— Guide to solutions for the assessed coursework —

Q1. Consider the KdV equation in the form

$$u_t + uu_x + u_{xxx} = 0\,.$$

The conservation laws for mass and momentum are

$$\begin{aligned} M_t + Q_x &= 0\,, \quad M = u\,, \quad Q = \tfrac{1}{2}u^2 + u_{xx} \\ I_t + S_x &= 0\,, \quad I = \tfrac{1}{2}u^2\,, \quad S = -\tfrac{1}{2}u_x^2 + uu_{xx} + \tfrac{1}{3}u^3\,. \end{aligned}$$

Show that there also exists a conservation law of the form

$$\frac{\partial}{\partial t}(xM - tI) + \frac{\partial}{\partial x}(\mathsf{Flux}) = 0\,.$$

Determine an expression for Flux.

S1. Differentiating

$$\begin{aligned} \tfrac{\partial}{\partial t}(xM - tI) &= xM_t - tI_t - I \\ &= -xQ_x + tS_x - I \\ &= -(xQ)_x + Q + (tS)_x - I \\ &= -(xQ - tS)_x + Q - I\,. \end{aligned}$$

But $Q - I = u_{xx}$ and so

$$\frac{\partial}{\partial t}(xM - tI) = -(xQ - tS)_x + u_{xx}\,,$$

giving

$$\mathsf{Flux} = xQ - tS - u_x\,.$$

Q2. Consider the nonlinear wave equation

$$u_{tt} + u_{xx} + u_{xxxx} + u + au^2 + bu^3 = 0\,, \tag{1}$$

for the scalar-valued function $u(x,t)$.

- Find the dispersion for the linear problem ($a = b = 0$),
- Let $u(x,t) = U(\theta)$, with $\theta = kx - \omega t$. Reduce the PDE (1) to an ODE for $U(\theta)$, with ω and k appearing in the equation as coefficients.

Take $k > 0$ to be fixed, and expand $U(\theta)$ and ω in a Taylor series in a small parameter ε,

$$\begin{aligned} U(\theta) &= \varepsilon U_1(\theta) + \varepsilon^2 U_2(\theta) + \varepsilon^3 U_3(\theta) + \cdots \\ \omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots\,. \end{aligned}$$

By requiring $U(\theta)$ to be a $2\pi-$periodic function of θ,

- solve for $\omega_0(k)$,
- show that $\omega_1 = 0$,
- determine ω_2 as a function of a and b, and
- determine the particular solution for $U_2(\theta)$, when

$$U_1(\theta) = A\mathrm{e}^{\mathrm{i}\theta} + \overline{A}\mathrm{e}^{-\mathrm{i}\theta},$$

where A is a complex constant of order unity.

S2. The dispersion relation for the linear problem is obtained by substituting a normal mode solution $u = A\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ into the linear equation

$$0 = u_{tt} + u_{xx} + u_{xxxx} + u = (-\omega^2 - k^2 + k^4 + 1)A\mathrm{e}^{\mathrm{i}(kx-\omega t)},$$

giving

$$\omega^2 = 1 - k^2 + k^4.$$

Let $u(x,t) = U(\theta)$ with $\theta = kx - \omega t$. Substitution into (1) gives

$$0 = u_{tt} + u_{xx} + u_{xxxx} + u + au^2 + bu^3 = \omega^2 U'' + k^2 U'' + k^4 U'''' + U + aU^2 + bU^3.$$

Take $k > 0$ to be fixed and expand $U(\theta)$ and ω in a Taylor series in a small parameter ε,

$$\begin{aligned} U(\theta) &= \varepsilon U_1(\theta) + \varepsilon^2 U_2(\theta) + \varepsilon^3 U_3(\theta) + \cdots \\ \omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots . \end{aligned}$$

Substitution into the ODE governing U,

$$\begin{aligned} &(\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2(\varepsilon U_1'' + \varepsilon^2 U_2'' + \varepsilon^3 U_3'' + \cdots) \\ &\quad + k^2(\varepsilon U_1'' + \varepsilon^2 U_2'' + \varepsilon^3 U_3'' + \cdots) \\ &\quad\quad + k^4(\varepsilon U_1'''' + \varepsilon^2 U_2'''' + \varepsilon^3 U_3'''' + \cdots) + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \cdots \\ &\quad\quad\quad + a(\varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \cdots)^2 + b(\varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \cdots)^3 \end{aligned}$$

Define

$$\mathbf{L}\phi = (\omega_0^2 + k^2)\phi'' + k^4\phi'''' + \phi.$$

Then the equations proportional to ε^n, for $n = 1, 2, 3$ are

$$\begin{aligned} \mathbf{L}U_1 &= 0 \\ \mathbf{L}U_2 &= -2\omega_0\omega_1 U_1'' - aU_1^2 \\ \mathbf{L}U_3 &= -\omega_1^2 U_1'' - 2\omega_0\omega_1 U_2'' - 2\omega_0\omega_2 U_1'' - 2aU_1U_2 - bU_1^3. \end{aligned}$$

Using the proposed form for $U_1(\theta)$,

$$0 = \mathbf{L}U_1 = (-\omega_0^2 - k^2 + k^4 + 1)U_1,$$

showing that $\omega_0(k)$ is determined by the dispersion relation of the linear problem

$$\omega_0(k) = \pm\sqrt{1 - k^2 + k^4}. \tag{2}$$

Now consider the equation for U_2 with U_1 substituted into the right-hand side

$$\mathbf{L}U_2 = 2\omega_0\omega_1(A\mathrm{e}^{\mathrm{i}\theta} + \overline{A}\mathrm{e}^{-\mathrm{i}\theta}) - a(A^2\mathrm{e}^{2\mathrm{i}\theta} + 2|A|^2 + \overline{A}^2\mathrm{e}^{-2\mathrm{i}\theta}). \tag{3}$$

There is a homogeneous solution U_2^h and a particular solution U_2^p. The homogeneous solution has the same form as U_1,

$$U_2^h = A_{21}\mathrm{e}^{\mathrm{i}\theta} + \overline{A_{21}}\mathrm{e}^{-\mathrm{i}\theta},$$

with A_{21} an arbitrary complex constant.

The particular solution has the form

$$U_2^p = A_{22}\theta\mathrm{e}^{\mathrm{i}\theta} + \overline{A_{22}}\theta\mathrm{e}^{-\mathrm{i}\theta} + A_{23}|A|^2 + A_{24}\mathrm{e}^{2\mathrm{i}\theta} + \overline{A_{24}}\mathrm{e}^{-2\mathrm{i}\theta}.$$

Substitution then gives

$$\mathbf{L}(A_{22}\theta\mathrm{e}^{\mathrm{i}\theta}) = 2\mathrm{i}(\omega_0^2 + k^2 - 2k^4)A_{22}\mathrm{e}^{\mathrm{i}\theta} = 2\omega_0\omega_1 A\mathrm{e}^{\mathrm{i}\theta},$$

and so

$$A_{22} = -\mathrm{i}\frac{\omega_0\omega_1 A}{\omega_0^2 + k^2 - 2k^4} = -\mathrm{i}\frac{\omega_0\omega_1 A}{(1-k^4)}.$$

Similarly,

$$A_{23} = -2a,$$

and

$$A_{24} = -\frac{aA^2}{1 - 4\omega_0^2 - 4k^2 + 16k^4} = \frac{a}{3}\frac{A^2}{(1-4k^4)}.$$

However, the requirement that $U_j(\theta)$ be $2\pi-$periodic in θ forces A_{22} to be zero, which can only be satisfied if $\omega_1 = 0$. In summary the general solution for $U_2(\theta)$ is

$$U_2 = A_{21}\mathrm{e}^{\mathrm{i}\theta} + \overline{A_{21}}\mathrm{e}^{-\mathrm{i}\theta} - 2a|A|^2 + \frac{a}{3}\frac{1}{(1-4k^4)}(A^2\mathrm{e}^{2\mathrm{i}\theta} + \overline{A}^2\mathrm{e}^{-2\mathrm{i}\theta}).$$

With A_{21} an arbitrary complex constant.

Now we are in a position to solve the equation for U_3. Substituting for U_1 and U_2 into the equation for U_3 gives

$$\begin{aligned}\mathbf{L}U_3 \;=\;& 2\omega_0\omega_2(A\mathrm{e}^{\mathrm{i}\theta} + \overline{A}\mathrm{e}^{-\mathrm{i}\theta})\\ &-2a(A\mathrm{e}^{\mathrm{i}\theta} + \overline{A}\mathrm{e}^{-\mathrm{i}\theta})\Big(A_{21}\mathrm{e}^{\mathrm{i}\theta} + \overline{A_{21}}\mathrm{e}^{-\mathrm{i}\theta} - 2a|A|^2 + \tfrac{a}{3}\tfrac{1}{(1-4k^4)}(A^2\mathrm{e}^{2\mathrm{i}\theta} + \overline{A}^2\mathrm{e}^{-2\mathrm{i}\theta})\Big)\\ &-b\big(A^3\mathrm{e}^{3\mathrm{i}\theta} + 3|A|^2A\mathrm{e}^{\mathrm{i}\theta} + 3|A|^2\overline{A}\mathrm{e}^{-\mathrm{i}\theta} + \overline{A}^3\mathrm{e}^{-3\mathrm{i}\theta}\big).\end{aligned}$$

To determine ω_2 only the terms on the right-hand side proportional to $\mathrm{e}^{\mathrm{i}\theta}$ need to be retained, giving

$$\mathbf{L}U_3 = \left(2\omega_0\omega_2 + 4a^2|A|^2 - \frac{2}{3}a^2\frac{1}{(1-4k^4)}|A|^2 - 3b|A|^2\right)A\mathrm{e}^{\mathrm{i}\theta} + \cdots.$$

The term on the right-hand side generates a particular solution for U_3 that is not $2\pi-$periodic. Setting it to zero then gives an expression for ω_2

$$\omega_2 = \frac{1}{2\omega_0}\left(-4a^2 + \frac{2}{3}a^2\frac{1}{(1-4k^4)} + 3b\right)|A|^2 \tag{4}$$

Hence, the frequency has the form

$$\omega = \omega_0 + \omega_2 \varepsilon^2 + \cdots ,$$

with ω_0 one of the roots of (2) and ω_2 given in (4).

Q3. Consider the NLS equation in the form

$$\mathrm{i} A_t + A_{xx} + |A|^2 A = 0 ,$$

for the complex-value function $A(x,t)$. Show that there exists a solitary wave solution of the form

$$A(x,t) = \mathrm{e}^{\mathrm{i}\omega t} A_0 \,\mathrm{sech}(Bx) ,$$

with ω, B and A_0 real parameters. Find expressions for B and A_0 as functions of ω.

S3. Starting with the assumed form for $A(x,t)$,

$$\begin{aligned} A_t &= \mathrm{i}\omega A \\ A_x &= -B \tanh(Bx)\, A \\ A_{xx} &= B^2 A - 2B^2 \mathrm{sech}^2(Bx)\, A \\ |A|^2 &= A_0^2 \,\mathrm{sech}^2(Bx) . \end{aligned}$$

Substituting into the NLS equation,

$$\begin{aligned} 0 &= \mathrm{i} A_t + A_{xx} + |A|^2 A \\ &= -\omega A + B^2 A - 2B^2 \,\mathrm{sech}^2(Bx) A + A_0^2 \,\mathrm{sech}^2(Bx)\, A \\ &= (B^2 - \omega) A + (A_0^2 - 2B^2)\,\mathrm{sech}^2(Bx) A . \end{aligned}$$

Hence there exists a solution of NLS of the form proposed if

$$B = \pm\sqrt{\omega} \quad \text{and} \quad A_0 = \pm\sqrt{2\omega} ,$$

with the additional requirement that $\omega > 0$. There are four solutions depending on the sign choices

$$A_{\pm}^{+}(x,t) = \sqrt{2\omega}\,\mathrm{sech}(\pm\sqrt{\omega}x) \quad \text{and} \quad A_{\pm}^{-}(x,t) = -\sqrt{2\omega}\,\mathrm{sech}(\pm\sqrt{\omega}x) ,$$

but they are related by $A_{\pm}^{-}(x,t) = -A_{\pm}^{+}(x,t)$, and the two sign choices for the argument are obtained by reversing the sign of x:

$$A_{-}^{\pm}(x,t) = A_{+}^{\pm}(-x,t) .$$

Q4. A weakly nonlinear dispersive wave is described by the equation

$$u_{tt} + u_{xx} + u_{xxxx} + u = \varepsilon u^3 . \tag{5}$$

Introduce variables $X = \varepsilon x$, $T = \varepsilon t$ and θ where

$$\theta_x = k(X,T) \quad \text{and} \quad \theta_t = -\omega(X,T) \quad \Rightarrow \quad k_T + \omega_X = 0 .$$

Seek a solution of (5) in the form

$$u = u_0(\theta, X, T) + \varepsilon u_1(\theta, X, T) + \cdots \quad \text{as} \quad \varepsilon \to 0\,.$$

Write $u_0 = A(X,T)\mathrm{e}^{\mathrm{i}\theta} + c.c.$ and obtain the equation for $A(X,T)$ at first order which ensures that u_1 is periodic in θ.

Using the dispersion relation of the linearised problem, simplify the solvability condition in order to show that

$$A_T + \omega'(k)A_X = \frac{3\mathrm{i}}{2\omega}A|A|^2 - \tfrac{1}{2}k_X\omega''(k)A\,. \tag{6}$$

From (6) derive the following form of conservation of wave action for (5),

$$\frac{\partial}{\partial T}\left(|A|^2\right) + \frac{\partial}{\partial X}\left(c_g|A|^2\right) = 0\,.$$

S4. With new variables X, T and θ, the derivatives transform to

$$\frac{\partial}{\partial x} = k\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial X} \quad \text{and} \quad \frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial T}\,.$$

Hence

$$\begin{aligned} u_{tt} &= \omega^2 u_{\theta\theta} - \varepsilon\omega_T u_\theta - 2\varepsilon\omega u_{\theta T} + \varepsilon^2 u_{TT} \\ u_{xx} &= k^2 u_{\theta\theta} + \varepsilon k_X u_\theta + 2\varepsilon k u_{\theta X} + \varepsilon^2 u_{XX} \\ u_{xxxx} &= k^4 u_{\theta\theta\theta\theta} + 4\varepsilon k^3 u_{\theta\theta\theta X} + 6\varepsilon k^2 k_X u_{\theta\theta\theta} + \mathcal{O}(\varepsilon^2)\,. \end{aligned}$$

Substitute into the governing equation,

$$\begin{aligned} &(\omega^2 + k^2)u_{\theta\theta} + k^4 u_{\theta\theta\theta\theta} + u - \varepsilon u^3 \\ &\quad -\varepsilon\big(\omega_T u_\theta + 2\omega u_{\theta T} - k_X u_\theta - 2k u_{\theta X}\big) \\ &\qquad +\varepsilon\big(4k^3 u_{\theta\theta\theta X} + 6k^2 k_X u_{\theta\theta\theta}\big) + \mathcal{O}(\varepsilon^2) = 0\,. \end{aligned} \tag{7}$$

Now expand u in a perturbation series in ε,

$$u(\theta, X, T, \varepsilon) = u_0(\theta, X, T) + \varepsilon u_1(\theta, X, T) + \mathcal{O}(\varepsilon^2)\,.$$

Substitute into (7) and then equate terms proportional to like powers of ε to zero. The equation proportional to ε^0 is

$$\mathbf{L}u_0 = 0\,,$$

where

$$\mathbf{L} := (\omega^2 + k^2)\frac{\partial^2}{\partial\theta^2} + k^4\frac{\partial^4}{\partial\theta^4} + 1\,.$$

At first order in ε,

$$\begin{aligned} -\mathbf{L}u_1 &= -\omega_T\frac{\partial u_0}{\partial\theta} - 2\omega\frac{\partial^2 u_0}{\partial\theta\partial T} + k_X\frac{\partial u_0}{\partial\theta} + 2k\frac{\partial^2 u_0}{\partial\theta\partial X} \\ &\qquad + 4k^3\frac{\partial^4 u_0}{\partial\theta^3\partial X} + 6k^2 k_X\frac{\partial^3 u_0}{\partial\theta^3} - u_0^3\,. \end{aligned}$$

The solution for u_0 is a normal mode solution

$$u_0(\theta, X, T) = A(X,T)\mathrm{e}^{\mathrm{i}\theta} + c.c.\,,$$

where $A(X,T)$ is to be determined. $\mathbf{L}u_0 = 0$ then gives

$$0 = \mathbf{L}u_0 = (-\omega^2 - k^2 + k^4 + 1)A\mathrm{e}^{\mathrm{i}\theta} + c.c.\,.$$

Hence the dispersion relation is

$$\omega^2 = 1 - k^2 + k^4\,.$$

Substituting u_0 into the right-hand side of the u_1 equation

$$-\mathbf{L}u_1 = \mathrm{e}^{\mathrm{i}\theta}\big(-\mathrm{i}\omega_T A - 2\mathrm{i}\omega A_T + \mathrm{i}k_X A + 2\mathrm{i}kA_X - 4\mathrm{i}k^3 A_X - 6\mathrm{i}k^2 k_X A\big) + c.c. - (A\mathrm{e}^{\mathrm{i}\theta} + \overline{A}\mathrm{e}^{\mathrm{i}\theta})^3\,.$$

In order for u_1 to be a $2\pi-$periodic function of θ, we require the term proportional to $\mathrm{e}^{\mathrm{i}\theta}$ to be zero

$$-\mathrm{i}\omega_T A - 2\mathrm{i}\omega A_T + \mathrm{i}k_X A + 2\mathrm{i}kA_X - 4\mathrm{i}k^3 A_X - 6\mathrm{i}k^2 k_X A - 3|A|^2 A = 0\,. \tag{8}$$

This equation can be simplified using the dispersion relation

$$2\omega\omega'(k) = -2k + 4k^3 \quad \text{and} \quad 2\omega\omega''(k) + 2\omega'\omega' = -2 + 12k^2\,.$$

Hence (8) simplifies to

$$\omega_T A + 2\omega A_T + 2\omega\omega'(k)A_X + (\omega\omega'' + \omega'\omega')k_X A - 3\mathrm{i}|A|^2 A = 0\,. \tag{9}$$

Now use the property

$$\omega_X + k_T = 0 \quad \Rightarrow \quad k_T + \omega'(k)k_X = 0\,,$$

and so

$$\omega_T + \omega'\omega' k_X = \omega_T + \omega'(-k_T) = \omega_T - \omega_T = 0\,.$$

Hence (9) simplies to

$$2\omega A_T + 2\omega\omega'(k)A_X + \omega\omega'' k_X A - 3\mathrm{i}|A|^2 A = 0\,.$$

Dividing by 2ω then gives the required form

$$A_T + \omega'(k)A_X = \frac{3\mathrm{i}}{2\omega}|A|^2 A - \tfrac{1}{2}\omega'' k_X A\,. \tag{10}$$

To determine conservation of wave action multiply (10) by $\overline{A}$,

$$\overline{A}A_T + \omega'(k)\overline{A}A_X = \frac{3\mathrm{i}}{2\omega}|A|^4 - \tfrac{1}{2}\omega'' k_X |A|^2\,.$$

The complex conjugate of this equation is

$$A\overline{A}_T + \omega'(k)A\overline{A}_X = -\frac{3\mathrm{i}}{2\omega}|A|^4 - \tfrac{1}{2}\omega'' k_X |A|^2\,.$$

Adding these two equations

$$\overline{A}A_T + A\overline{A}_T + \omega'(k)(\overline{A}A_X + A\overline{A}_X) = -\omega''(k)k_X|A|^2\,,$$

or

$$\frac{\partial}{\partial T}|A|^2 + \omega'(k)\frac{\partial}{\partial X}|A|^2 + \omega''(k)k_X|A|^2 = 0\,.$$

The second and third terms combine to give

$$\frac{\partial}{\partial T}\big(|A|^2\big) + \frac{\partial}{\partial X}\big(c_g|A|^2\big) = 0\,,$$

which is the required form of the conservation of wave action.