1. Let $\mathbf{N}(u,v)$ be the unit normal associated with a regular coordinate patch for a surface S. Show that at every point $p \in S$,

$$\mathbf{N}_u \times \mathbf{N}_v = K(p)\,\mathbf{x}_u \times \mathbf{x}_v\,,$$

where $K(p)$ is the Gaussian curvature at the point p.

2. Suppose $\alpha(u)$ is a unit speed curve in $\mathbb{R}^3$ with positive curvature $\kappa(u)$ at each point, and normal $\mathbf{N}(u)$ and binormal $\mathbf{B}(u)$. Let

$$U = \{(u,v) \in \mathbb{R}^2 \;:\; -\infty < u < +\infty\,, \quad 0 \le v < 2\pi\}\,,$$

and define $\mathbf{x} : U \to \mathbb{R}^3$ by

$$\mathbf{x}(u,v) = \alpha(u) + r(\cos v\,\mathbf{N}(u) + \sin v\,\mathbf{B}(u))\,,$$

where r is a positive constant. Determine conditions such that this *tubular surface* is regular, and find an explicit expression for the normal vector at each point on the surface.

3. For the tubular surface in question 2, determine an expression for the Gaussian curvature at each point, and show that for each fixed u there are exactly two points on the surface where the Gaussian curvature vanishes.

4. Consider a surface of revolution with coordinate chart

$$\mathbf{x}(u,v) = (g(u), h(u)\cos v, h(u)\sin v)\,, \quad (u,v) \in U$$

where $U = \{(u,v) \in \mathbb{R}^2 \;:\; -\infty < u < \infty\,, \quad 0 \le v < 2\pi\}$. Give conditions on $g(u)$, $h(u)$ and $u(t)$ such that every meridian, that is, every curve of constant v:

$$\alpha(t) = \mathbf{x}(u(t), v_0)\,, \quad v_0 \text{ fixed}\,,$$

is a geodesic.

5. For the surface of revolution in question 4, determine conditions on $h(u)$ so that curves of constant u:

$$\alpha(t) = \mathbf{x}(u_0, v(t))\,, \quad u_0 \text{ fixed}\,,$$

are also geodesics.