MATM016: Theory of Water Waves Autumn 2010

— Class Test : Wednesday 24 November —

1. For a normal mode solution, of a linear wave equation, with frequency ω and wavenumber vector $\mathbf{k} = (k.\ell)$, define the *group velocity vector* $\mathbf{c}_g$. [6]

2. Consider the linear Zakharov-Kuznetsov equation

$$u_t + u_0 u_x + u_{xxx} + u_{xyy} = 0\,,$$

for the scalar-valued function $u(x,t)$ where u_0 is a nonzero constant. For a normal-mode solution of the form $u(x,t) = \widehat{u}\mathrm{e}^{\mathrm{i}(kx+\ell y=\omega t)} + c.c.$, with $\widehat{u}$ a complex constant, find (a) the dispersion relation and (b) the group velocity vector. [18]

3. For the dispersion relation in Question 2, show that the group velocity vector $\mathbf{c}_g$ and the wavenumber vector $\mathbf{k}$ are not in general parallel. [8]

4. Consider the nonlinear wave equation

$$u_{tt} - u_{xx} + 4u - 6u^2 = 0\,, \tag{1}$$

for the scalar-valued function $u(x,t)$. Show that there is a stationary solitary wave solution of the steady problem of the form

$$u(x,t) := \widehat{u}(x) = A\,\mathrm{sech}^2(Bx)\,,$$

and determine A and B. [10]

5. For the nonlinear wave equation (1), show that there is a conservation law of the form

$$I_t + S_x = 0 \quad \text{with} \quad I = u_t u_x\,.$$

Determine the flux $S(x,t)$ for this conservation law. [10]

6. Consider the Euler equations for an inviscid fluid in two space dimensions and time,

$$\begin{aligned} u_t + uu_x + vu_y + \rho^{-1}p_x &= 0 \\ v_t + uv_x + vv_y + \rho^{-1}p_y + g &= 0\,, \end{aligned}$$

where $g > 0$ is the gravitational constant, ρ is the constant fluid density, $p(x, y, t)$ is the pressure and $(u(x, y, t), v(x, y, t))$ is the velocity field. With the assumptions $u = \phi_x$ and $v = \phi_y$, show that these equations can be reduced to Bernoulli's equation

$$\phi_t + \tfrac{1}{2}\big(\phi_x^2 + \phi_y^2\big) + gy + \frac{p}{\rho} = f(t)\,.$$

where $f(t)$ is an arbitrary function of time. [18]

7. Suppose a wave ray is defined by the graph $(X, Y(X))$ and

$$\frac{dY}{dX} = \frac{\ell}{k}\,,$$

where (k, ℓ) are components of the wavenumber vector $\mathbf{k} = (k, \ell)$. Define

$$\sigma^2 = k^2 + \ell^2\,, \quad k = \frac{\partial \Theta}{\partial X} \quad \text{and} \quad \ell = \frac{\partial \Theta}{\partial Y}\,.$$

Show that

$$\frac{\sigma}{\sqrt{1 + Y_X^2}} = \frac{\partial \Theta}{\partial X}\,.$$

Derive the ray equation

$$\frac{d}{dX}\left(\frac{\sigma Y_X}{\sqrt{1 + Y_X^2}}\right) - \sigma_Y \sqrt{1 + Y_X^2} = 0\,.$$

[20]

8. Consider a normal mode solution in one space dimension and time, with frequency ω and wavenumber k. Prove the following relationship between the group velocity c_g and the phase velocity c_p

$$c_g - c_p = k\frac{dc_p}{dk}\,.$$

[10]