— Assessed Coursework —

1. Consider the KdV equation in the form

$$u_t + uu_x + u_{xxx} = 0\,.$$

The conservation laws for mass and momentum are

$$\begin{aligned} M_t + Q_x &= 0\,, \quad M = u\,, \quad Q = \tfrac{1}{2}u^2 + u_{xx} \\ I_t + S_x &= 0\,, \quad I = \tfrac{1}{2}u^2\,, \quad S = -\tfrac{1}{2}u_x^2 + uu_{xx} + \tfrac{1}{3}u^3\,. \end{aligned}$$

Show that there also exists a conservation law of the form

$$\frac{\partial}{\partial t}(xM - tI) + \frac{\partial}{\partial x}(\mathsf{Flux}) = 0\,.$$

Determine an expression for Flux. [10]

2. Consider the nonlinear wave equation

$$u_{tt} + u_{xx} + u_{xxxx} + u + au^2 + bu^3 = 0\,, \tag{1}$$

for the scalar-valued function $u(x,t)$.

- Find the dispersion for the linear problem ($a = b = 0$), [5]
- Let $u(x,t) = U(\theta)$, with $\theta = kx - \omega t$. Reduce the PDE (1) to an ODE for $U(\theta)$, with ω and k appearing in the equation as coefficients. [5]

Take $k > 0$ to be fixed, and expand $U(\theta)$ and ω in a Taylor series in a small parameter ε,

$$\begin{aligned} U(\theta) &= \varepsilon U_1(\theta) + \varepsilon^2 U_2(\theta) + \varepsilon^3 U_3(\theta) + \cdots \\ \omega &= \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots\,. \end{aligned}$$

By requiring $U(\theta)$ to be a $2\pi-$periodic function of θ,

- solve for $\omega_0(k)$, [5]
- show that $\omega_1 = 0$, [5]
- determine ω_2 as a function of a and b, and [15]
- determine the particular solution for $U_2(\theta)$, when [10]

$$U_1(\theta) = A\mathrm{e}^{\mathrm{i}\theta} + \overline{A}\mathrm{e}^{-\mathrm{i}\theta}\,,$$

where A is a complex constant of order unity. [5]

3. Consider the NLS equation in the form

$$\mathrm{i}A_t + A_{xx} + |A|^2 A = 0\,,$$

for the complex-value function $A(x,t)$. Show that there exists a solitary wave solution of the form

$$A(x,t) = \mathrm{e}^{\mathrm{i}\omega t} A_0 \,\mathrm{sech}(Bx)\,,$$

with ω, B and A_0 real parameters. Find expressions for B and A_0 as functions of ω. [10]

4. (This question is based on **Q1.54** on page 59 of JOHNSON[1])

A weakly nonlinear dispersive wave is described by the equation

$$u_{tt} + u_{xx} + u_{xxxx} + u = \varepsilon u^3\,. \qquad (2)$$

Introduce variables $X = \varepsilon x$, $T = \varepsilon t$ and θ where

$$\theta_x = k(X,T) \quad \text{and} \quad \theta_t = -\omega(X,T),,$$

which implies

$$k_T + \omega_X = 0 \quad \text{and} \quad k_T + c_g k_X = 0\,.$$

Seek a solution of (2) in the form

$$u = u_0(\theta, X, T) + \varepsilon u_1(\theta, X, T) + \cdots \quad \text{as} \quad \varepsilon \to 0\,.$$

Write $u_0 = A(X,T)\mathrm{e}^{\mathrm{i}\theta} + c.c.$ and obtain the equation for $A(X,T)$ at first order which ensures that u_1 is periodic in θ. [5]

Using the dispersion relation of the linearised problem, simplify the solvability condition in order to show that [15]

$$A_T + \omega'(k) A_X = \frac{3\mathrm{i}}{2\omega} A|A|^2 - \tfrac{1}{2} k_X \omega''(k) A\,. \qquad (3)$$

From (3) derive the following form of conservation of wave action for (2), [10]

$$\frac{\partial}{\partial T}\left(|A|^2\right) + \frac{\partial}{\partial X}\left(c_g |A|^2\right) = 0\,.$$

[1]R.S. Johnson. *A modern introduction to the mathematical theory of water waves*, CUP (1997)