— Unassessed Coursework 2 —

► Consider the KdV equation with general coefficients

$$a_0 \frac{\partial u}{\partial t} + a_1 u \frac{\partial u}{\partial x} + a_2 \frac{\partial^3 u}{\partial x^3} = 0\,,$$

where a_0, a_1, a_2 are arbitrary nonzero parameters. Introduce a scaling of the dependent and independent variables leading to dimensionless variables

$$\widetilde{u} = \frac{u}{A}\,, \quad \widetilde{x} = \frac{x}{\alpha}\,, \quad t = \frac{t}{\beta}\,.$$

Determine expressions for the scaling parameters A, α and β (as functions of a_0, a_1, a_2) so that the KdV equation reduces to

$$\frac{\partial \widetilde{u}}{\partial \widetilde{t}} + u \frac{\partial \widetilde{u}}{\partial \widetilde{x}} + \delta \frac{\partial^3 \widetilde{u}}{\partial \widetilde{x}^3} = 0\,,$$

where

$$\delta = \mathrm{sign}(a_0 a_2)\,.$$

What is the significance of sign of δ?

► The *Benjamin-Bona-Mahoney equation* (BBM equation) is a model equation for shallow water waves that is equivalent to the KdV equation. The u_{xxx} term is replaced by a u_{xxt} term. Here is a generalisation of it to include $x-$dependent dispersion

$$u_t + u u_x - \alpha(x)^2 u_{xxt} = 0\,, \tag{1}$$

where α is a given nonzero function of x.

- Suppose α is constant, say $\alpha := \alpha_0$. Find the dispersion relation of the linearized version
$$u_t + u_0 u_x - \alpha_0^2 u_{xxt} = 0\,,$$
for normal-mode solutions of the form $u(x,t) = \widehat{u}\mathrm{e}^{\mathrm{i}(kx - \omega t)} + \text{c.c.}$.
- Show that the group velocity associated with the linear waves in part (a) is negative for large k and find the value, k_0, such that $c_g < 0$ for $k > k_0$.
- Consider the nonlinear problem with $\alpha = \alpha_0$ constant. Show that there exists solitary wave solutions of the form
$$u(x,t) = A\,\mathrm{sech}^2(B\xi)\,, \quad \xi = x - ct\,, \quad \text{taking} \quad c > 0\,,$$
and find expressions for A and B.

- For the above solitary waves, does the speed increase or decrease with increasing amplitude?

- Now, consider the case where $\alpha(x)$ varies with x. Show that solutions of BBM have the following perturbed conservation law

$$E_t + F_x = \alpha'(x)\, R(x,t)\,, \quad \text{with} \quad E = \tfrac{1}{2}u^2 + \tfrac{1}{2}\alpha(x)^2 u_x^2\,,$$

  and find expressions for F and R.

► Consider a fluid configuration with two layers of fluid with different densities and irrotational flow. The lower layer has density ρ_1 and extends from $y = 0$ to $y = h_0$. The upper layer has density ρ_2 and extends from $y = h_0$ to $y = +\infty$.

The governing equation and boundary conditions for the linear version of this problem are

$$\begin{aligned} \Delta\phi_1 &= 0 \qquad \text{for } 0 < y < h_0\,, \quad \tfrac{\partial \phi_1}{\partial y} = 0 \text{ at } y = 0\,,\\ \Delta\phi_2 &= 0 \qquad \text{for } h_0 < y < +\infty\,, \quad \tfrac{\partial \phi_2}{\partial y} \to 0 \text{ as } y \to +\infty\,, \end{aligned}$$

where Δ is the Laplacian. The boundary conditions at the interface $y = h_0$ are

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y}\,, \quad \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y}\,, \quad \rho_1 \frac{\partial \phi_1}{\partial t} - \rho_2 \frac{\partial \phi_2}{\partial t} + (\rho_1 - \rho_2) g\eta = 0\,,$$

where g is the positive gravitational constant.

- Consider normal mode solutions of the form $\eta(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c.$ and

$$\phi_1(x,y,t) = B_1(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c. \quad \text{and} \quad \phi_2(x,y,t) = B_2(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c.\,,$$

  where A is a complex constant. Determine expressions for $B_1(y)$ and $B_2(y)$ satisfying the Laplace equation in interior and the boundary conditions at $y = 0$ and $y \to \infty$.

- Using the boundary conditions at $y = h_0$ determine a relationship between $B_1(h_0)$ and A and $B_2(h_0)$ and A.

- Show that the dispersion relation for the system is

$$\omega^2 = \frac{(\rho_1 - \rho_2) gk \tanh(kh_0)}{\rho_1 + \rho_2 \tanh(kh_0)}\,.$$

- Suppose $k > 0$ and $\rho_1 < \rho_2$. What happens to the frequencies? What does this mean physically? Using the dispersion relation as a guide, discuss the implication for the time evolution.