Differential Geometry • Solutions for Exercise Sheet 3

Question 1. For any regular coordinate patch, we have

$$\mathbf{N}_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \qquad \text{and} \qquad \mathbf{N}_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v$$

so that

$$\begin{aligned}\mathbf{N}_u \times \mathbf{N}_v &= (a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v) \times (a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v) \\ &= a_{11}a_{21}(\mathbf{x}_u \times \mathbf{x}_v) + a_{12}a_{21}(\mathbf{x}_v \times \mathbf{x}_u) = (a_{11}a_{22} - a_{12}a_{21})(\mathbf{x}_u \times \mathbf{x}_v) \\ &= \det d\mathbf{N}_p\,(\mathbf{x}_u \times \mathbf{x}_v) = K(p)\,(\mathbf{x}_u \times \mathbf{x}_v)\,.\end{aligned}$$

Question 2. Starting with $\mathbf{x}(u,v) = \alpha(u) + r\cos v\mathbf{N}(u) + r\sin v\mathbf{B}(u)$, we calculate

$$\begin{aligned}\mathbf{x}_u &= (1 - r\kappa\cos v)\mathbf{T}(u) - r\tau\sin v\mathbf{N}(u) + r\tau\cos v\mathbf{B}(u)\,, \\ \mathbf{x}_v &= -r\sin v\mathbf{N}(u) + r\cos v\mathbf{B}(u)\,, \\ \mathbf{x}_u \times \mathbf{x}_v &= r(1 - r\kappa\cos v)\cos v\mathbf{N}(u) + r(1 - r\kappa\cos v)\sin v\mathbf{B}(u)\,, \\ ||\mathbf{x}_u \times \mathbf{x}_v|| &= r(1 - r\kappa\cos v) \qquad \text{and so} \qquad \mathbf{N}(u,v) = \cos v\mathbf{N}(u) + \sin v\mathbf{B}(u)\,.\end{aligned}$$

The tubular surface $\mathbf{x}(u,v)$ is regular if the vector $\mathbf{x}_u \times \mathbf{x}_v$ is non-vanishing, i.e. when $(1 - r\kappa\cos v) > 0$. However since this must be true for all v such that $0 \le v < 2\pi$ it is required that $r > \rho(u)$ for each point on the curve $\alpha(u)$, where $\rho(u) = \frac{1}{\kappa(u)}$.

Question 3. Evaluating the coefficients of the first and second fundamental forms for the tubular surface $\mathbf{x}(u,v)$, we have

$$\begin{aligned}E &= \mathbf{x}_u \cdot \mathbf{x}_u = ((1 - r\kappa\cos v)^2 + (r\tau)^2\ ,\ F = \mathbf{x}_u \cdot \mathbf{x}_v = r^2\tau\ ,\ G = \mathbf{x}_v \cdot \mathbf{x}_v = r^2 \\ \mathbf{N}_u &= -\kappa\cos v\mathbf{T}(u) - \tau\sin v\mathbf{N}(u) + \tau\cos v\mathbf{B}(u)\ ,\ \mathbf{N}_v = -\sin v\mathbf{N}(u) + \cos v\mathbf{B}(u)\,, \\ e &= -\mathbf{N}_u \cdot \mathbf{x}_u = -\kappa(1 - r\kappa\cos v)\cos v + r\tau^2\ ,\ f = -\mathbf{N}_v \cdot \mathbf{x}_u = -r\tau, \\ g &= -\mathbf{N}_v \cdot \mathbf{x}_v = -r\,, \\ EG - F^2 &= r^2(1 - r\kappa\cos v)^2\ , \qquad eg - f^2 = -r\kappa(1 - r\kappa\cos v)\cos v\end{aligned}$$

and hence

$$K(p) = -\frac{\kappa(u)\cos v}{r(1 - r\kappa(u)\cos v)}\,.$$

For each fixed u, $K(p)$ vanishes at exactly two points; when $v = \frac{\pi}{2}$ and when $v = \frac{3\pi}{2}$.

Question 4. For each fixed $v_0 \in [0, 2\pi)$, the curve $\alpha(t)$ on the surface of revolution $\mathbf{x}(u,v)$ is given by $\alpha(t) = \mathbf{x}(u(t), v_0)$. Hence

$$\alpha'(t) = \mathbf{x}_u u_t(t), \qquad \alpha''(t) = \mathbf{x}_{uu}u_t^2(t) + \mathbf{x}_u u_{tt}(t)\,.$$

$\alpha(t)$ is a geodesic if and only if $\mathbf{x}_u \cdot \alpha''(t) = \mathbf{x}_v \cdot \alpha''(t) = 0$ for all t. That is, if and only if

$$\mathbf{x}_u \cdot \mathbf{x}_{uu}u_t^2(t) + \mathbf{x}_u \cdot \mathbf{x}_u u_{tt}(t) = 0 \qquad \text{and} \qquad \mathbf{x}_v \cdot \mathbf{x}_{uu}u_t^2(t) + \mathbf{x}_v \cdot \mathbf{x}_u u_{tt}(t) = 0\,.$$

But, the second equation is satisfied identically, since $\langle \mathbf{x}_v, \mathbf{x}_{uu} \rangle = \langle \mathbf{x}_v, \mathbf{x}_u \rangle = 0$. For the first equation we have $\langle \mathbf{x}_u, \mathbf{x}_{uu} \rangle = g_u g_{uu} + h_u h_{uu}$ and $\langle \mathbf{x}_u, \mathbf{x}_u \rangle = g_u^2 + h_u^2$. Therefore multiplication of the first equation by $2u_t$ leads to

$$0 = 2(g_u g_{uu} + h_u h_{uu}) u_t^3 + 2(g_u^2 + h_u^2) u_t u_{tt} = \frac{d}{dt}\left((g_u^2 + h_u^2) u_t^2\right)$$

which implies that $(g_u^2 + h_u^2) u_t^2 = \|\alpha'(t)\|^2$ is constant; that is, any *constant* speed curve on the meridian is a geodesic.

Question 5. For this curve,

$$\alpha'(t) = \mathbf{x}_v v_t \qquad \text{and} \qquad \alpha''(t) = \mathbf{x}_v v_{tt} + \mathbf{x}_{vv} v_t^2 \, .$$

By definition, $\alpha(t)$ is a geodesic if $\mathbf{x}_u \cdot \alpha''(t) = \mathbf{x}_v \cdot \alpha''(t) = 0$, which results in

$$\mathbf{x}_u \cdot \mathbf{x}_{vv} \, v_t^2(t) + \mathbf{x}_u \cdot \mathbf{x}_v \, v_{tt}(t) = 0 \qquad \text{and} \qquad \mathbf{x}_v \cdot \mathbf{x}_{vv} \, v_t^2(t) + \mathbf{x}_v \cdot \mathbf{x}_v \, v_{tt}(t) = 0 \, .$$

However the second equation is satisfied exactly since $\mathbf{x}_v \cdot \mathbf{x}_{vv} = 0$ and $\mathbf{x}_v \cdot \mathbf{x}_u = 0$. For the first equation,

$$\mathbf{x}_u \cdot \mathbf{x}_{vv} = -h h_u \qquad \text{and} \qquad \mathbf{x}_u \cdot \mathbf{x}_v = 0 \, ,$$

and so the first equation reduces to

$$-h h_u v_t^2 = 0 \, .$$

But since $v_t \neq 0$ and $h \neq 0$, it is required that $h_u = 0$. In other words, a curve of constant u is a geodesic only if $h(u_0)$ is a (nonvanishing) constant, independent of the value of u_0.