— Unassessed Coursework 1 —

Consider the linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} + \beta u = 0\,, \tag{1}$$

for the scalar-valued function $u(x,t)$, where c_0 and β are given positive constants.

1. Determine the dispersion relation, $D(\omega, k)$, associated with plane wave solutions of the form
$$u(x,t) = \mathrm{Re}\left(A\mathrm{e}^{\mathrm{i}(kx-\omega t)}\right),$$
where k is a wavenumber and ω the frequency.

2. Determine the phase velocity c_p and the group velocity c_g of the plane wave solutions. Show that the group velocity is slower than the phase velocity.

3. Show that $u(x,t) = \mathrm{Re}(u^c(x,t))$, where
$$u^c(x,t) = \int_{-\infty}^{\infty} A(k)\,\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\,\mathrm{d}k\,,$$
is a solution of (1) when $\omega(k)$ satisfies the dispersion relation. You may assume without proof that the integral is absolutely convergent. Suppose
$$u(x,0) = \cos(k_0 x)\,,$$
for some fixed $k_0 > 0$. What is $A(k)$?

4. The energy density for (1) is
$$E = \tfrac{1}{2}u_t^2 + \tfrac{1}{2}c_0^2 u_x^2 + \tfrac{1}{2}\beta u^2\,.$$
Determine the energy flux density F and show that the energy conservation law is
$$E_t + F_x = 0\,.$$

5. Let
$$u(x,t) = A\,\mathrm{e}^{\mathrm{i}\theta} + \overline{A}\,\mathrm{e}^{-\mathrm{i}\theta}\,, \quad \theta = kx - \omega t\,,$$
(here $\overline{A}$ denotes complex conjugate) and substitute it into the energy density and energy flux and average over θ
$$\overline{E} = \frac{1}{2\pi}\int_0^{2\pi} E(\theta)\,\mathrm{d}\theta\,, \quad \overline{F} = \frac{1}{2\pi}\int_0^{2\pi} F(\theta)\,\mathrm{d}\theta\,,$$
(here $\overline{E}$ denotes average). Show that $\overline{F} = c_g\,\overline{E}$ where c_g is the group velocity.

6. Answer questions 1 and 2 for the linear wave equation
$$\frac{\partial^2 u}{\partial t^2} - c_0^2\,\frac{\partial^2 u}{\partial x^2} - \beta\,\frac{\partial^4 u}{\partial t^2 \partial x^2} = 0\,.$$