Degenerate conservation laws, bifurcation of solitary waves

— and criticality of internal waves —

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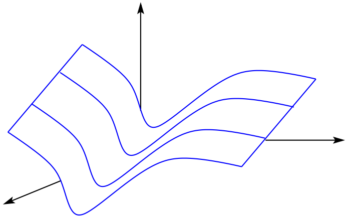

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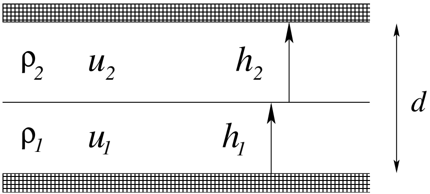
Two-layer flow with a rigid lid

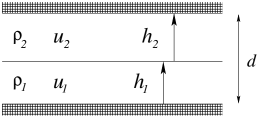


$$
\begin{aligned}
R(h_1, u_1, u_2) &= \tfrac{1}{2}\rho_1 u_1^2 - \tfrac{1}{2}\rho_2 u_2^2 + (\rho_1 - \rho_2) g h_1 \\
Q_1(h_1, u_1, u_2) &= \rho_1 h_1 u_1 \\
Q_2(h_1, u_1, u_2) &= \rho_2 (d - h_1) u_2 \,.
\end{aligned}
$$

Criticality:

$$
F_1^2 + r F_2^2 = 1\,, \quad r = \frac{\rho_2}{\rho_1}\,, \quad F_j^2 := \frac{u_j^2}{g(1-r)h_j}\,.
$$

Criticality and the KdV equation



Near criticality, the appropriate model for weakly nonlinear behaviour is the KdV equation

$$a_1\, A_T + a_2\, AA_X + a_3\, A_{XXX} = 0\,, \quad T = \varepsilon^3 t\,, \quad X = \varepsilon x\,.$$

The coefficients a_1 and a_3 are determined from the dispersion relation associated with linearization about the uniform flow. The coefficient a_2 is generally perceived to the be the difficult coefficient to calculate: it requires evaluation of eigenfunctions, solvability and evaluation of integrals.

Claim: a_2 is the easy coefficient to compute.

Criticality and geometry of $(h_1, u_1, u_2) \mapsto (R, Q_1, Q_2)$

Define the mapping $\mathbf{P}(\mathbf{c}) := (R(\mathbf{c}), Q_1(\mathbf{c}), Q_2(\mathbf{c}))$, with

$$
\begin{aligned}
R(\mathbf{c}) &= \tfrac{1}{2}\rho_1 u_1^2 - \tfrac{1}{2}\rho_2 u_2^2 + (\rho_1 - \rho_2) g h_1 \\
Q_1(\mathbf{c}) &= \rho_1 h_1 u_1 \\
Q_2(\mathbf{c}) &= \rho_2 (d - h_1) u_2 \,,
\end{aligned}
$$

where $\mathbf{c} = (h_1, u_1, u_2)$. Then

$$
\mathrm{D}\mathbf{P}(\mathbf{c}) = \begin{bmatrix} (\rho_1 - \rho_2) g & \rho_1 u_1 & -\rho_2 u_2 \\ \rho_1 u_1 & \rho_1 h_1 & 0 \\ -\rho_2 u_2 & 0 & \rho_2 (d - h_1) \end{bmatrix} ,
$$

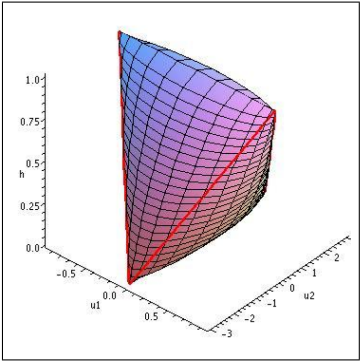
and criticality corresponds to when $f(\mathbf{c}) = 0$ where

$$
f(\mathbf{c}) := \det(\mathrm{D}\mathbf{P}(\mathbf{c})) = \rho_1 \rho_2 (\rho_1 - \rho_2) g h_1 (d - h_1) \left[1 - F_1^2 - r F_2^2\right] ,
$$

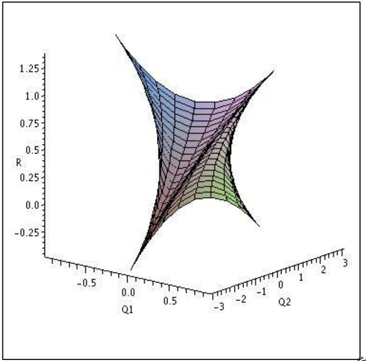
Plot the surface $f(\mathbf{c}) = 0$ and its image in the (R, Q_1, Q_2) plane.

Criticality surfaces for two-layer flow

$\mathfrak{g}$



$\mathfrak{g}^*$

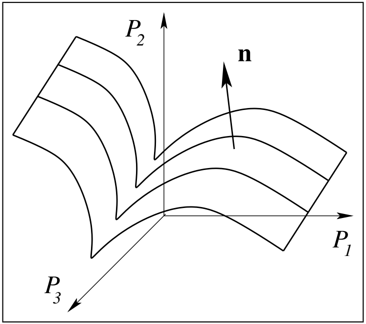


The normal vector on the criticality $\mathbf{P}-$surface

The condition

$$\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0\,,$$

defines a hypersurface in $\mathbf{c}-$space with image in $\mathbf{P}-$space.
There exists $\mathbf{n}$ satisfying $\mathrm{D}\mathbf{P}(\mathbf{c})\mathbf{n} = 0$. It is a normal vector to the $\mathbf{P}-$surface.



Criticality and $\mathrm{d}f(\mathbf{c}) \cdot \mathbf{n}$

Now

$$f(\mathbf{c}) := \det[\mathrm{D}\mathbf{P}(\mathbf{c})] = \mathrm{C}\left[(1-r) - \frac{u_1^2}{gh_1} - r\frac{u_2^2}{gh_2}\right], \quad \mathrm{C} = \rho_1^2\rho_2 g h_1 h_2 .$$

The criticality surface in (h_1, u_1, u_2) space is defined by $f^{-1}(0)$ and a vector $\mathbf{v}$ is tangent to this surface if $\mathrm{d}f \cdot \mathbf{v} = 0$. Now,

$$\mathrm{d}f = \frac{\mathrm{C}}{g}\left(\frac{u_1^2}{h_1^2} - \frac{ru_2^2}{h_2^2}, -\frac{2u_1}{h_1}, -\frac{2ru_2}{h_2}\right),$$

and so

$$\langle \mathrm{d}f, \mathbf{n}\rangle = \frac{3\mathrm{C}}{\rho_1 g}\left(\rho_1\frac{u_1^2}{h_1^2} - \rho_2\frac{u_2^2}{h_2^2}\right).$$

The coefficient a_2 in the KdV equation is proportional to $\langle \mathrm{d}f, \mathbf{n}\rangle$.

Curvature of the **P** mapping and the coefficient a_2

Look at the second derivative of **P** in the direction **n**

$$\frac{d^2}{ds^2}\bigg|_{s=0} \mathbf{P}(\mathbf{c} + s\mathbf{n}) = D^2\mathbf{P}(\mathbf{c})(\mathbf{n}, \mathbf{n}) \, .$$

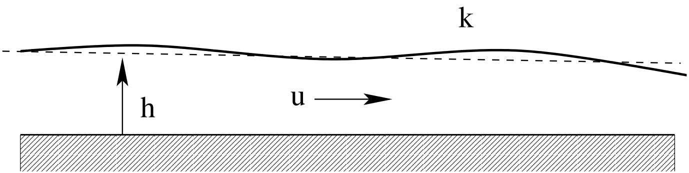
The projection of this second derivative is the coefficient a_2

$$a_2 = \frac{d^2}{ds^2}\bigg|_{s=0} \langle \mathbf{n}, \mathbf{P}(\mathbf{c} + s\mathbf{n}) \rangle$$

modulo a positive multiple, and this second derivative is proportional to $\langle \mathrm{d}f, \mathbf{n} \rangle$.

The coefficent a_2 is determined by the geometry of the uniform flow.

Criticality of Stokes travelling waves
– mean flow, secondary criticality and dark solitary waves



Consider Stokes travelling waves in shallow water coupled to a mean flow (uniform flow). They can be parameterized by

$$(h, u, k) \mapsto (R, Q, B)$$

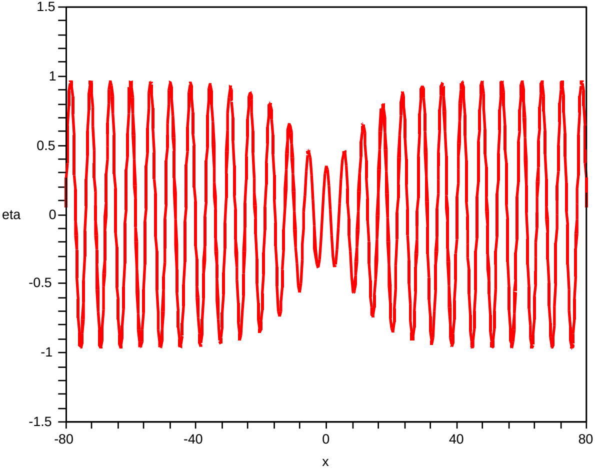
Where R is the Bernoulli function, Q the mass flux and B is the action flux. This family of waves is *critical* (or secondary critical) when

$$\det\left[\frac{\partial(R, Q, B)}{\partial(h, u, k)}\right] = 0\,.$$

Implications of criticality? a class of solitary waves is generated: steady "dark solitary waves".

(cf. B & DONALDSON, J. Fluid Mech. 2006)

Schematic of steady dark solitary waves



κ calc required a few days – versus months/year for direct calc!

Criticality and internal solitary waves

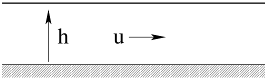
- Two-layer flow with a rigid lid
  - uniform flows = 3D RE, critical surface is 2D
  - $\langle \mathrm{d}f, \mathbf{n} \rangle = 0$ separates solitary waves of elevation from solitary waves of depression.
  - 3D mean flow (uniform flow) coupled to a periodic wave = 4D RE, 3D critical surface, bif. to internal steady DSWs
- Two-layer flow with a free surface
  - uniform flow = 4D RE, critical surface is 3D
  - $\langle \mathrm{d}f, \mathbf{n} \rangle = 0$ is a 2D manifold
  - uniform flow (mean flow) coupled to a periodic wave = 5D RE, 4D critical surface, bif. to internal steady DSWs

Theory predicts manifold of bifurcating solitary waves from each family of degenerate RE. The bifurcating SWs may have exponentially small tails in the case of two layers with free surface.

cf. B & DONALDSON, Phys Fluids (2007), Eur J Mech B/Fluids (2008)

Criticality in hydraulics

In classical hydraulics, a uniform flow with velocity u and depth h,



is said to be critical when $u^2 = gh$ (Froude number unity).
Other characterizations of criticality used in hydraulics such as: *for any fixed* $R = gh + \frac{1}{2}u^2$*, the uniform flow which maximises* $Q = uh$ *(when* $u > 0$*) is critical*.

Generalize criticality to nontrivial flows? (e.g. Benjamin 1971, Gill 1977, Killworth 1992, Clark & Johnson 2001).

New observation: Uniform flows are relative equilibria (RE), and critical uniform flows are degenerate RE – symmetry is central.

Some consequences:
– can generalize criticality to other fluid flows (characterize as RE)
– criticality leads to new solitary waves

Dispersive conservation laws and criticality

Consider conservation laws with dispersive regularization

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = \mathbf{D}\mathbf{U}_{xxx} \,,$$

where $\mathbf{U} \in \mathbb{R}^n$, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth mapping (the flux vector), and $\mathbf{D}$ is an $n \times n$ matrix.

Let $\mathbf{U}_0 \in \mathbb{R}^n$ be any constant vector.

- zero is a simple eigenvalue of $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$, then locally, in the direction $\mathbf{n}$ is KdV dynamics
- if zero is an eigenvalue $\det[\mathrm{D}\mathbf{F}(\mathbf{U}_0)] = 0$ of algebraic multiplicity two but geometric multiplicity one, then locally, in the direction $\mathbf{n}$, is Boussinesq dynamics
- if zero is an eigenvalue $\det[\mathrm{D}\mathbf{F}(\mathbf{U}_0)] = 0$ of algebraic multiplicity two and geometric multiplicity two, then locally, in the direction $\mathbf{n}$, is coupled KdV dynamics

A criticality view of the theory in Hickernell (1983), Helfrich & Pedlosky (1993) and Grimshaw et al (1998,2000,2002).

Degenerate conservation laws

Consider conservation laws with dispersive regularisation

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = \mathbf{D}\mathbf{U}_{xxx}\,.$$

The conservation law is *degenerate* at $\mathbf{U}_0$ if the Jacobian $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ is singular

$$\det[\mathrm{D}\mathbf{F}(\mathbf{U}_0)] = 0 \quad \text{criticality ?}$$

$\mathrm{D}\mathbf{F}$ is not necessarily symmetric. However, for models of stratified (layered) flow,

$$\mathbf{F}(\mathbf{U}) = \mathbf{M}\nabla\mathrm{E}(\mathbf{U})\,, \quad \mathbf{M}^T = \mathbf{M}\,, \quad \mathbf{M} \text{ invertible}$$

with $\mathbf{M}^{-1}\mathbf{D}$ symmetric, where $\nabla\mathrm{E}$ is the specific energy and mass flux vector (was called $\mathbf{P}$ earlier).

KdV model near criticality

Let $X = \varepsilon x$ and $T = \varepsilon^3 t$ and decompose

$$\mathbf{U}(x,t) = \mathbf{U}_0 + \varepsilon^2 A(X,T,\varepsilon)\boldsymbol{\xi} + \varepsilon^3 V(X,T,\varepsilon)\,, \quad \boldsymbol{\eta}^T V = 0\,,$$

with $\mathrm{D}\mathbf{F}(\mathbf{U}_0)\boldsymbol{\xi} = 0$ and $\mathrm{D}\mathbf{F}(\mathbf{U}_0)^T\boldsymbol{\eta} = 0$. Formally,

$$A_T + \kappa\, A A_X - \nu A_{XX} = \varepsilon\, \mathbf{R}_1$$
$$\tfrac{d}{dX}\left(\mathbf{P} D\mathbf{F}(\mathbf{U}_0)V + \tfrac{1}{2}\mathbf{P} D^2\mathbf{F}(\mathbf{U}_0)u^2 - \mathbf{P}\mathbf{D}(\mathbf{U}_0)u_X\right) = \varepsilon \mathbf{R}_2\,,$$

where

$$\kappa = \frac{d^2}{ds^2}\bigg|_{s=0} \langle \boldsymbol{\eta}, \mathbf{F}(\mathbf{U}_0 + s\boldsymbol{\xi})\rangle\,,$$

and

$$\nu = \langle \boldsymbol{\eta}, \mathbf{D}\boldsymbol{\xi}\rangle\,.$$

Double criticality and a Boussinesq model

Consider conservation laws with regularisation

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = \mathbf{D}\mathbf{U}_{xxx} \,,$$

and suppose it is doubly degenerate

$$D\mathbf{F}(\mathbf{U}_0)\xi_1 = 0 \quad \text{and} \quad D\mathbf{F}(\mathbf{U}_0)\xi_2 = \boldsymbol{\xi}_1 \,,$$

with appropriate left eigenvectors $\boldsymbol{\eta}_j$ satisfying $\langle \boldsymbol{\eta}_i, \boldsymbol{\xi}_j \rangle = \delta_{ij}$.

Reduction to a Boussinesq model

Let $X = \varepsilon x$ and $T = \varepsilon^2 t$ and decompose

$$\mathbf{U}(x,t) = \mathbf{U}_0 + \varepsilon^2 A(X,T,\varepsilon)\boldsymbol{\xi}_1 + \varepsilon^3 B(X,T,\varepsilon)\boldsymbol{\xi}_2 + \varepsilon^4 W(X,T,\varepsilon)\,,$$

with $\boldsymbol{\eta}_1^T W = 0$ and $\boldsymbol{\eta}_2^T W = 0$. Then formally,

$$\begin{aligned} A_T + B_X &= \varepsilon\,\mathbf{R}_1 \\ B_T + \kappa\, A A_X &= \nu A_{XXX} + \varepsilon\,\mathbf{R}_2 \end{aligned}$$

for some remainder terms $\mathbf{R}_j$ where

$$\kappa = \left.\frac{d^2}{ds^2}\right|_{s=0} \langle \boldsymbol{\eta}_2, \mathbf{F}(\mathbf{U}_0 + s\boldsymbol{\xi}_1)\rangle\,.$$

Formally taking the limit as $\varepsilon \to 0$ and combining gives

$$A_{TT} - \left(\tfrac{1}{2}\kappa\, A^2\right)_{XX} + \nu\, A_{XXXX} = 0\,.$$

Double criticality and coupled KdV equations

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = \mathbf{D}\mathbf{U}_{xxx}\,.$$

Suppose $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ has a double zero eigenvalue of algebraic multiplicity two and geometric multiplicity two. Then a similar argument as above with $X = \varepsilon x$, $T = \varepsilon^3 t$ and

$$\mathbf{U} = \mathbf{U}_0 + \varepsilon^2 A(X,T,\varepsilon)\boldsymbol{\xi}_1 + \varepsilon^2 B(X,T,\varepsilon)\boldsymbol{\xi}_2 + \cdots$$

gives

$$\begin{aligned}
\frac{\partial A}{\partial T} + \Gamma^1_{11}\, A\frac{\partial A}{\partial X} + \Gamma^1_{12}\,\frac{\partial(AB)}{\partial X} + \Gamma^1_{22} B\frac{\partial B}{\partial X} &= \nu_{11}\,\frac{\partial^3 A}{\partial X^3} + \nu_{12}\,\frac{\partial^3 B}{\partial X^3}\\
\frac{\partial B}{\partial T} + \Gamma^2_{11}\, A\frac{\partial A}{\partial X} + \Gamma^2_{12}\,\frac{\partial(AB)}{\partial X} + \Gamma^2_{22} B\frac{\partial B}{\partial X} &= \nu_{21}\,\frac{\partial^3 A}{\partial X^3} + \nu_{22}\,\frac{\partial^3 B}{\partial X^3}\,,
\end{aligned}$$

$$\Gamma^k_{ij} := \langle \boldsymbol{\eta}_k, D^2\mathbf{F}(\mathbf{U}_0)(\boldsymbol{\xi}_i,\boldsymbol{\xi}_j)\rangle\,.$$

Double criticality and coupled Burgers equations

$$\mathbf{U}_t + (\mathbf{F}(\mathbf{U}))_x = \mathbf{D}\mathbf{U}_{xx}\,.$$

Suppose $\mathrm{D}\mathbf{F}(\mathbf{U}_0)$ has a double zero eigenvalue of algebraic multiplicity two and geometric multiplicity two. Then a similar argument as above with $X = \varepsilon x$, $T = \varepsilon^2 t$ and

$$\mathbf{U} = \mathbf{U}_0 + \varepsilon A(X,T,\varepsilon)\boldsymbol{\xi}_1 + \varepsilon B(X,T,\varepsilon)\boldsymbol{\xi}_2 + \cdots$$

gives

$$\begin{aligned} A_T + \Gamma^1_{11}\,AA_X + \Gamma^1_{12}\,(AB)_X + \Gamma^1_{22}\,BB_X &= \nu_{11}\,A_{XX} + \nu_{12}\,B_{XX} \\ B_T + \Gamma^2_{11}\,AA_X + \Gamma^2_{12}\,(AB)_X + \Gamma^2_{22}BB_X &= \nu_{21}\,A_{XX} + \nu_{22}\,B_{XX}\,, \end{aligned}$$

Under appropriate hypotheses, coupled Burger's is valid (B & Zelik, work in progress).

Validity of the reduced models

What can one say rigorously about these reduced models?

Suppose the conservation law is hyperbolic ($D\mathbf{F}(\mathbf{U}_0)$ diagonalisable with real eigenvalues), and suppose **D** *is symmetric and positive definite. Then there exists $T_0 > 0$ such that*

$$\|\mathbf{U} - \varepsilon A^{\text{Burgers}} \xi\|_{W_b^{1,2}(\mathbb{R})} \leq C\varepsilon^{3/2} e^{KT_0} ,$$

where C and K depend on the norm of the initial data (initial data in $W_b^{2,2}(\mathbb{R})$), but are independent of ε. Here, $W_b^{1,2}(\mathbb{R})$ is the Sobolev space based on the uniformly local space $L_b^p(\mathbb{R})$ with norm

$$\|u\|_{L_b^p(\mathbb{R})} := \sup_{s\in\mathbb{R}} \|u\|_{L^p([s,s+1])} .$$

(B & Zelik, in preparation).

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