Differential Geometry • Solutions for Exercise Sheet 1

Question 1. For the smooth curve $\alpha(t) = (e^t \cos t, e^t \sin t, e^t),\ t \in \mathbb{R},$ the tangent vector $\alpha'(t)$ to the curve is

$$\alpha'(t) = (e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t)$$

for $t \in \mathbb{R}$, and hence the arc length $s(t)$ of the curve is

$$s(t) = \int_{-\infty}^{t} ||\alpha'(r)||\, dr = \sqrt{3}\int_{-\infty}^{t} e^r\, dr = \sqrt{3}e^t\,.$$

Thus the arc-length parametrization of the curve $\alpha(t)$ is

$$\alpha(s) = \left(\frac{s}{\sqrt{3}}\cos\left(\ln\frac{s}{\sqrt{3}}\right), \frac{s}{\sqrt{3}}\sin\left(\ln\frac{s}{\sqrt{3}}\right), \frac{s}{\sqrt{3}}\right), \quad s > 0\,.$$

Question 2. For the smooth curve $\alpha(t) = (3\cosh 2t, 3\sinh 2t, 6t),\ t \in [0,\pi],$ the tangent vector $\alpha'(t)$ to the curve is

$$\alpha'(t) = (6\sinh 2t, 6\cosh 2t, 6)$$

for $t \in [0,\pi]$, and hence the arc length s is given by

$$s = \int_0^{\pi} \sqrt{6^2\sinh^2 2t + 6^2\cosh^2 2t + 6^2}\, dt = 6\sqrt{2}\int_0^{\pi} \sqrt{\cosh^2 2t}\, dt = 3\sqrt{2}\sinh 2\pi\,.$$

Question 3. The curve $\alpha(t) = (a\cos(t), a\sin(t), f(t))\,, a > 0$ will lie in a plane in $\mathbb{R}^3$ if for some real constants α, β, γ and δ with $\gamma \neq 0$, the equation

$$\alpha\, a\cos t + \beta\, a\sin t + \gamma\, f(t) = \delta$$

is satisfied for all t. In other words, the function $f(t)$ must take the form

$$f(t) = \frac{\delta}{\gamma} - a\left(\frac{\alpha}{\gamma}\cos t + \frac{\beta}{\gamma}\sin t\right)\,,$$

for some real constants α, β, γ and δ with the above properties. Another way to prove this is to recall that a regular curve in $\mathbb{R}^3$ is planar if and only if the torsion $\tau(t)$ vanishes. But

$$\tau(t) = 0 \quad \Leftrightarrow \quad \alpha'(t) \times \alpha''(t) \cdot \alpha'''(t) = 0\,.$$

Substituting $\alpha(t)$ into this expression shows that $f(t)$ must satisfy the differential equation $f''' + f' = 0$. The general solution of this differential equation is $f(t) = A + B\cos s + C\sin s$ for arbitrary constants A, B and C, as in the expression for $f(t)$ above.

Question 4.

(*a*)

$$\begin{aligned}
\alpha(t) &= (at(3-t^3), 3at^2, at(3+t^2)) \\
\alpha'(t) \times \alpha''(t) &= (18a^2(t^2-1), -36a^2 t, 18a^2(t^2+1)) \\
||\alpha'(t) \times \alpha''(t)|| &= 18\sqrt{2}a^2(t^2+1) \\
(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t) &= (6a)^3\,.
\end{aligned}$$

Therefore,
$$\kappa(t) = \frac{1}{3a(t^2+1)^2} \qquad \text{and} \quad \tau(t) = \frac{1}{3a(t^2+1)^2}\,.$$

$$\begin{aligned}
(b) \qquad \alpha(t) &= (a(t-\sin t), a(1-\cos t), bt)\\
\alpha'(t) &= (a(1-\cos t), a\sin t, b)\\
\|\alpha'(t)\| &= \sqrt{b^2+4a^2\sin^2\tfrac{1}{2}t}\\
\alpha'(t)\times\alpha''(t) &= (-ab\cos t, ab\sin t, -a^2+a^2\cos t)\\
||\alpha'(t)\times\alpha''(t)|| &= a\sqrt{a^2(\cos t-1)^2+b^2}\\
(\alpha'(t)\times\alpha''(t))\cdot\alpha'''(t) &= -a^2b\,.
\end{aligned}$$

$$\text{Therefore}, \quad \kappa(t) = \frac{a}{b^2}\frac{(1+4\frac{a^2}{b^2}\sin^4\frac{1}{2}t)^{1/2}}{(1+4\frac{a^2}{b^2}\sin^2\frac{1}{2}t)^{3/2}} \quad \text{and} \quad \tau(t) = \frac{-b}{(b^2+a^2(\cos t-1)^2)}\,.$$

$$\begin{aligned}
(c) \qquad \alpha(t) &= \left(a(1+\cos t), a\sin t, 2a\sin\frac{t}{2}\right)\\
\alpha'(t)\times\alpha''(t) &= \left(\frac{a^2}{2}(\sin\frac{t}{2}+\sin t\cos\frac{t}{2}), -a^2\cos^3\frac{t}{2}, a^2\right)\\
||\alpha'(t)\times\alpha''(t)|| &= \frac{a^2}{2}\sqrt{5+3\cos^2\frac{t}{2}}\\
(\alpha'(t)\times\alpha''(t))\cdot\alpha'''(t) &= \frac{3}{4}a^3\cos\frac{t}{2}\,.
\end{aligned}$$

$$\text{Therefore}, \quad \kappa(t) = \frac{1}{2a}\frac{\sqrt{5+3\cos^2\frac{t}{2}}}{(1+\cos^2\frac{t}{2})^{3/2}} \quad \text{and} \quad \tau(t) = \frac{3}{a}\frac{\cos\frac{t}{2}}{5+3\cos^2\frac{t}{2}}\,.$$

Question 5. A curve $\alpha(t)$ lies on a sphere of radius r if $\alpha(t)\cdot\alpha(t) = r^2$ for all t. For the curve $\alpha(t) = (a\sin^2 t, a\sin t\cos t, a\cos t)$, we see that

$$||\alpha(t)||^2 = a^2\sin^4 t + a^2\sin^2 t\cos^2 t + a^2\cos^2 t = a^2\,.$$

Hence, $\alpha(t)$ lies on a sphere in $\mathbb{R}^3$ of radius a and centred at the origin.

Question 6. Given the smooth curve $\alpha(t) = (a\cos t, a\sin t, bt),\ a,b>0$ in $\mathbb{R}^3$.

(b) Using the reparametrization $g(t) = -t$, which is clearly smooth and has a smooth inverse, we have

$$(\alpha\circ g)(t) = (a\cos t, -a\sin t, -bt)\,.$$

Hence, the trace of $\alpha\circ g$ is the same as that of α.

Furthermore, since $g'(t) = -1 < 0$, we see that this parametrization reverses the orientation of the curve.

Finally, we have that $(\alpha\circ g)'(t) = (-a\sin t, -a\cos t, -b) \neq 0$, for all t. Hence the reparametrization is regular.

(c) Similarly, take the reparametrization $g(t) = \frac{\pi}{2} - t$. We get

$$(\alpha \circ g)(t) = (a \sin t, a \cos t, b(\frac{\pi}{2} - t)) .$$

Again, $g'(t) = -1 < 0$, so the parametrization $\alpha \circ g$ reverses the orientation of the curve. Similar to (b) above, this reparametrization is regular.

(d) Take the reparametrization $g(t) = 2t$. $g'(t) > 0$, so preserves the orientation of the curve. This reparametrization is regular.

(e) The trace of this curve is only half the trace of the curve in (a) because be^{-t} is always positive (assuming $t \in [-\infty, +\infty]$ in (a)). So this function is not a reparametrization of the curve in (a).

(f) There is no reparametrization which takes this curve to the curve in (a). The function $g(t) = t^3 - t$ is not even one-one, and since $g'(t)$ vanishes when $t = \pm\frac{1}{\sqrt{3}}$, it is not regular. Therefore, this function is not a reparametrization of the curve in (a).

Question 7. Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, $t \in (a, b)$, and let $c > 0$ be an arbitrary constant. Then we have (with $\langle \alpha(t), \alpha(t) \rangle = \alpha(t) \cdot \alpha(t)$),

$$\begin{aligned}
\langle \alpha(t), \alpha(t) \rangle = c \quad &\Leftrightarrow \quad \alpha_1^2(t) + \alpha_2^2(t) + \alpha_3^2(t) = c \\
&\Leftrightarrow \quad 2\left(\alpha_1(t)\alpha_1^{'}(t) + \alpha_2(t)\alpha_2^{'}(t) + \alpha_3(t)\alpha_3^{'}(t)\right) = 0 \\
&\Leftrightarrow \quad \langle \alpha(t), \alpha^{'}(t) \rangle = 0
\end{aligned}$$

Question 8. The *logarithmic spiral* is given by $\alpha(t) = (e^{-t} \cos t, e^{-t} \sin t)$, $t \in [1, +\infty]$.

Since $\sin t$ and $\cos t$ are bounded functions and $e^{-t} \to 0$ as $t \to \infty$, we see that both $e^{-t} \sin t$ and $e^{-t} \cos t$ tend to 0 as $t \to \infty$. Hence, $\alpha(t) \to 0$ as $t \to \infty$.

Furthermore, $\alpha^{'}(t) = \left(-e^{-t}(\sin t + \cos t), -e^{-t}(\sin t - \cos t)\right) \to 0$ as $t \to \infty$, for the same reasons.

The arc length $s(t)$ of the spiral is given by

$$s(t) = \int_1^t ||\alpha^{'}(r)||\, dr = \sqrt{2} \int_1^t e^{-r}\, dr = \sqrt{2}\left(e^{-1} - e^{-t}\right) .$$

Hence $s(t) \to \sqrt{2}e^{-1}$ as $t \to \infty$.

Question 9. Since the vectors T, N and B form a right-handed orthonormal frame for $\mathbb{R}^3$, the Darboux vector $\omega(s)$ takes the form

$$\omega(s) = a(s)T(s) + b(s)N(s) + c(s)B(s)$$

for some scalar-valued functions $a(s), b(s), c(s)$. Hence

$$\omega \times T = bN \times T + cB \times T = -bB + cB \times T = -bB + cN ,$$

and the requirement that $\omega \times T = T' = \kappa N$ means that $b = 0$ and $c = \kappa$. So $\omega = aT + \kappa B$ for some scalar-valued function $a(s)$, and we have

$$\omega \times N = (aT + \kappa B) \times N = aT \times N + \kappa B \times N = aB - \kappa T\,.$$

Now, requiring that $\omega \times N = N' = -\kappa T + \tau B$ means that $a = \tau$, so the Darboux vector is given by $\omega(s) = \tau T + \kappa B$.

Finally, we check that $\omega(s) \times B = B'$: $(\tau T + \kappa B) \times B = \tau T \times B = -\tau N = B'$.

It is easy to see that

$$||\omega|| = \sqrt{\langle \omega, \omega \rangle} = \sqrt{\langle \tau T + \kappa B, \tau T + \kappa B \rangle} = \sqrt{\kappa^2 + \tau^2}$$

Question 10. (a) $\Leftrightarrow$ (b): Using the arc-length parametrization for $\alpha(t)$, we see that $\kappa = ||\alpha''(t)|| = 0 \Leftrightarrow \alpha'(t) = c$, for some constant vector c.

(a) $\Leftrightarrow$ (c): $\kappa = 0$ for all t implies that $\alpha'(t) \times \alpha''(t) = 0$ for all t. Hence $\alpha'(t)$ and $\alpha''(t)$ are linearly dependent for all t. The converse is also clear.