1. Reparametrize the following curve in terms of the arc length $\alpha(t) = (e^t \cos t, e^t \sin t, e^t)$, $t \in \mathbb{R}$.

2. Compute the length of the following curve $\alpha(t) = (3\cosh 2t, 3\sinh 2t, 6t)$ $0 \leq t \leq \pi$.

3. Determine the function $f(t)$ such that the following curve lies in a plane

$$\alpha(t) = (a\cos(t), a\sin(t), f(t))\,, \quad a > 0\,.$$

4. Find the curvature and the torsion of the curves given by

$$\begin{aligned} &(a) \quad \alpha(t) = (a(3t - t^3), 3at^2, a(3t + t^3)) \\ &(b) \quad \alpha(t) = (a(t - \sin t), a(1 - \cos t), bt) \\ &(c) \quad \alpha(t) = (a(1 + \cos t), a\sin t, 2a\sin\tfrac{1}{2}t) \end{aligned}$$

5. Prove that the curve given by $\alpha(t) = (a\sin^2 t, a\sin t\cos t, a\cos t)$ lines on a sphere.

6. Consider the following curves in $\mathbb{R}^3$ where $a, b > 0$

$$\begin{aligned} &(a) \quad \alpha(t) = (a\cos t, a\sin t, bt) \\ &(b) \quad \alpha(t) = (a\cos t, -a\sin t, -bt) \\ &(c) \quad \alpha(t) = (a\sin t, a\cos t, b(\tfrac{1}{2}\pi - t)) \\ &(d) \quad \alpha(t) = (a\cos 2t, a\sin 2t, 2bt) \\ &(e) \quad \alpha(t) = (a\cos e^{-t}, a\sin e^{-t}, be^{-t}) \\ &(f) \quad \alpha(t) = (a\cos(t^3 - t), a\sin(t^3 - t), b(t^3 - t)) \end{aligned}$$

where a and b are positive real parameters. Determine whether $(b) - (f)$ are reparametrizations of (a) and represent the same curve. Determine which parametrizations are oriented the same as (a) and which have an opposite orientation. Determine which reparametrizations are regular.

7. Let $\alpha : (a, b) \to \mathbb{R}^3$ be a regular differentiable parametrized curve. Show that $\|\alpha(t)\|$ is a constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in (a, b)$.

8. Let $\alpha(t) = (e^{-t}\cos t, e^{-t}\sin t)$ for $1 \leq t < +\infty$ Sketch the trace of this curve (this curve is called a *logarithmic spiral*). Show that $\alpha(t) \to 0$ as $t \to +\infty$. Show that $\alpha'(t) \to 0$ as $t \to +\infty$ and compute

$$s(t) = \int_1^t \|\alpha'(r)\|\, dr\,,$$

and show that $\lim_{t\to\infty} s(t)$ is finite. In other words, show that the infinite spiral has finite length!

9. If $\alpha(s)$ is a unit-speed curve with nonzero curvature, find a vector $\omega(s)$ such that

$$T' = \omega \times T\,, \quad N' = \omega \times N\,, \quad B' = \omega \times B\,,$$

where $\times$ is the vector cross product. This vector $\omega(s)$ is called the **Darboux vector**. Determine its length.

10. Show that each of the following leads to an equivalent characterization of curves which are straight lines in $\mathbb{R}^3$,

- (a) The curvature κ is identically zero.
- (b) The tangent at each point is parallel to a fixed line.
- (c) The curve $\alpha(t)$ has the property that $\alpha'(t)$ and $\alpha''(t)$ are linearly dependent.