## Module MS322/422 DIFFERENTIAL GEOMETRY

# Solutions to Autumn Semester 1999 Examination Paper

**Question 1.**

(a) With $\alpha(t) = \mathbf{x}(u(t), v(t))$ and $\alpha(t) = \mathbf{x}(\overline{u}(t), \overline{v}(t))$,

$$\alpha'(0) = \mathbf{x}_u u' + \mathbf{x}_v v' \quad \text{and} \quad \beta'(0) = \mathbf{x}_u \overline{u}' + \mathbf{x}_v \overline{v}' \,.$$

Hence

$$\alpha'(0) \cdot \beta'(0) = \mathbf{x}_u \cdot \mathbf{x}_u u' \overline{u}' + \mathbf{x}_u \cdot \mathbf{x}_v \left( u' \overline{v}' + \overline{u}' v' \right) + \mathbf{x}_v \cdot \mathbf{x}_v \overline{v}' v' \,.$$

Letting $\quad E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v \quad$ and $\quad G = \mathbf{x}_v \cdot \mathbf{x}_v, \quad$ and

$$\mathbf{a} = \begin{pmatrix} u_t \\ v_t \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} \overline{u}_t \\ \overline{v}_t \end{pmatrix},$$

the above expression becomes

$$I_p(\mathbf{a}, \mathbf{b}) = \mathbf{a}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \mathbf{b}$$

the first fundamental form.

(b) Let $L\,(\alpha; t_0, t)$ denote the length of the curve $\alpha(t)$ from $\alpha(t_0)$ to $\alpha(t)$. Then

$$L\,(\alpha; t_0, t) = \int_{t_0}^{t} \parallel \alpha'(r) \parallel \, dr.$$

But $\quad \alpha(t) = \mathbf{x}\,(u(t), v(t)) \quad$ and so

$$L\,(\alpha; t_0, t) = \int_{t_0}^{t} \sqrt{E u_r^2 + 2F u_r v_r + G v_r^2} \;\; dr.$$

(c) Using part (a), we need to find the coefficients $E$, $F$ and $G$. Now,

$$\mathbf{x}_u = (-\sin u \sin v, \cos u \sin v, 0)\,; \quad \mathbf{x}_v = (\cos u \cos v, \sin u \cos v, -\sin v)\,.$$

Hence $\quad E = \mathbf{x}_u \cdot \mathbf{x}_u = \sin^2 v, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v = 0, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v = 1, \quad$ and so

$$I_p(\mathbf{a}, \mathbf{b}) = \mathbf{a}^T \begin{pmatrix} \sin^2 v & 0 \\ 0 & 1 \end{pmatrix} \mathbf{b} = a_1 b_1 \sin^2 v + a_2 b_2 \,.$$

(d) By part (b),

$$L\,(\alpha; 0, 2\pi) = \int_{0}^{2\pi} \sqrt{E u_r^2 + 2F u_r v_r + G v_r^2} \;\; dr.$$

By part (c), $F = 0, \quad G = 1 \quad \text{and} \quad E = \sin^2 v$:

$$L(\alpha; 0, 2\pi) = \int_0^{2\pi} \sqrt{\sin^2 v(r) u_r^2 + v_r^2}\ dr.$$

Now,

$$\alpha(t) = \mathbf{x}\left(t, \frac{3\pi}{4}\right) \qquad \Rightarrow \qquad u(t) = t \quad \text{and} \quad v(t) = \frac{3\pi}{4}.$$

Hence

$$L(\alpha; 0, 2\pi) = \int_0^{2\pi} \sqrt{\sin^2\left(\frac{3\pi}{4}\right)}\ dr = 2\pi \sin\frac{3\pi}{4} = \pi\sqrt{2}.$$

**Question 2.**

(a) Since $\alpha(s)$ is a unit speed curve, $\| \mathbf{T}(s) \| = \| \alpha'(s) \| = 1$. Also, from the definition of $\kappa(s)$, we have that $\| \mathbf{N} \| = 1$. The orthogonality of $\mathbf{T}$ and $\mathbf{N}$ follows from

$$0 = \frac{1}{2}\frac{d}{ds}\mathbf{T}(s) \cdot \mathbf{T}(s) = \mathbf{T}' \cdot \mathbf{T} = \kappa(s)\mathbf{N}(s) \cdot \mathbf{T}(s),$$

since $\kappa(s) > 0$. $\mathbf{B}(s)$ can be constructed using the vector cross product: $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$, which is orthogonal to $\mathbf{T}(s)$ and $\mathbf{N}(s)$. $\mathbf{B}(s)$ has unit length since

$$\| \mathbf{B}(s) \|^2 = \| \mathbf{T}(s) \times \mathbf{N}(s) \|^2 = \| \mathbf{T}(s) \|^2 \| \mathbf{N}(s) \|^2 - (\mathbf{T} \cdot \mathbf{N})^2 = 1.$$

(b) $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ by the definition given in part (a), which implies also that $\mathbf{B} \cdot \mathbf{T}' = 0$. Now $\mathbf{N}'(s) = a_1\mathbf{T}(s) + a_2\mathbf{N}(s) + a_3\mathbf{B}(s)$. But

$$\mathbf{N} \cdot \mathbf{N} = 1 \quad \Rightarrow \quad a_2 = 0, \qquad \mathbf{N} \cdot \mathbf{T} = 0 \quad \Rightarrow \quad a_1 = -\kappa(s),$$

and $a_3 = \tau$ by definition. Hence $\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$. Finally,

$$\mathbf{B}'(s) = b_1\mathbf{T} + b_2\mathbf{N} + b_3\mathbf{B}.$$

But

$$\mathbf{B} \cdot \mathbf{B} = 1 \quad \Rightarrow \quad b_3 = 0, \qquad \mathbf{B} \cdot \mathbf{T} = 0 \quad \text{and} \quad \mathbf{B} \cdot \mathbf{T}' = 0 \quad \Rightarrow \quad b_1 = 0,$$

and

$$\mathbf{B} \cdot \mathbf{N} = 0 \quad \Rightarrow \quad \mathbf{B}' \cdot \mathbf{N} = -\mathbf{B} \cdot \mathbf{N}' = -\tau.$$

Hence, $\mathbf{B}' = -\tau\mathbf{N}$. This completes the derivation of the Frenét-Serret equations.

(c) When $\tau(s) \equiv 0$, then $\mathbf{B}'(s) = 0$ by the Frenét-Serret equations; hence $\mathbf{B}(s) = \mathbf{B}_0$ with $\| \mathbf{B} \| = 1$. Let

$$\gamma(s) = (\alpha(s) - \alpha(a)) \cdot \mathbf{B}_0.$$

Then $\gamma'(s) = \mathbf{T}(s) \cdot \mathbf{B}_0 = 0$ implies that $\gamma(s)$ is constant. But $\gamma(a) = 0$ implies that the constant is zero; this completes the proof.

(d) If $\tau(s) \equiv 0$, then by part (c), $\alpha(s)$ lies in the plane orthogonal to $\mathbf{B}_0$. Conversely, suppose that $\alpha(s)$ lies in a plane. Let $\mathbf{q} \in \mathbb{R}^3$ be normal to the plane. Then $(\alpha(s) - \alpha(a)) \cdot \mathbf{q} = 0$ and so

$$0 = \alpha'(s) \cdot \mathbf{q} = \mathbf{T}(s) \cdot \mathbf{q} \quad \Rightarrow \quad \mathbf{q} = a(s)\mathbf{N}(s) + b(s)\mathbf{B}(s),$$

for some functions $a(s)$, $b(s)$. This further implies that

$$a'\mathbf{N} + b'\mathbf{B} + a\mathbf{N}' + b\mathbf{B}' = (a' - \tau b)\mathbf{N} - a\kappa\mathbf{T} + (b' + a\tau)\mathbf{B} = 0.$$

Hence $a = 0$, $b = b_0$ (constant) and $\tau b_0 = 0$. Since $b_0$ cannot be zero (it would imply $\mathbf{q} = 0$), it follows that $\tau = 0$ and so $\mathbf{q} = b_0\mathbf{B}_0$.

**<u>Question 3.</u>**

(a) $\quad \alpha(t) = \mathbf{x}(u(t), v(t)) \quad \Rightarrow \quad \alpha'(t) = u'\mathbf{x}_u + v'\mathbf{x}_v$. But

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\| \mathbf{x}_u \times \mathbf{x}_v \|},$$

and by the properties of the vector cross product, $\mathbf{N} \perp \mathbf{x}_u$ and $\mathbf{N} \perp \mathbf{x}_v$. Hence $\alpha'(t) \perp \mathbf{N}$.

(b) A curve $\alpha(t)$ in $S$ is a geodesic if the acceleration $\alpha''(t)$ is always normal to $S$. Other acceptable definitions: (i) a unit-speed curve is geodesic if $\kappa_g$, the geodesic curvature, vanishes; (ii) an arbitrary regular curve is a geodesic if $\kappa_g = 0$ and it has constant speed.

(c) Given any curve in $S$, $\alpha'(t) \in \mathbf{T}_{\alpha(t)}(S)$ for each $t$. Now, from part (b), we know that $\alpha''(t) = a(t)\mathbf{N}(t)$ for some scaler function $a(t)$. Hence

$$\alpha'(t) \cdot \alpha''(t) = a(t)\alpha'(t) \cdot \mathbf{N}(t) = 0,$$

since $\alpha' \perp \mathbf{N}$. But $\alpha' \cdot \alpha'' = 0 \Rightarrow \alpha' \cdot \alpha'$ is constant. Hence $\| \alpha'(t) \|$ is constant.

(d) With $\alpha(t) = \mathbf{x}(u(t), v(t))$ and $\alpha'(t) = \mathbf{x}_u\dot{u} + \mathbf{x}_v\dot{v}$, we have that

$$\alpha''(t) = \mathbf{x}_u\ddot{u} + \mathbf{x}_v\ddot{v} + (\mathbf{x}_{uu}\dot{u} + \mathbf{x}_{uv}\dot{v})\,\dot{u} + (\mathbf{x}_{uv}\dot{u} + \mathbf{x}_{vv}\dot{v})\,\dot{v}.$$

From the definition of a geodesic, we also have that

$$\mathbf{x}_u \cdot \alpha'' = 0 \quad \text{and} \quad \mathbf{x}_v \cdot \alpha'' = 0\,.$$

Hence the geodesic equations are:

$$\mathbf{x}_u \cdot \left(\mathbf{x}_u \ddot{u} + \mathbf{x}_v \ddot{v} + \left(\mathbf{x}_{uu}\dot{u} + \mathbf{x}_{uv}\dot{v}\right)\dot{u} + \left(\mathbf{x}_{uv}\dot{u} + \mathbf{x}_{vv}\dot{v}\right)\dot{v}\right) = 0; \tag{0.1}$$

$$\mathbf{x}_v \cdot \left(\mathbf{x}_u \ddot{u} + \mathbf{x}_v \ddot{v} + \left(\mathbf{x}_{uu}\dot{u} + \mathbf{x}_{uv}\dot{v}\right)\dot{u} + \left(\mathbf{x}_{uv}\dot{u} + \mathbf{x}_{vv}\dot{v}\right)\dot{v}\right) = 0. \tag{0.2}$$

It remains only to simplify these two equations. First note that

$$\begin{aligned} E = \mathbf{x}_u \cdot \mathbf{x}_u \quad &\Rightarrow \quad E_u = 2\mathbf{x}_u \cdot \mathbf{x}_{uu} \quad \text{and} \quad E_v = 2\mathbf{x}_u \cdot \mathbf{x}_{uv}; \\ G = \mathbf{x}_v \cdot \mathbf{x}_v \quad &\Rightarrow \quad G_u = 2\mathbf{x}_v \cdot \mathbf{x}_{uv} \quad \text{and} \quad G_v = 2\mathbf{x}_v \cdot \mathbf{x}_{vv}; \end{aligned}$$

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = 0 \quad \Rightarrow \quad \mathbf{x}_{uu} \cdot \mathbf{x}_v + \mathbf{x}_u \cdot \mathbf{x}_{uv} = 0 \quad \text{and} \quad \mathbf{x}_{uv} \cdot \mathbf{x}_v + \mathbf{x}_u \cdot \mathbf{x}_{vv} = 0;$$

and so

$$G_u = -2\mathbf{x}_u \cdot \mathbf{x}_{vv} \qquad \text{and} \qquad E_v = -2\mathbf{x}_v \cdot \mathbf{x}_{uu}.$$

Expanding (0.1) and (0.2) and using the above identities, we have

$$\begin{aligned} E\ddot{u} + \frac{1}{2}E_u\dot{u}^2 + E_v\dot{u}\dot{v} - \frac{1}{2}G_u\dot{v}^2 &= 0, \\ G\ddot{v} - \frac{1}{2}E_v\dot{u}^2 + G_u\dot{u}\dot{v} + \frac{1}{2}G_v\dot{v}^2 &= 0, \end{aligned}$$

which, after dividing by $E$ and $G$ respectively, yield the required result.

**Question 4.**

(a) A necessary condition for a surface $S$ to be non-orientable is that it requires at least two coordinate charts to cover $S$. Since the surface here is covered by one coordinate chart, it is orientable.

(b) $\mathbf{x}_u = (1, h'(u)\cos v, h'(u)\sin v)\,; \quad \mathbf{x}_v = (0, -h(u)\sin v, h(u)\cos v).$

Hence $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1 + h_u^2; \quad F = \mathbf{x}_u \cdot \mathbf{x}_v = 0; \quad G = \mathbf{x}_v \cdot \mathbf{x}_v = h^2.$

Now, $F = 0$ implies that $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal. But $\mathbf{x}_u$ and $\mathbf{x}_v$ are not orthonormal since $\| \mathbf{x}_u \| = \sqrt{1 + h_u^2} > 1$.

(c) A minimal surface is one in which the mean curvature is zero at each point on the surface.

(d) Let $e = \mathbf{N} \cdot \mathbf{x}_{uu}$, $f = \mathbf{N} \cdot \mathbf{x}_{uv}$ and $g = \mathbf{N} \cdot \mathbf{x}_{vv}$. Then the mean curvature is

$$H = \frac{Ge + Eg - 2Ff}{2\left(EG - F^2\right)},$$

and $H = 0$ implies that $Ge + Eg - 2Ff = 0$. Put $F = 0$ by part (b) and compute

$$\mathbf{x}_{uu} = \left(0, h''(u)\cos v, h''(u)\sin v\right); \quad \mathbf{x}_{vv} = \left(0, -h(u)\cos v, -h(u)\sin v\right);$$

$$\mathbf{N} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\| \mathbf{x}_u \times \mathbf{x}_v \|} = \frac{(h'(u), -\cos v, -\sin v)}{\sqrt{1+h_u^2}}.$$

Hence

$$e = -\frac{h''}{\sqrt{1+h_u^2}} \qquad \text{and} \qquad g = \frac{h}{\sqrt{1+h_u^2}},$$

and so

$$0 = Ge + Eg = -\frac{h''h^2}{\sqrt{1+h_u^2}} + \frac{h\left(1+h_u^2\right)}{\sqrt{1+h_u^2}}$$

implies that $hh'' = 1 + h_u^2$.

**Question 5.**

(a) $\sigma(t) = (1-t)\mathbf{p} + t\mathbf{q} \quad \Rightarrow \quad \sigma'(t) = \mathbf{q} - \mathbf{p}$. Hence

$$\int_0^1 \| \sigma'(t) \| \, dt \;=\; \| \mathbf{q} - \mathbf{p} \| \int_0^1 dt \;=\; \| \mathbf{q} - \mathbf{p} \| \, .$$

(b) By the Cauchy-Schwartz inequality,

$$\alpha'(t) \cdot \mathbf{u} \;\leq\; \left|\alpha'(t) \cdot \mathbf{u}\right| \;\leq\; \| \alpha'(t) \| \, \| \mathbf{u} \| \;=\; \| \alpha'(t) \| \, .$$

Now using this inequality in

$$L(\alpha) = \int_a^b \| \alpha'(r) \| \, dr \geq \int_a^b \alpha'(r) \cdot \mathbf{u} \, dr = \int_a^b \frac{d}{dr} \left(\alpha(r) \cdot \mathbf{u}\right) \, dr$$

$$= \; \alpha(r) \cdot \mathbf{u}|_a^b \;=\; (\mathbf{q} - \mathbf{p}) \cdot \mathbf{u} \;= \| \mathbf{q} - \mathbf{p} \|$$

then implies that $L(\alpha) \geq \| \mathbf{q} - \mathbf{p} \|$.

(c) Let $\alpha'(t) = \gamma(t)\mathbf{u} + Y(t)$ where $\gamma(t) = \alpha'(t) \cdot \mathbf{u}$ and $\mathbf{u} \cdot Y = 0$. By part (b), we have

$$\int_a^b \gamma(t) dt \;=\; \| \mathbf{q} - \mathbf{p} \| \, .$$

Hence

$$L(\alpha) = \int_a^b \| \alpha'(r) \| \, dr = \int_a^b \sqrt{\gamma^2 + Y \cdot Y} dr \;= \| \mathbf{q} - \mathbf{p} \| = \int_a^b \gamma(r) dr.$$

Therefore

$$\int_a^b \left(\sqrt{\gamma^2 + Y \cdot Y} - \gamma(r)\right) dr \;=\; 0.$$

Since the integrand is non-negative and continuous, the integral vanishes if $\sqrt{\gamma^2 + Y \cdot Y} - \gamma = 0$. But this implies that $\| Y \| = 0$. Hence $L(\alpha) = \| \mathbf{q} - \mathbf{p} \| \;\Rightarrow\; \alpha'(t) = \gamma(t)\mathbf{u}$ with $\alpha(a) = p$. Thus $\alpha(t)$ is a straight line connecting $\mathbf{p}$ and $\mathbf{q}$.

(d) By part (a), a straight line connecting $\mathbf{p}$ and $\mathbf{q}$ has length $\| \mathbf{q} - \mathbf{p} \|$. By part (b), any regular curve connecting $\mathbf{p}$ and $\mathbf{q}$ has length greater than or equal to the straight line distance. But by part (c), equality occurs only if the curve $\alpha(t)$ is a straight line. Therefore, the shortest distance between two points in $\mathbb{R}^n$ with $n \geq 2$ is a straight line.

**Question 6.**

(a) The tangent space at each point on $S$ is given by span $\{\mathbf{x}_u, \mathbf{x}_v\}$; or equivalently, as the set of vectors in $\mathbb{R}^3$ which are orthogonal to $\mathbf{N}$. By definition, $\mathbf{N}(u,v)$ is a unit vector: $\mathbf{N}(u,v) \cdot \mathbf{N}(u,v) = 1$. Hence

$$2\,\frac{\partial \mathbf{N}}{\partial u} \cdot \mathbf{N} = 0 \quad \text{and} \quad 2\,\frac{\partial \mathbf{N}}{\partial v} \cdot \mathbf{N} = 0;$$

that is $\mathbf{N}$ is orthogonal to $\mathbf{N}_u$ and $\mathbf{N}_v$. Thus $\mathbf{N}_u$, $\mathbf{N}_v \in T_p(S)$.

(b) $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal to $\mathbf{N}$. Hence

$$\begin{aligned}
\mathbf{x}_u \cdot \mathbf{N} = 0 \quad &\Rightarrow \quad \mathbf{x}_{uu} \cdot \mathbf{N} = -\mathbf{x}_u \cdot \mathbf{N}_u \\
\mathbf{x}_u \cdot \mathbf{N} = 0 \quad &\Rightarrow \quad \mathbf{x}_{uv} \cdot \mathbf{N} = -\mathbf{x}_u \cdot \mathbf{N}_v \\
\mathbf{x}_v \cdot \mathbf{N} = 0 \quad &\Rightarrow \quad \mathbf{x}_{uv} \cdot \mathbf{N} = -\mathbf{x}_v \cdot \mathbf{N}_u \\
\mathbf{x}_v \cdot \mathbf{N} = 0 \quad &\Rightarrow \quad \mathbf{x}_{vv} \cdot \mathbf{N} = -\mathbf{x}_v \cdot \mathbf{N}_v.
\end{aligned}$$

(c) Take the dot product of each equation with $\mathbf{x}_u$ and $\mathbf{x}_v$ and use the fact that $\mathbf{x}_u \cdot \mathbf{x}_v = 0$:

$$\begin{aligned}
\mathbf{x}_u \cdot \mathbf{N}_u &= a_{11}\,\mathbf{x}_u \cdot \mathbf{x}_u & \qquad \mathbf{x}_u \cdot \mathbf{N}_v &= a_{12}\,\mathbf{x}_u \cdot \mathbf{x}_u \\
\mathbf{x}_v \cdot \mathbf{N}_u &= a_{21}\,\mathbf{x}_v \cdot \mathbf{x}_v & \qquad \mathbf{x}_v \cdot \mathbf{N}_v &= a_{22}\,\mathbf{x}_v \cdot \mathbf{x}_v
\end{aligned}$$

Now use part (b) to transform the left hand side of each equation. The result is

$$a_{11} = -\frac{\mathbf{N} \cdot \mathbf{x}_{uu}}{E}, \qquad a_{12} = -\frac{\mathbf{N} \cdot \mathbf{x}_{uv}}{E}, \qquad a_{21} = -\frac{\mathbf{N} \cdot \mathbf{x}_{uv}}{G}, \qquad a_{22} = -\frac{\mathbf{N} \cdot \mathbf{x}_{vv}}{G}.$$

(d) By part (c), we have $\mathbf{N}_u = a_{11}\,\mathbf{x}_u + a_{21}\,\mathbf{x}_v$ and $\mathbf{N}_v = a_{12}\,\mathbf{x}_u + a_{22}\,\mathbf{x}_v$. Hence

$$\begin{aligned}
\mathbf{N}_u \times \mathbf{N}_v &= (a_{11}\,\mathbf{x}_u + a_{21}\,\mathbf{x}_v) \times (a_{12}\,\mathbf{x}_u + a_{22}\,\mathbf{x}_v) \\
&= a_{11}a_{12}\,\mathbf{x}_u \times \mathbf{x}_u + a_{11}a_{22}\,\mathbf{x}_u \times \mathbf{x}_v + a_{21}a_{12}\,\mathbf{x}_v \times \mathbf{x}_u + a_{21}a_{22}\,\mathbf{x}_v \times \mathbf{x}_v.
\end{aligned}$$

But $\mathbf{x}_u \times \mathbf{x}_u = 0 = \mathbf{x}_v \times \mathbf{x}_v$ and $\mathbf{x}_v \times \mathbf{x}_u = -\mathbf{x}_u \times \mathbf{x}_v$; hence

$$\mathbf{N}_u \times \mathbf{N}_v = (a_{11}a_{22} - a_{12}a_{21})\;\mathbf{x}_u \times \mathbf{x}_v,$$

and so

$$K(p) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$