Differential Geometry • Solutions for Exercise Sheet 2

Question 1. (a) is not a regular coordinate chart since $\mathbf{x}_u \times \mathbf{x}_v = (2v, -2uv, -2u)$ which vanishes when $u = v = 0$. (b) is a regular coordinate chart: $\mathbf{x}(u,v) = (\sin \pi u, \cos \pi u, v)$ is differentiable for all $(u,v) \in U$ and the Jacobian

$$\begin{pmatrix} \pi \cos \pi u & 0 \\ -\pi \sin \pi u & 0 \\ 0 & 1 \end{pmatrix}$$

clearly has rank 2 for all $u \in (-1,1)$. Regularity can also be checked by showing that $\mathbf{x}_u \times \mathbf{x}_v \neq 0$. (c) can be verified to be a regular coordinate chart following one of the arguments in (b).

Question 2. Clearly, $\mathbf{x}(u,v) = (u-v, u+v, u^2-v^2)$ is differentiable (i.e. the component functions of $\mathbf{x}(u,v)$ are differentiable and the Jacobian

$$J(u,v) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 2u & -2v \end{pmatrix}$$

has rank 2. Hence S is a regular surface. Regularity can also be checked by showing that $\mathbf{x}_u \times \mathbf{x}_v \neq 0$. The surface S is a saddle of the form: $x_3 = x_1 x_2$.

Question 3. The area of a parallelogram is equal to the product of the length of the sides times the sin of the acute angle between two adjacent sides. Let θ be the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$, where $0 \leq \theta \leq \pi$. Then,

$$\begin{aligned} ||\mathbf{v} \times \mathbf{w}||^2 &= ||\mathbf{v}||^2||\mathbf{w}||^2 - (\langle \mathbf{v}, \mathbf{w} \rangle)^2 \\ &= ||\mathbf{v}||^2||\mathbf{w}||^2 - ||\mathbf{v}||^2||\mathbf{w}||^2 \cos^2 \theta \\ &= ||\mathbf{v}||^2||\mathbf{w}||^2 \sin^2 \theta \,, \end{aligned}$$

so that $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| \, ||\mathbf{w}|| \, \sin \theta$, which is the area of a parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$.

Question 4. Differentiating $\mathbf{x}(u_1,u_2) = \mathbf{y}(h_1(u_1,u_2), h_2(u_1,u_2))$ with respect to u_1 and u_2,

$$\frac{\partial \mathbf{x}}{\partial u_1} = \frac{\partial h_1}{\partial u_1}\frac{\partial \mathbf{y}}{\partial v_1} + \frac{\partial h_2}{\partial u_1}\frac{\partial \mathbf{y}}{\partial v_2} \qquad \text{and} \qquad \frac{\partial \mathbf{x}}{\partial u_2} = \frac{\partial h_1}{\partial u_2}\frac{\partial \mathbf{y}}{\partial v_1} + \frac{\partial h_2}{\partial u_2}\frac{\partial \mathbf{y}}{\partial v_2} \,.$$

$$\text{So} \quad \frac{\partial \mathbf{x}}{\partial u_1} \times \frac{\partial \mathbf{x}}{\partial u_2} = \left(\frac{\partial h_1}{\partial u_1}\frac{\partial h_2}{\partial u_2} - \frac{\partial h_2}{\partial u_1}\frac{\partial h_1}{\partial u_2} \right) \frac{\partial \mathbf{y}}{\partial v_1} \times \frac{\partial \mathbf{y}}{\partial v_2} = \begin{vmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} \\ \frac{\partial h_2}{\partial u_1} & \frac{\partial h_2}{\partial u_2} \end{vmatrix} \frac{\partial \mathbf{y}}{\partial v_1} \times \frac{\partial \mathbf{y}}{\partial v_2} \,.$$

Question 5. Let $\mathbf{x}(u,v) = (f(u,v), g(u,v), h(u,v))$ where f, g and h are the component functions of $\mathbf{x}$. Then $\mathbf{x}_u = (f_u, g_u, h_u)$ and $\mathbf{x}_v = (f_v, g_v, h_v)$ and so

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = f_u^2 + g_u^2 + h_u^2 \,, \qquad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = f_u f_v + g_u g_v + h_u h_v \,, \qquad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = f_v^2 + g_v^2 + h_v^2 \,.$$

and therefore

$$EG = f_u^2 f_v^2 + f_u^2 g_v^2 + f_u^2 h_v^2 + g_u^2 f_v^2 + g_u^2 g_v^2 + g_u^2 h_v^2 + h_u^2 f_v^2 + h_u^2 g_v^2 + h_u^2 h_v^2 \,,$$

$$F^2 = f_u^2 f_v^2 + f_u f_v g_u g_v + f_u f_v h_u h_v + g_u g_v f_u f_v + g_u^2 g_v^2 + g_u g_v h_u h_v + h_u h_v f_u f_v + h_u h_v g_u g_v + h_u^2 h_v^2 \,.$$

On the other hand,

$$\mathbf{x}_u \times \mathbf{x}_v = (g_u h_v - h_u g_v, -f_u h_v + h_u f_v, f_u g_v - g_u f_v)\,,$$

and so

$$\begin{aligned} ||\mathbf{x}_u \times \mathbf{x}_v||^2 &= (g_u h_v - h_u g_v)^2 + (-f_u h_v + h_u f_v)^2 + (f_u g_v - g_u f_v)^2 \\ &= g_u^2 h_v^2 - 2 g_u h_v h_u g_v + h_u^2 g_v^2 + f_u^2 h_v^2 - 2 f_u h_v h_u f_v + h_u^2 f_v^2 \\ &\quad + f_u^2 g_v^2 - 2 f_u g_v g_u f_v + g_u^2 f_v^2 \,. \end{aligned}$$

Subtracting F^2 from EG, the terms $f_u^2 f_v^2$, $g_u^2 g_v^2$ and $h_u^2 h_v^2$ cancel, and hence we have $||\mathbf{x}_u \times \mathbf{x}_v||^2 = EG - F^2$, from which the required result follows.

Question 6. $\quad \mathbf{x}(u,v) = (\cosh u \cos v, \cosh u \sin v, u)\;.$

$$\mathbf{x}_u = (\sinh u \cos v, \sinh u \sin v, 1)\;\;,\;\; \mathbf{x}_v = (-\cosh u \sin v, \cosh u \cos v, 0)\,,$$
$$\mathbf{x}_u \times \mathbf{x}_v = (-\cosh u \cos v, -\cosh u \sin v, \sinh u \cosh u)\;,\;\; ||\mathbf{x}_u \times \mathbf{x}_v|| = \cosh^2 u\,,$$

$$N(u,v) \;=\; \left(-\frac{\cos v}{\cosh u}, -\frac{\sin v}{\cosh u}, \frac{\sinh u}{\cosh u}\right)\,,$$

$$N_u = \left(\frac{\sinh u \cos v}{\cosh ru}, \frac{\sinh u \sin v}{\cosh^2 u}, \frac{1}{\cosh^2 u}\right) \quad , \quad N_v = \left(\frac{\sin v}{\cosh u}, -\frac{\cos v}{\cosh u}, 0\right)\,.$$

Hence $$N_u = \frac{1}{\cosh^2 u}\mathbf{x}_u \qquad , \qquad N_v = -\frac{1}{\cosh^2 u}\mathbf{x}_v$$

and so $$dN_p \;=\; \begin{pmatrix} \frac{1}{\cosh^2 u} & 0 \\ 0 & -\frac{1}{\cosh^2 u} \end{pmatrix}\,.$$

Thus the mean curvature $H = -\frac{1}{2}\,\mathrm{Trace}\,(dN_p) = 0$ for all $(u,v) \in \mathbb{R}^2$ and hence, for all points on the surface. The surface is therefore a minimal surface.

Question 7. For the curve α with these properties, the Frenet-Serret equations give

$$T' = \frac{N}{s} \qquad \text{and} \qquad N' = -\frac{T}{s}\,,$$

so that

$$s^2 T'' + sT' + T = 0\,.$$

The general solution of this equation is

$$T(s) = \mathbf{A}\cos(\ln s) + \mathbf{B}\sin(\ln s)\,.$$

where $\mathbf{A}$ and $\mathbf{B}$ are vectors in $\mathbb{R}^3$. Let $\alpha(s_0) \in \mathbb{R}^3$ be any specified vector, then the Fundamental Theorem of Curves now gives

$$\begin{aligned}
\alpha(s) &= \alpha(s_0) + \int_{s_0}^{s} T(r)\, dr \\
&= \alpha(s_0) + \int_{s_0}^{s} (\mathbf{A}\cos(\ln r) + \mathbf{B}\sin(\ln r))\, dr \\
&= \alpha(s_0) + \frac{1}{2}(\mathbf{A} - \mathbf{B})\, s\cos(\ln s) + \frac{1}{2}(\mathbf{A} + \mathbf{B})\, s\sin(\ln s)\,.
\end{aligned}$$

Question 8. Since $\mathbf{x}(u,v) = (u, v, f(u,v))$,

$$\mathbf{x}_u = (1, 0, f_u), \qquad \mathbf{x}_v = (0, 1, f_v), \qquad \mathbf{x}_u \times \mathbf{x}_v = (-f_u, -f_v, 1), \qquad ||\mathbf{x}_u \times \mathbf{x}_v||^2 = 1 + f_u^2 + f_v^2\,.$$

Therefore,

$$\text{Area}\,(\mathbf{x}(W)) = \int\int_W ||\mathbf{x}_u \times \mathbf{x}_v||\, du\, dv = \int\int_W \sqrt{1 + f_u^2 + f_v^2}\, du\, dv\,.$$

Question 9. For the coordinate chart $(U, \mathbf{x})$ with $\mathbf{x}(u,v) = (u, v, f(u,v))$,

$$\begin{gathered}
\mathbf{x}_u = (1, 0, f_u),\ \mathbf{x}_v = (0, 1, f_v),\ \mathbf{x}_{uu} = (0, 0, f_{uu}),\ \mathbf{x}_{uv} = (0, 0, f_{uv}),\ \mathbf{x}_{vv} = (0, 0, f_{vv}), \\
\mathbf{x}_u \times \mathbf{x}_v = (-f_u, -f_v, 1) \quad , \quad ||\mathbf{x}_u \times \mathbf{x}_v|| = \sqrt{1 + f_u^2 + f_v^2}\,,
\end{gathered}$$

and so

$$N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}\,.$$

Therefore,

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = 1 + f_u^2 \ , \ F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = f_u f_v \ , \ G = \mathbf{x}_v, \mathbf{x}_v \rangle = 1 + f_v^2\,,$$

$$e = \langle N, \mathbf{x}_{uu} \rangle = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}} \quad , \quad f = \langle \mathbf{x}_{uv}, N \rangle = -\frac{f_{vu}}{\sqrt{1 + f_u^2 + f_v^2}}\,,$$

$$g = \langle N, \mathbf{x}_{vv} \rangle = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}\,.$$

where E, F, G, e, f, g are the coefficients in the first and second fundamental forms of the surface. The Gaussian K and mean curvature H are then computed to be,

$$K = \frac{eg - f^2}{EG - F^2} = \frac{f_{uu} f_{vv} - (f_{vu})^2}{(1 + f_u^2 + f_v^2)^2}$$

and

$$H = -\frac{1}{2}\left(\frac{2fF - (eG - gE)}{EG - F^2}\right) = \frac{2(f_{uv} - f_{vu}) + f_{uu}(1 + f_v^2) + f_{vv}(1 + f_u^2)}{2\,(1 + f_u^2 + f_v^2)^{\frac{3}{2}}}\,.$$

Question 10. The ellipsoid S is defined as the set

$$S = \left\{ (x_1, x_2, x_3) \ : \ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \right\} ,$$

with the condition $a_1 > a_2 > a_3 > 1$. Let

$$U = \{(\theta, \phi) \in \mathbb{R}^2 \ : \ 0 \leq \theta < 2\pi \quad \text{and} \quad 0 < \phi < \pi\}$$

and define $\mathbf{x} : U \to \mathbb{R}^3 \cap S$ by

$$\mathbf{x}(\theta, \phi) = (a_1 \sin\phi \cos\theta, a_2 \sin\phi \sin\theta, a_3 \cos\phi) .$$

The coordinate patch $(U, \mathbf{x})$ is regular at each point on the ellipsoid except the poles where $\phi = 0, \pi$. Therefore introduce a second chart which covers the points excluded by $(U, \mathbf{x})$. Let

$$V = \{(u, v) \in \mathbb{R}^2 \ : \ 0 \leq u < 2\pi \quad \text{and} \quad 0 < v < \pi\}$$

and define $\mathbf{y} : V \to \mathbb{R}^3 \cap S$ by

$$\mathbf{y}(u, v) = (a_1 \sin u \sin v, a_2 \cos v, a_3 \sin v \cos u) .$$

The coordinate patch $(V, \mathbf{y})$ is regular at each point on the ellipsoid except the poles where $v = 0, \pi$. Since $\mathbf{x}(U) \cup \mathbf{y}(V)$ cover S, this pair of coordinate charts provides an atlas for S.

The principal curvatures at each point can now be computed be using a coordinate chart valid in the neighbourhood of each point. For example consider a point on S in the image of the first coordinate chart $(U, \mathbf{x})$. We can compute $\mathbf{x}_\theta$, $\mathbf{x}_\phi$, N, $\mathbf{x}_{\theta\theta}$, $\mathbf{x}_{\theta\phi}$ and $\mathbf{x}_{\phi\phi}$. From these expressions we can compute E, F, G, e, f, g and then the entries of dN_p. The Gaussian K and mean curvature H can be computed at each point and then the principal curvatures can be expressed in terms of K and H using the formula: $k_{1,2} = H \pm \sqrt{H^2 - K}$.