UNIVERSITY OF SURREY ©

Faculty of Engineering and Physical Sciences
Department of Mathematics

M.Math Undergraduate Programme in Mathematical Studies

– Examination –

MMath Module MSM.TWW – THEORY OF WATER WAVES

Time allowed - 2.5 hrs Spring Semester 2008

Attempt FOUR questions.
If any candidate attempts more than FOUR questions,
only the best FOUR solutions will be taken into account.

Question 1

(Linear sloshing with surface tension)
Consider the irrotational linear 2D water-wave problem in finite depth – bewteen two fixed walls – with both gravitational and surface tension forces acting on the fluid.

The velocity potential $\phi(x,y,t)$ is required to satisfy Laplace's equation in the fluid interior and

$$\phi_y = 0 \quad \text{at} \quad y = -h\,,$$
$$\phi_x = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L\,.$$

At the free surface the boundary conditions are

$$\eta_t = \phi_y \quad \text{and} \quad \phi_t + g\eta - \tau\eta_{xx} = 0\,,$$

where $\tau > 0$ is the coefficient of surface tension.

(a) Consider solutions of the form

$$\eta(x,t) = A(x)\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.} \quad \text{and} \quad \phi(x,y,t) = B(x)\cosh k(y+h)\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}$$

Find an expression for $B(x)$ and show that the boundary conditions require $\frac{kL}{\pi}$ to be an integer. [5]

(b) Find a relationship between $A(x)$ and $B(x)$. [5]

(c) Use the free surface boundary conditions to find the infinite set of frequencies ω_n^2 for sloshing. [7]

(d) Show that, in the limit as $h \to \infty$, the set of frequencies are

$$\omega_n^2 = g\frac{n\pi}{L} + \tau\frac{n^3\pi^3}{L^3}\,.$$

[3]

(e) For the case $h \to \infty$ show that there exists values of g, L and τ where resonance occurs: $\omega_2 = 2\omega_1$. [5]

Question 2

(The Kadomtsev-Petviashvili equation)
Consider the KP equation which is a generalization of the KdV equation

$$u_t + uu_x + u_{xxx} + v_y = 0 \quad u_y = v_x \,. \tag{1}$$

(a) Consider the linearized version

$$u_t + u_0 u_x + u_{xxx} + v_y = 0 \quad u_y = v_x \,,$$

where u_0 is a constant. Taking solutions of the form

$$(u(x,y,t), v(x,y,t)) = (\widehat{u}, \widehat{v})\mathrm{e}^{\mathrm{i}(kx+\ell y-\omega t)} + \mathsf{c.c.}\,,$$

with $k \neq 0$, find the dispersion relation. [5]

(b) Consider the nonlinear equation and assume a form for the solution

$$u(x,y,t) = \widehat{u}(\xi)\,, \quad \xi = x - ct + \ell y$$

for some $c > 0$ and $\ell > 0$. Show that there exists a solution of the form

$$\widehat{v}(\xi) = \ell\widehat{u}(\xi)\,, \quad \widehat{u}(\xi) = A\,\mathrm{sech}^2(B\xi)\,,$$

and find expressions for A and B. [9]

(c) What condition does c need to satisfy for existence of the solitary wave solutions found in part (b)? [3]

(d) The KP equation has conservation law for momentum of the form

$$I_t + S_x + F_y = 0\,, \quad \text{with} \quad I = \tfrac{1}{2}u^2\,,$$

when u satisfies KP. Find expressions for the fluxes S and F. [8]

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Question 3

(Wave refraction)
Consider the steady propagation of waves in shallow water governed by the ray equations. The graph of a ray is given by $(X, Y(X))$ with $Y(X)$ satisfying

$$\frac{d}{dX}\left(\frac{\sigma Y_X}{\sqrt{(1+Y_X^2)}}\right) - \sigma_Y\sqrt{(1+Y_X^2)} = 0\,. \qquad (2)$$

where $D(X, Y)$ is determined by bottom topography, and $\sigma(X, Y)$ is determined from the equation

$$(g\sigma(X,Y) + \tau\sigma(X,Y)^3)\tanh(\sigma(X,Y)D(X,Y)) = \mathsf{constant}\,,$$

where $\tau > 0$ is the coefficient of surface tension.

(a) Suppose $\sigma_Y = 0$. Derive Snell's Law, $\sigma(X)\sin\theta = \mathsf{constant}$, where $\theta = \tan^{-1}(Y_X)$ with $0 < \theta < \pi/2$. [6]

(b) In cylindrical polar coordinates, (r, θ), suppose the wave rays depend only on r. Then Fermat's integral becomes

$$L(\theta) = \int_{r_1}^{r_2} \sigma(r)\sqrt{1 + r^2\,\theta_r^2}\,\mathrm{d}r\,,$$

where $\theta_r = \frac{d\theta}{dr}$. Show that the Euler-Lagrange equation associated with this integral leads to the following equation for θ_r

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{K^2}{r^2(\sigma^2 r^2 - K^2)}\,,$$

where K is a constant. [11]

(c) By defining an angle α

$$\sin\alpha = \frac{r\theta_r}{\sqrt{1 + r^2\theta_r^2}}\,,$$

derive a form of Snell's law in polar coordinates. [8]

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Question 4

(Conservation of wave action for modified KdV)
Consider the modified[1] KdV equation in standard form

$$u_t + au_x + \varepsilon\, u^2 u_x + u_{xxx} = 0\,, \tag{3}$$

where a is a constant and ε is a small parameter.

(a) For the linear equation $u_t + au_x + u_{xxx} = 0$, determine the dispersion relation and group velocity for normal-mode solutions: $u(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c.$ [5]

(b) Let $X = \varepsilon x$ and $T = \varepsilon t$ and define a phase by $\theta_x = k(X,T)$ and $\theta_t = -\omega(X,T)$. Using the chain rule:

$$\frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial T}\,, \quad \frac{\partial}{\partial x} = k\frac{\partial}{\partial \theta} + \varepsilon\frac{\partial}{\partial X}\,,$$

transform the PDE into an equation for $u(\theta, X, T, \varepsilon)$. [6]

(c) Let

$$u(\theta,X,T,\varepsilon) = u_0(\theta,X,T) + \varepsilon\, u_1(\theta,X,T) + \cdots\,,$$

and take $u_0 = A(X,T)\mathrm{e}^{\mathrm{i}\theta} + \overline{A(X,T)}\mathrm{e}^{-\mathrm{i}\theta}$ By requiring u_1 to be $2\pi-$periodic in θ, show that the solvability condition requires $A(X,T)$ to satisfy

$$A_T + c_g\, A_X - 3kk_X A + \mathrm{i}k|A|^2 A = 0\,. \tag{4}$$

[8]

(d) Show that *conservation of wave action*

$$\frac{\partial}{\partial T}\left(|A|^2\right) + \frac{\partial}{\partial X}\left(c_g|A|^2\right) = 0\,,$$

follows from (4). [6]

[1] "Modified" means that the nonlinearity is cubic rather than quadratic.

Question 5

(The Benjamin-Bona-Mahoney equation)
The BBM equation is a model equation for shallow water waves of the form

$$u_t + uu_x - \alpha(x)^2 u_{xxt} = 0\,, \qquad (5)$$

where α is a given nonzero function of x.

(a) Suppose α is constant, say $\alpha := \alpha_0$. Find the dispersion relation of the linearized version

$$u_t + u_0 u_x - \alpha_0^2 u_{xxt} = 0\,,$$

for normal-mode solutions of the form $u(x,t) = \widehat{u}\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}$. [4]

(b) Show that the group velocity associated with the linear waves in part (a) is negative for large k and find the value, k_0, such that $c_g < 0$ for $k > k_0$. [5]

(c) Consider the nonlinear problem with α constant. Show that there exists solitary wave solutions of the form

$$u(x,t) = A\,\mathrm{sech}^2(B\xi)\,, \quad \xi = x - ct\,, \quad \text{taking} \quad c > 0\,,$$

and find expressions for A and B. [8]

(d) For the solitary waves in part (c), does the speed increase or decrease with increasing amplitude? [2]

(e) Now, consider the case where $\alpha(x)$ varies with x. Show that solutions of BBM have the following perturbed conservation law

$$E_t + F_x = \alpha'(x)\,R(x,t)\,, \quad \text{with} \quad E = \tfrac{1}{2}u^2 + \tfrac{1}{2}\alpha(x)^2 u_x^2\,,$$

and find expressioins for F and R. [6]

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