1. Let $U = \{(u,v) \in \mathbb{R}^2 \ : \ -1 < u < 1,\ -1 < v < 1\}$, and determine if the following functions $\mathbf{x} : U \to \mathbb{R}^3$ qualify as regular coordinate charts for a surface S in $\mathbb{R}^3$,

$$(a)\ \mathbf{x}(u,v) = (u^2, u-v, v^2) \quad (b)\ \mathbf{x}(u,v) = (\sin \pi u, \cos \pi u, v) \quad (c)\ \mathbf{x}(u,v) = (u, u+v, v^{3/2})\,.$$

2. Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\mathbf{x}(u,v) = (u-v, u+v, u^2-v^2)$. Show that the image of $\mathbf{x}$ is a regular surface S in $\mathbb{R}^3$ and describe S.

3. Let $\mathbf{v} \in \mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^3$ be linearly independent vectors. Prove that $\|\mathbf{v} \times \mathbf{w}\|$ is the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$.

4. Let $\mathbf{x}(u_1,u_2)$ and $\mathbf{y}(v_1,v_2)$ be two regular coordinate charts for all $u \in U \subset \mathbb{R}^2$ and $v \in V \subset \mathbb{R}^2$ and suppose that in the region $\mathbf{x}(U) \cap \mathbf{y}(V)$ there is a differentiable function $h(u)$ such that $h = \mathbf{y}^{-1} \circ \mathbf{x}$, that is

$$\mathbf{y}(h_1(u_1,u_2), h_2(u_1,u_2)) = \mathbf{x}(u_1,u_2)\,.$$

Prove that

$$\frac{\partial \mathbf{x}}{\partial u_1} \times \frac{\partial \mathbf{x}}{\partial u_2} = \det \begin{bmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} \\ \frac{\partial h_2}{\partial u_1} & \frac{\partial h_2}{\partial u_2} \end{bmatrix} \frac{\partial \mathbf{y}}{\partial v_1} \times \frac{\partial \mathbf{y}}{\partial v_2}\,.$$

5. Let $E = \mathbf{x}_u \cdot \mathbf{x}_u$, $F = \mathbf{x}_u \cdot \mathbf{x}_v$ and $G = \mathbf{x}_v \cdot \mathbf{x}_v$, and show that

$$\|\mathbf{x}_u \times \mathbf{x}_v\| = \sqrt{EG - F^2}\,.$$

6. Show that the mean curvature is zero at every point on the surface of revolution

$$\mathbf{x}(u,v) = (\cosh u\ \cos v, \cosh u\ \sin v, u)\,, \quad -\infty < u < \infty\,,\ 0 \le v < 2\pi\,.$$

7. Let $\kappa(s) = 1/s$ and $\tau(s) = 0$ for all $s > 0$. Find the unit speed curve in $\mathbb{R}^3$ associated with this curvature and torsion.

8. Show that the surface area for a piece of surface $\mathbf{x}(W)$ with coordinate patch $\mathbf{x}(u,v) = (u, v, f(u,v))$ for $(u,v) \in \mathbb{R}^2$ and $f(u,v)$ any smooth function, is

$$\mathrm{Area}(\mathbf{x}(W)) = \int\!\!\int_W \sqrt{1 + f_u^2 + f_v^2}\ du\, dv\,.$$

9. Determine the coefficients E, F, G, e, f, g in the first and second fundamental forms, and expressions for the Gaussian and mean curvature at each point of the surface defined by $\mathbf{x}(u,v) = (u, v, f(u,v))$ for $(u,v) \in \mathbb{R}^2$ and $f(u,v)$ any smooth function.

10. Find a set of regular coordinate charts that covers the ellipsoid:

$$S = \left\{ (x_1, x_2, x_3) \ : \ \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \right\}$$

where $a_1 > a_2 > a_3 > 1$. Use the coordinate charts to sketch an argument for how one would calculate the principal curvatures at each point on the ellipsoid.