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M. Math. Honours Course in Mathematical Studies

Level 3/4 Examination

Module MS322/422 DIFFERENTIAL GEOMETRY

Time allowed – $2\frac{1}{2}$ hrs

Autumn Semester 1999

Attempt FOUR questions. If any candidate attempts more than FOUR questions only the best FOUR solutions will be taken into account.

Question 1

Let $(U, \mathbf{x})$ where $U \subset \mathbf{R}^2$ and $\mathbf{x} : U \to \mathbf{R}^3 \cap S$ be a regular coordinate chart for a surface S. Let $\alpha(t)$ and $\beta(t)$ be regular curves in $\mathbf{x}(U)$ with $\mathbf{p} = \alpha(0) = \beta(0)$.

(a) Write $\alpha(t) = \mathbf{x}(u(t), v(t))$ and $\beta(t) = \mathbf{x}(\overline{u}(t), \overline{v}(t))$ and use $\alpha'(0) \cdot \beta'(0)$ to construct the first fundamental form for S at $\mathbf{p}$. [9]

(b) Use the first fundamental form in part (a) to define the length of a regular curve $\alpha(t)$ in $\mathbf{x}(U)$ from $\alpha(t_0)$ to $\alpha(t)$. [5]

(c) Let $\mathbf{x}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$ be a coordinate patch for a part of the unit sphere: $U = \{(u, v) \in \mathbf{R}^2 \ : \ 0 \leq u < 2\pi, \ 0 < v < \pi\}$. Find the first fundamental form on $\mathbf{x}(U)$. [7]

(d) Compute the length of the curve $\alpha(t) = \mathbf{x}(t, 3\pi/4)$ for $0 \leq t < 2\pi$ on the unit sphere, using the first fundamental form derived in part (c). [4]

Question 2

Let $\alpha(s)$, $a \leq s \leq b$, be a regular unit speed curve in $\mathbf{R}^3$. Define $\kappa(s)$ to be the positive function $\kappa(s) = \|\mathbf{T}'(s)\| = \|\alpha''(s)\|$, for all $s \in [a, b]$. Take $\mathbf{N}(s) = \mathbf{T}'(s)/\kappa(s)$ to be the unit normal.

(a) Construct a vector $\mathbf{B}(s)$ so that $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is an orthonormal frame field at $\alpha(s)$ for each s. [4]

(b) Let $\tau(s) = \mathbf{B}(s) \cdot \mathbf{N}'(s)$ be the torsion. Derive the governing Frenet-Serret differential equations for $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$. [7]

(c) Suppose $\tau(s) = 0$ for all $s \in [a, b]$. Prove that $\mathbf{B}(s) = \mathbf{B}_0$, where $\mathbf{B}_0$ is a constant unit vector, and that

$$(\alpha(s) - \alpha(a)) \cdot \mathbf{B}_0 = 0 \,, \quad \forall \ s \in [a, b] \,.$$

[7]

(d) Prove that $\tau(s) = 0$ if and only if $\alpha(s)$ lies in a plane. [7]

Question 3

Let $\mathbf{x} : U \to \mathbf{R}^3 \cap S$ be a smooth regular coordinate patch for a surface S where U is an open subset of $\mathbf{R}^2$ with coordinates (u, v). The unit normal to S is $\mathbf{N}(u, v) = \mathbf{x}_u \times \mathbf{x}_v / \|\mathbf{x}_u \times \mathbf{x}_v\|$. Let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a smooth regular curve in S.

(a) Show that $\alpha'(t)$ is orthogonal to $\mathbf{N}(u(t), v(t))$. [4]

(b) State the conditions $\alpha(t)$ must satisfy to be a *geodesic* curve in S. [3]

(c) Prove that a geodesic curve has constant speed: $\|\alpha'(t)\|$ is constant. [5]

(d) Suppose that $\mathbf{x}(u, v)$ corresponds to an orthogonal coordinate patch, that is, $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$, and let $E = \mathbf{x}_u \cdot \mathbf{x}_u$ and $G = \mathbf{x}_v \cdot \mathbf{x}_v$. Prove that the curve $\alpha(t)$ is a geodesic if $u(t)$ and $v(t)$ satisfy

$$\ddot{u} + \frac{1}{2E}(E_u \dot{u}^2 + 2E_v \dot{u}\dot{v} - G_u \dot{v}^2) = 0\,,$$

$$\ddot{v} + \frac{1}{2G}(-E_v \dot{u}^2 + 2G_u \dot{u}\dot{v} + G_v \dot{v}^2) = 0\,.$$

[13]

Question 4

Consider the following surface in $\mathbf{R}^3$, where $h(u) > 0$ and $h'(u) > 0$,

$$S = \{\, \mathbf{x}(u, v) = (u, h(u)\cos v, h(u)\sin v)\,, \quad 0 \le u < 2\pi,\ v > 0\,\}$$

(a) Sketch an argument that this surface is orientable. [4]

(b) Show that the pair of vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ form an *orthogonal* basis for the tangent space of S, but do not form an *orthonormal* basis. [8]

(c) Define a *minimal surface.* [3]

(d) Suppose that $h(u)$ satisfies $hh_{uu} = 1 + h_u^2$. Show that S is a minimal surface. You may use without proof that the mean curvature can be expressed as

$$H = \tfrac{1}{2}(Ge + Eg - 2Ff)/(EG - F^2)\,,$$

where $e = \mathbf{N} \cdot \mathbf{x}_{uu}$, $f = \mathbf{N} \cdot \mathbf{x}_{uv}$ and $g = \mathbf{N} \cdot \mathbf{x}_{vv}$. [10]

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Question 5

Let $\alpha(t)$, $a \le t \le b$ be any smooth regular curve in $\mathbf{R}^n$, $n \ge 2$, joining the two points

$$\alpha(a) = \mathbf{p} \in \mathbf{R}^n \quad \text{and} \quad \alpha(b) = \mathbf{q} \in \mathbf{R}^n \,,$$

and let $\sigma : [0,1] \to \mathbf{R}^n$ be the straight line connecting $\mathbf{p}$ and $\mathbf{q}$:

$$\sigma(t) = (1-t)\mathbf{p} + t\mathbf{q}\,, \quad 0 \le t \le 1\,.$$

Define the length of a curve $\alpha : [a,b] \to \mathbf{R}^n$ by $L(\alpha) = \int_a^b \|\alpha'(r)\|\,dr$.

(a) Show that $\|\mathbf{q}-\mathbf{p}\| = \int_0^1 \|\sigma'(t)\|\,dt$. [3]

(b) Let $\mathbf{u} = \dfrac{\mathbf{q}-\mathbf{p}}{\|\mathbf{q}-\mathbf{p}\|}$, and prove that

$$\|\alpha'(t)\| \ge \alpha'(t)\cdot\mathbf{u} \quad \text{and} \quad L(\alpha) \ge \|\mathbf{p}-\mathbf{q}\|\,.$$

[7]

(c) Prove that if $L(\alpha) = \|\mathbf{q}-\mathbf{p}\|$ then $\alpha(t)$ is a straight line. [10]

(d) Prove that the shortest distance between two points in $\mathbf{R}^n$, $n \ge 2$, is a straight line. [5]

Question 6

Let $\mathbf{x} : U \to \mathbf{R}^3 \cap S$ be a smooth regular coordinate chart for a surface S where U is an open subset of $\mathbf{R}^2$ with coordinates (u,v). The unit normal to S is $\mathbf{N}(u,v) = \mathbf{x}_u \times \mathbf{x}_v / \|\mathbf{x}_u \times \mathbf{x}_v\|$.

(a) Prove that $\mathbf{N}_u$ and $\mathbf{N}_v$ lie in the tangent space of S for each $p \in \mathbf{x}(U)$. [5]

(b) Show that

$$\mathbf{x}_u\cdot\mathbf{N}_u = -\mathbf{N}\cdot\mathbf{x}_{uu}\,, \quad \mathbf{x}_u\cdot\mathbf{N}_v = \mathbf{x}_v\cdot\mathbf{N}_u = -\mathbf{N}\cdot\mathbf{x}_{uv}\,, \quad \mathbf{x}_v\cdot\mathbf{N}_v = -\mathbf{N}\cdot\mathbf{x}_{vv}\,.$$

[6]

(c) Suppose $\mathbf{x}_u\cdot\mathbf{x}_v = 0$ for each $p \in \mathbf{x}(U)$, and let

$$\mathbf{N}_u = a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v \quad \mathbf{N}_v = a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v\,.$$

Determine expressions for a_{11}, a_{12}, a_{21} and a_{22} in terms of $\mathbf{N}$, $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$, $E = \mathbf{x}_u\cdot\mathbf{x}_u$ and $G = \mathbf{x}_v\cdot\mathbf{x}_v$ only. [9]

(d) Using the relationship $\mathbf{N}_u \times \mathbf{N}_v = \mathrm{K}(p)\,\mathbf{x}_u \times \mathbf{x}_v$ or otherwise, express the Gaussian curvature $\mathrm{K}(p)$ at each point $p \in \mathbf{x}(U)$ in terms of the coefficients a_{11}, a_{12} a_{21} and a_{22} of part (c). [5]

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