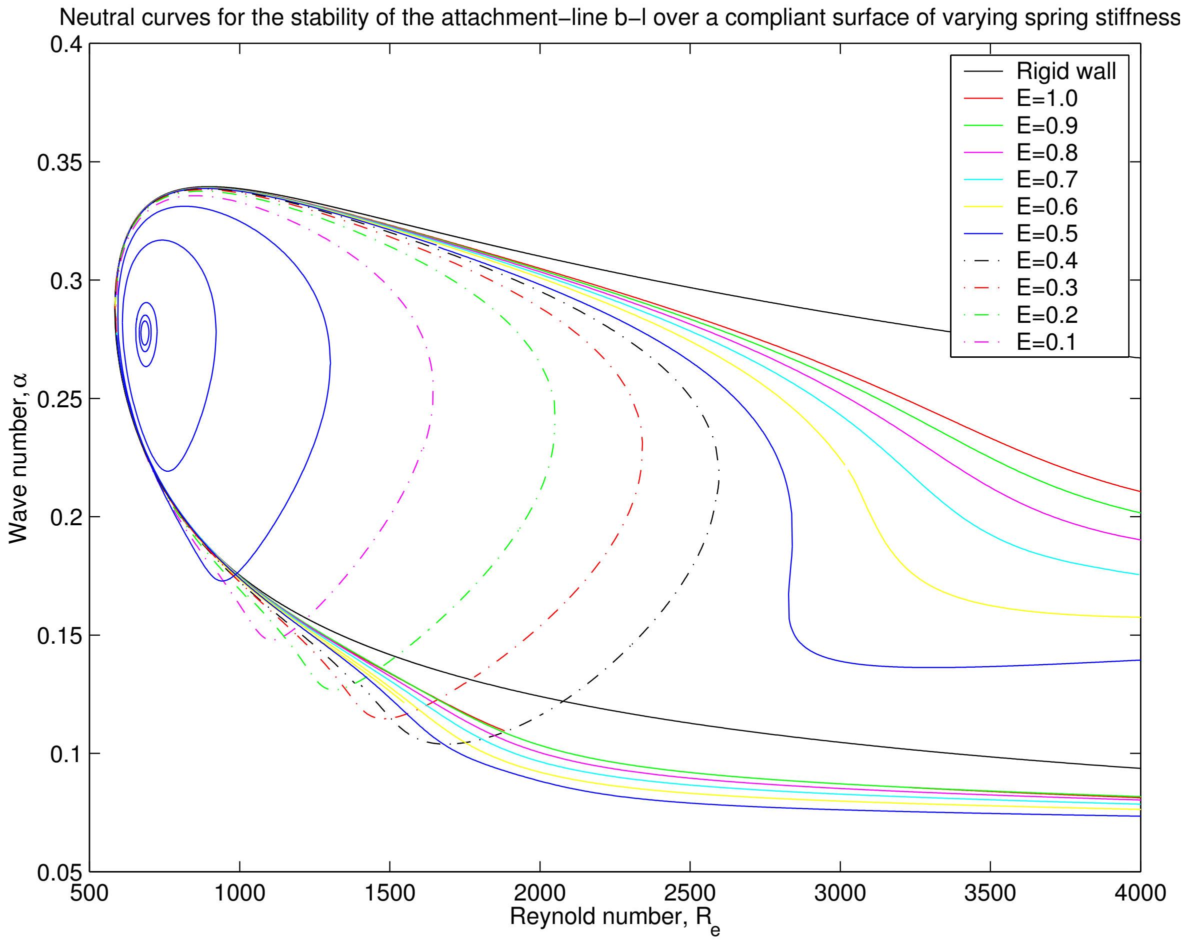
$$\mathbf{u}_x = \mathbf{A}(x, \lambda, \mathbf{p})\mathbf{u} \quad \mathbf{u} \in \mathbb{C}^6 \quad 0 < x < +\infty$$

with boundary conditions at $x = 0$, where $\mathbf{p}$ represents parameters.

The system at infinity $\mathbf{A}_\infty(\lambda)$ has three eigenvalue with positive real part and three with negative, and there are three boundary conditions at $x = 0$. Hence the natural space to integrate on is $\bigwedge^3(\mathbb{C}^6)$.

In this case the effect of compliance is more pronounced, showing a substantial stabilizing effect with increased passive compliance. Indeed, the qualitative affect of stabilization is very similar to the stabililization of the Blasius boundary layer shown by CARPENTER & GARRAD (1985).

In other words the effect of compliance on fins is qualitatively the same as the effect compliance on the body.


Neutral curves for the stability of the attachment−line b−l over a compliant surface of varying spring stiffness
Wave number, α
Reynold number, R_e
Rigid wall
E=1.0
E=0.9
E=0.8
E=0.7
E=0.6
E=0.5
E=0.4
E=0.3
E=0.2
E=0.1
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
500
1000
1500
2000
2500
3000
3500
4000

Geometric Integration and the Grassmannian

Consider the general problem of integrating the induced equation on $k-$dimensional subspaces of $n-$dimensional vector spaces

$$\mathbf{U}_x = \mathbf{A}^{(k)}(x,\lambda)\mathbf{U}\,, \quad \mathbf{U} \in \bigwedge^k(V)\,.$$

For a path of $k-$dimensional subspaces, it is required to stay on the Grassmannian, $G_{k,n}$ for all x.

Open Questions.

- Find a good characterization of $G_{k,n}$
- Show that $G_{k,n}$ is an invariant manifold.
- Design/choose an integrator that preserves $G_{k,n}$
- Under what conditions can the Grassmannian be attracting?

Remark.

- Magnus integrators are showing excellent properties when applied to integration of the ODEs on $\bigwedge^k(V)$.
- N.D. APARICIO, S.J.A. MALHAM & M. OLIVER. *Numerical evaluation of the Evans function by Magnus integration*, BIT (in press, 2005).

Induced systems for $\mathbf{u}_x = \mathbf{A}\mathbf{u} + \mathbf{f}$

For illustration, take $V = \mathbb{C}^4$ and consider the inhomogeneous ODE

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u} + \mathbf{f}(x)\,, \quad \mathbf{u} \in \textstyle\bigwedge^1(V) \tag{13}$$

with $\mathbf{f}$ given. Suppose that with $\mathbf{f} = 0$ it is natural to integrate the homogeneous solution on $\bigwedge^2(V)$.

$$\mathbf{U}_x = \mathbf{A}^{(2)}(x,\lambda)\mathbf{U}\,, \quad \mathbf{U} \in \textstyle\bigwedge^2(V)\,. \tag{14}$$

How does one obtain the solution to the inhomogeneous problem?

NG & REID suggest integration on $\bigwedge^3(V)$,

$$\mathbf{W}_x = \mathbf{A}^{(3)}(x,\lambda)\mathbf{W} + \mathbf{U} \wedge \mathbf{f}\,, \quad \mathbf{U} \in \textstyle\bigwedge^3(V)\,. \tag{15}$$

and then to extract the particular solution of (13) from $\mathbf{W}$.

Open questions.

- How to obtain the particular solution of (13) when it is difficult (impossible) to integrate (13), but possible to integrate (14)?
- What about solvability? If there are non-trivial solutions of the homogeneous equation, then (13) will only be solvable for a subclass of $\mathbf{f}$. But (15) will be solvable for *any* $\mathbf{f}$. How to reconcile these two issues?
- Generalize to inhomogeneous systems when V is any $n-$dimension vector space and the homogeneous euqation is natural on $\bigwedge^k(V)$.

- B.S. NG & W.H. REID, *The compound matrix method for ordinary differential systems*, J. Comp. Phys. **58** (1985) 209-228.

Gap solitary waves, and bifurcation from the continuous spectrum

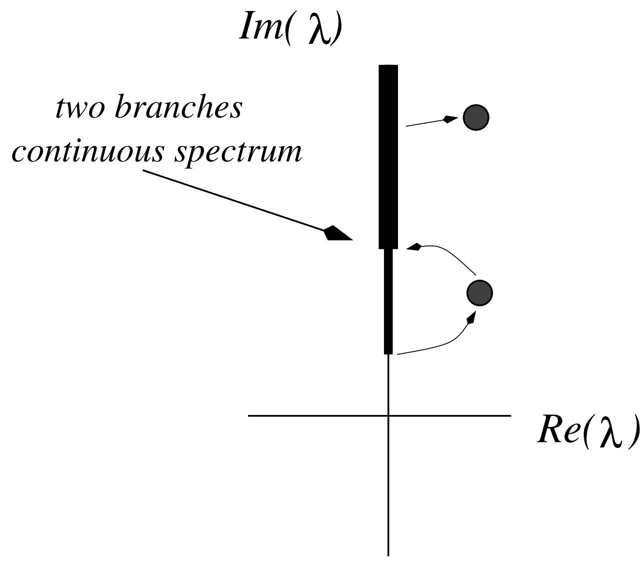
An example of the class of spinor-type equations, nonlinear Dirac equations, gap models in geophysical fluid dynamics, and the massive Thirring models is

$$\begin{aligned} \mathrm{i}(A_t + A_x) + B + (|B|^2 + \rho|A|^2)A &= 0 \\ \mathrm{i}(B_t - B_x) + A + (|A|^2 + \rho|B|^2)B &= 0 \end{aligned}$$

Linearisation about gap solitary waves ... leads to system

$$\mathbf{u}_x = \mathbf{A}(x,\lambda)\mathbf{u} \quad \mathbf{u} \in \mathbb{C}^4$$

This system has been studied by DERKS & GOTTWALD and they show that there are interesting bifurcations from the continuous spectrum.



- G. DERKS & G. GOTTWALD [2004] *A robust numerical method to study oscillatory instability of gap solitary waves*, SIAM J. Applied Dynam. Sys. (in press)

Krein signature in Geometric Integration

Consider the linear constant coefficient Hamiltonian system

$$\mathbf{J}\mathbf{u}_t = \mathbf{A}\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{R}^2$$

with $\det(\mathbf{A}) > 0$. Then the spectrum of $\mathbf{J}^{-1}\mathbf{A}$ is $\{\pm\mathtt{i}\omega\}$ with $\omega^2 = \det(\mathbf{A})$,

$$\mathbf{A}\xi = \mathtt{i}\omega\mathbf{J}\xi\,.$$

The *signature* associated with the eigenspace $\{\pm\mathtt{i}\omega\}$ is

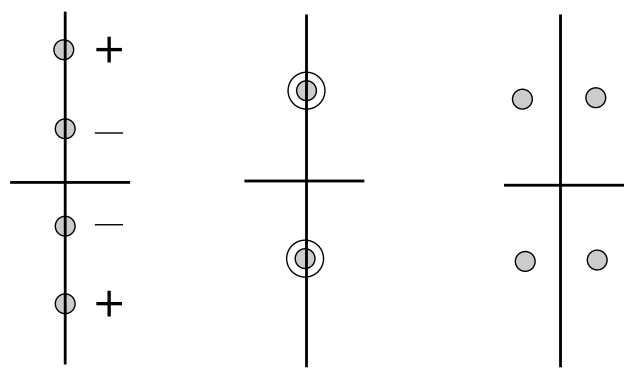
$$s = \mathrm{sign}\left(\mathtt{i}\omega(\overline{\xi},\xi)\right)$$

where ω is the symplectic form, e.g. $\omega(\mathbf{u},\mathbf{v}) = \langle \mathbf{J}\mathbf{u},\mathbf{v}\rangle$. It determines the rotation of the periodic orbit (clockwise or counterclockwise), and can not be reversed by a symplectic change of coordinates.

History.

- Nineteenth Century: WEIERSTRASS, THOMPSON & TAIT.
- Twentieth Century: WILLIAMSON, KREIN, MACKAY.

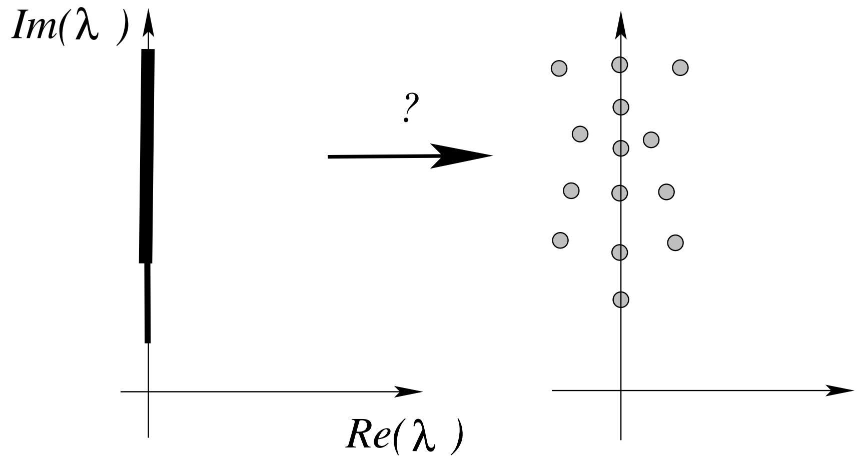
Signature is important for Hamiltonian instabilities. A necessary condition – for a pair of eigenvalues which collide on the imaginary axis to become unstable – is that the eigenvalues have *opposite signature*.



Krein signature in Geometric Integration

Suppose that the exact system has two branches of continuous spectrum and the two branches have opposite Krein signature.

What happens to the spectra under perturbation by discretization?



Open problems.

- Show that the two branches of continuous spectrum in the massive Thirring model have opposite Krein signature.
- Explain the explosion of unstable eigenvalues in the numerical work of BARASHENKOV & ZEMLYANAYA (2000).
- Explain why the use of exterior algebra for this problem as in DERKS & GOTTWALD (2005) does not lead to spurious eigenvalues.
- Geometric integration problem: what are the general principles involved with designing a numerical discretization so that the discretized continuous spectrum lies entirely in the exact continuous spectrum.
- I.V. BARASHENKOV & E.V. ZEMLYANAYA. *Oscillatory instabilities of gap solitons: a numerical study*, Comp. Phys. Comm. **126** 22–27 (2000).

Numerics of the Maslov index

Consider a linear time-dependent Hamiltonian system

$$\mathbf{J}\mathbf{u}_t = \mathbf{A}(t)\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{R}^{2n}$$

and suppose that $\mathbf{A}(t+T) = \mathbf{A}(t)$ for all t. Let $\Phi(t)$ be the fundamental solution matrix. It has a symplectic QR factorization

$$\Phi(t) = \mathbf{Q}(t)\mathbf{R}(t)$$

But every symplectic orthogonal matrix (*orthosymplectic*) is of the form

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{Q}_1(t) & -\mathbf{Q}_2(t) \\ \mathbf{Q}_2(t) & \mathbf{Q}_1(t) \end{bmatrix}\,,$$

with $\mathbf{Q}_j(t)$ $n \times n$ matrices satisfying

$$\mathbf{Q}_1(t) + \mathtt{i}\mathbf{Q}_2(t) \quad \textit{is unitary}\,.$$

Define

$$\mathrm{e}^{\mathtt{i}2\pi\theta(t)} = \det[\mathbf{Q}_1(t) + \mathtt{i}\mathbf{Q}_2(t)]$$

Then $m = \theta(T) - \theta(0)$ is an integer: *the Maslov index*.

Every symplectic periodic orbit has a Maslov index.

The numerical algorithm of LEIMKUHLER & VAN VLECK is designed to compute $\mathbf{Q}_1$ and $\mathbf{Q}_2$, and hence can also be used to compute the Maslov index.

- B.J. LEIMKUHLER & E.S. VAN VLECK. *Orthosymplectic integration of linear Hamiltonian systems*, Numer. Math. **77** 269–282.

The Maslov index on $\bigwedge^2(\mathbb{R}^4)$

Consider a linear time-dependent Hamiltonian system

$$\mathbf{J}\mathbf{u}_t = \mathbf{A}(t)\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{R}^4$$

and suppose that $\mathbf{A}(t+T) = \mathbf{A}(t)$ for all t.

Another way to characterize the Maslov index is as an index of rotation for Lagrangian planes. In the exterior algebra setting this leads to a path in the *Lagrangian Grassmannian*

In standard coordinates, the Lagrangian Grassmannian is the $3-$dimensional submanifold of $\mathbb{P}(\bigwedge^2(\mathbb{R}^2))$ defined by

$$u_1u_6 - u_2u_5 + u_3u_4 = 0 \quad \text{and} \quad u_2 + u_5 = 0\,.$$

In terms of standard coordinates on $\bigwedge^2(\mathbb{R}^4)$ the Maslov index is determined from (see JONES (1988))

$$\mathrm{e}^{2\mathrm{i}\theta(t)} = \frac{u_1 - u_6 - \mathrm{i}u_3 + \mathrm{i}u_4}{u_1 - u_6 + \mathrm{i}u_3 - \mathrm{i}u_4}\,.$$

This expression involves a choice of coordinates, which is required for the numerical integration, but it can be recast in a coordinate-free way.

Apply to compute the Maslov index of solitary waves. This leads to self-adjoint eigenvalue problems of the form

$$\phi_{xxxx} + a\phi_{xx} + b(x)\phi = \lambda\phi\,, \quad x \in \mathbb{R}$$

where $b(x) \to 0$ exponentially as $x \to \pm\infty$.

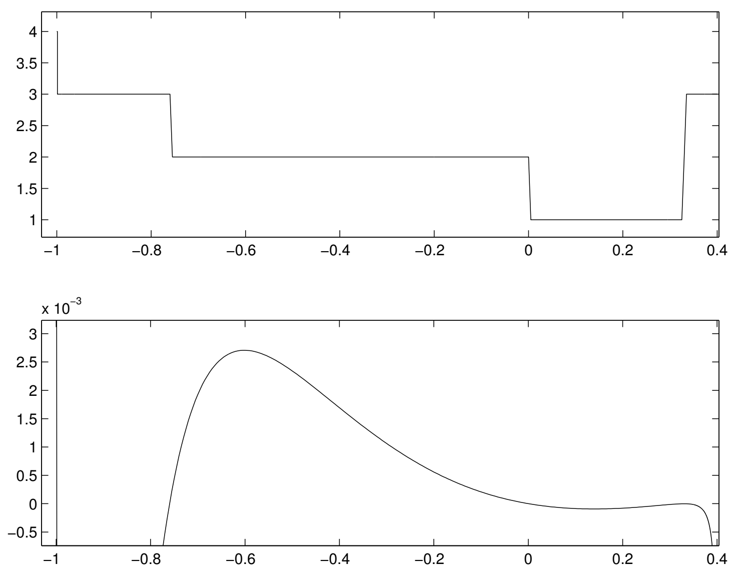
Reformulate as

$$\mathbf{u}_x = \mathbf{J}\mathbf{A}(x)\mathbf{u}\,, \quad \mathbf{u} \in \mathbb{R}^4\,,$$

and then construct the induced system on $\bigwedge^2(\mathbb{R}^4)$. The induced system has the Lagrangian Grassmannian as an invariant manifold. Using the formula, the Maslov index of a solitary wave is defined by

$$\lim_{L\to\infty} \frac{\theta(+L)-\theta(-L)}{2\pi} .$$

An example of the computation is shown below in the upper figure. The lower figure shows the Evans function for the self-adjoint ODE.



One can also derive formulae for the Maslov index on $\bigwedge^3(\mathbb{R}^6)$ and other wedge spaces, and we have computed on $\bigwedge^3(\mathbb{R}^6)$, but still many open theoretical questions.

- TJB, F. CHARDARD & F. DIAS. *Computing the Maslov index of solitary waves*, in preparation (2005).