UNIVERSITY OF SURREY©

School of Electronics and Physical Sciences

Department of Mathematics

M.Math Undergraduate Programme in Mathematical Studies

Examination

Module MSM.TWW THEORY OF WATER WAVES

Time allowed - 2.5 hrs Spring Semester 2007

Attempt FOUR questions.
If any candidate attempts more than FOUR questions,
only the best FOUR solutions will be taken into account.

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Question 1

(Quintic nonlinear Schrödinger equation)
The nonlinear Schrödinger equation with quintic nonlinearity, for the complex-valued function $A(x,t)$, is

$$\mathrm{i}\,A_t + A_{xx} + a\,|A|^4 A = 0\,, \quad a = \pm 1\,. \tag{1}$$

Associated with quintic NLS are the conserved densities

$$M = |A|^2\,, \quad I = -\mathrm{i}\,(\overline{A}A_x - A\overline{A_x})\,, \quad E = |A_x|^2 - \frac{a}{3}|A|^6\,.$$

(a) Find fluxes Q, S and F such that when A is a solution of (1) then

$$M_t + Q_x = 0\,, \quad I_t + S_x = 0\,, \quad E_t + F_x = 0\,.$$

[6]

(b) Suppose $A(x,t) \to 0$ as $x \to \pm\infty$. Show that

$$\int_{-\infty}^{+\infty} S(x,t)\,\mathrm{d}x = C \int_{-\infty}^{+\infty} E(x,t)\,\mathrm{d}x\,,$$

and determine the value of the constant C. [5]

(c) Suppose $A(x,t) \to 0$ as $x \to \pm\infty$ and consider the moment of inertia of a solution,

$$J(t) = \int_{-\infty}^{+\infty} x^2 |A(x,t)|^2\,\mathrm{d}x\,.$$

By differentiating $J(t)$ and using the conservation laws show that

$$\frac{d^2 J}{dt^2} = 2\int_{-\infty}^{+\infty} S(x,t)\,\mathrm{d}x\,.$$

[9]

(d) What can one conclude about existence of solutions of (1)? Consider the cases $a = -1$ and $a = +1$ separately. [5]

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Question 2

(Conservation of wave action)
Consider the nonlinear wave equation

$$u_{tt} - u_{xx} + u = \varepsilon\, u^3\,, \tag{2}$$

where ε is a small parameter.

(a) For the linear equation, $u_{tt} - u_{xx} + u = 0$, determine the dispersion relation and group velocity for normal-mode solutions: $u(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c.$ [3]

(b) Let $X = \varepsilon x$ and $T = \varepsilon t$ and define a phase by $\theta_x = k(X,T)$ and $\theta_t = -\omega(X,T)$. Using the chain rule:

$$\frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial T}\,, \quad \frac{\partial}{\partial x} = k\frac{\partial}{\partial\theta} + \varepsilon\frac{\partial}{\partial X}\,,$$

transform the PDE into an equation for $u(\theta, X, T, \varepsilon)$. [4]

(c) Let

$$u(\theta, X, T, \varepsilon) = u_0(\theta, X, T) + \varepsilon\, u_1(\theta, X, T) + \cdots\,,$$

and take $u_0 = A(X,T)\mathrm{e}^{\mathrm{i}\theta} + \overline{A(X,T)}\mathrm{e}^{-\mathrm{i}\theta}$ By requiring u_1 to be $2\pi-$periodic in θ, show that the solvability condition requires $A(X,T)$ to satisfy

$$\omega_T A + 2\omega A_T + k_X A + 2kA_X - 3\mathrm{i}\,|A|^2 A = 0\,. \tag{3}$$

[10]

(d) Show that *conservation of wave action*

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega}\right) + \frac{\partial}{\partial X}\left(c_g\frac{E}{\omega}\right) = 0\,, \quad \text{where} \quad E = \omega^2|A|^2\,,$$

follows from (3). [8]

Question 3

(Wave refraction and ray theory)
Consider the steady propagation of waves in shallow water governed by the ray equations. The graph of a ray is given by $(X, Y(X))$ with $Y(X)$ satisfying

$$\frac{d}{dX}\left(\frac{\sigma Y_X}{\sqrt{(1+Y_X^2)}}\right) - \sigma_Y\sqrt{(1+Y_X^2)} = 0\,. \qquad (4)$$

where $\sigma(X, Y)$ is determined from the equation

$$g\sigma(X,Y)\tanh(\sigma(X,Y)D(X,Y)) = \mathsf{constant}\,,$$

and $D(X, Y)$ is determined by bottom topography. You may assume throughout that $\frac{dY}{dX} > 0$.

(a) Suppose $\sigma_Y = 0$ and $\sigma'(X) > 0$ (signifying decreasing depth). Show that

$$\frac{d}{dX}\left(\frac{dY}{dX}\right) < 0\,.$$

[5]

(b) Let $Y_X = \tan(\theta)$ with $0 < \theta < \pi/2$ and $\sigma_Y = 0$. Show that rays satisfy Snell's Law

$$\sigma(X)\sin\theta = \mathsf{constant}\,.$$

[6]

(c) Suppose $\sigma(Y)$; that is, a function of Y only. Show that rays satisfy

$$\frac{\sigma(Y)}{\sqrt{1+Y_X^2}} = \mathsf{constant}\,.$$

[8]

(d) Suppose as in part (c) that σ depends on Y only, and $\sigma_Y > 0$. Does

$$\frac{d}{dX}\left(\frac{dY}{dX}\right)$$

increase or decrease along rays? [6]

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Question 4

(The Korteweg-DeVries equation)
Consider the KdV equation with variable dispersion

$$u_t + uu_x + a(x)u_{xxx} = 0\,. \tag{5}$$

(a) Suppose $a(x) = -1$ for all x (negative dispersion) and consider travelling wave solutions of (5) of the form $u(x,t) = \eta(\xi)$ with $\xi = x + ct$ with positive wave speed $c > 0$. Show that η satisfies

$$-\eta_{\xi\xi} + \tfrac{1}{2}\eta^2 + c\eta = \mathsf{constant}\,. \tag{6}$$

[3]

(b) Take $a = -1$, $c > 0$ and $\mathsf{constant} = 0$. Show that there is a solitary wave solution $\eta(\xi)$ of (6) with $\eta(\xi) \to 0$ as $\xi \to \pm\infty$. Give an explicit expression for $\eta(\xi)$. How is this solitary wave different from the classical KdV solitary wave where $a(x) := +1$? [8]

(c) Consider (5) with $a(x)$ a given smooth function which is not identically zero and let $M(x,t) := u(x,t)$ and $Q(x,t) = \frac{1}{2}u^2 + a(x)u_{xx}$. Show that

$$M_t + Q_x = R_1(x,t)\,,$$

where $R_1(x,t)$ *is not zero* when $a(x)$ is nonconstant, even when $u(x,t)$ satisfies (5). Give an expression for $R_1(x,t)$. [6]

(d) Similar to part (c) show that the momentum conservation law for (5) is perturbed to

$$I_t + S_x = R_2(x,t)\,, \quad I(x,t) = \tfrac{1}{2}u(x,t)^2\,,$$

when $u(x,t)$ is a solution of (5). Give expressions for $S(x,t)$ and $R_2(x,t)$. [8]

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Question 5

(Dispersion properties of interfacial waves)

Consider a fluid configuration with two layers of fluid with different densities and irrotational flow. The lower layer has density ρ_1 and extends from $y = 0$ to $y = h$. The upper layer has density ρ_2 and extends from $y = h$ to $y = +\infty$.

The governing equation and boundary conditions are

$$\begin{aligned} \Delta\phi_1 = 0 \qquad & \text{for } 0 < y < h\,, \quad \tfrac{\partial \phi_1}{\partial y} = 0 \text{ at } y = 0\,, \\ \Delta\phi_2 = 0 \qquad & \text{for } h < y < +\infty\,, \quad \tfrac{\partial \phi_2}{\partial y} \to 0 \text{ as } y \to +\infty\,, \end{aligned}$$

where Δ is the Laplacian. The boundary conditions at the interface $y = h$ are

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y}\,, \quad \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y}\,, \quad \rho_1 \frac{\partial \phi_1}{\partial t} - \rho_2 \frac{\partial \phi_2}{\partial t} + (\rho_1 - \rho_2) g \eta = 0\,,$$

where g is the positive gravitational constant.

(a) Consider normal mode solutions of the form $\eta(x,t) = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c.$ and

$$\phi_1(x,y,t) = B_1(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c. \quad \text{and} \quad \phi_2(x,y,t) = B_2(y)\mathrm{e}^{\mathrm{i}(kx-\omega t)} + c.c.\,,$$

where A is a complex constant. Determine expressions for $B_1(y)$ and $B_2(y)$ satisfying the Laplace equation in interior and the boundary conditions at $y = 0$ and $y \to \infty$. [6]

(b) Using the boundary conditions at $y = h$ determine a relationship between $B_1(h)$ and A and $B_2(h)$ and A. [7]

(c) Show that the dispersion relation for the system is

$$\omega^2 = \frac{(\rho_1 - \rho_2) g k \tanh(kh)}{\rho_1 + \rho_2 \tanh(kh)}\,.$$

[8]

(d) Suppose $k > 0$ and $\rho_1 < \rho_2$. What does this mean physically? Using the dispersion relation as a guide, discuss the implication for the time evolution. [4]

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