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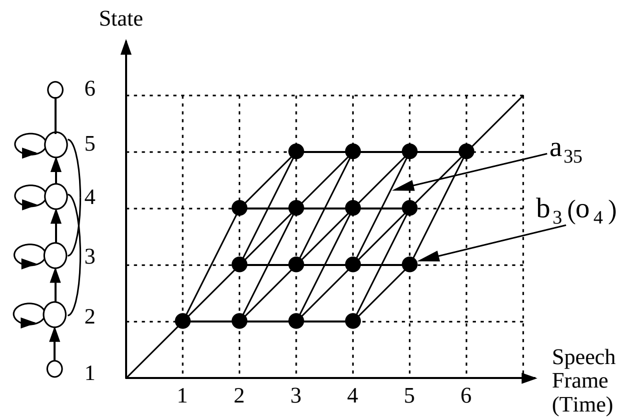
HMM tutorial 2

by Dr Philip Jackson

- Recap. of MMs and HMMs
- Computing likelihoods
  - Markov model
  - Hidden Markov model
- Finding best state sequence
  - Viterbi algorithm
  - Trellis diagram
- Summary

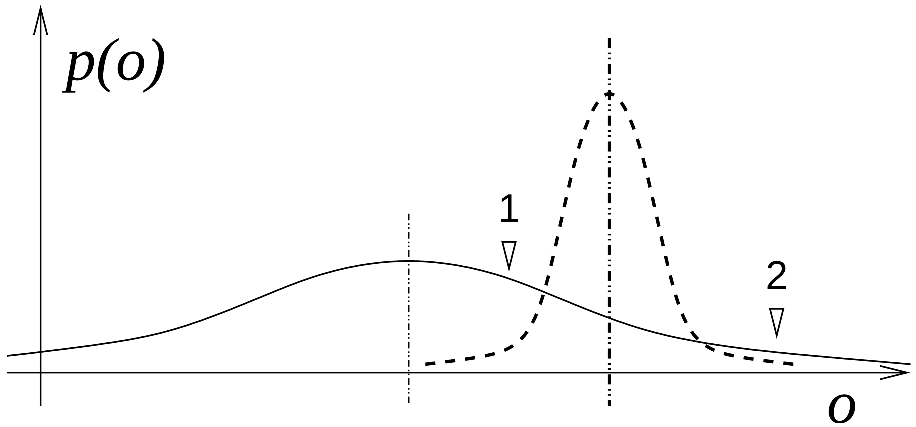


from (Young et al. 1997)

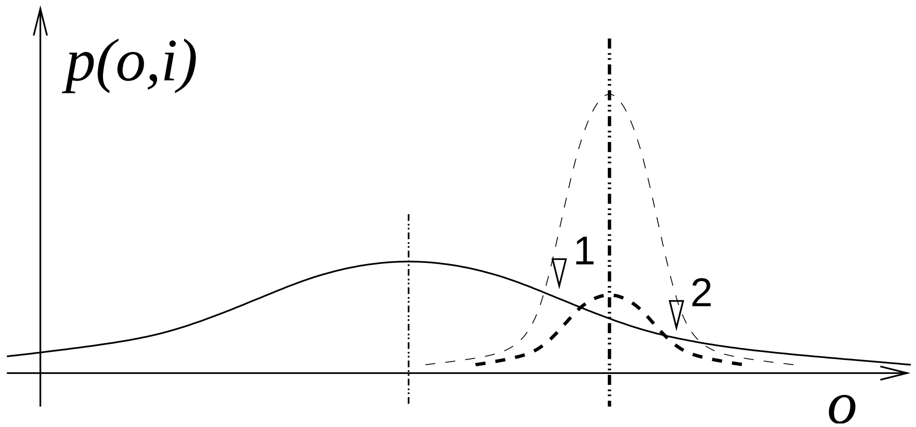
http://www.ee.surrey.ac.uk/Personal/P.Jackson/tutorial/

Recapitulation: fundamentals

Maximum likelihood estimation



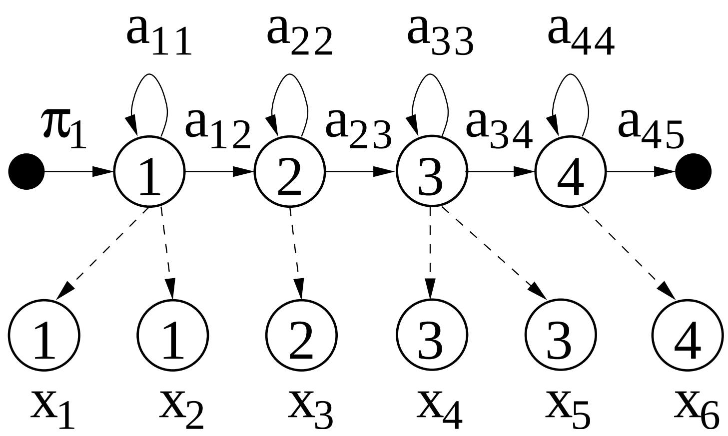
Bayesian estimation



Markov Model, $\mathcal{M}$

(a) Initial-state probabilities,
$\pi = \{\pi_i\} = \{P(x_1 = i)\}$ for $1 \le i \le N$;

(b) State-transition probabilities,
$A = \{a_{ij}\} = \{P(x_t = j | x_{t-1} = i)\}$ for $1 \le i, j \le N$;



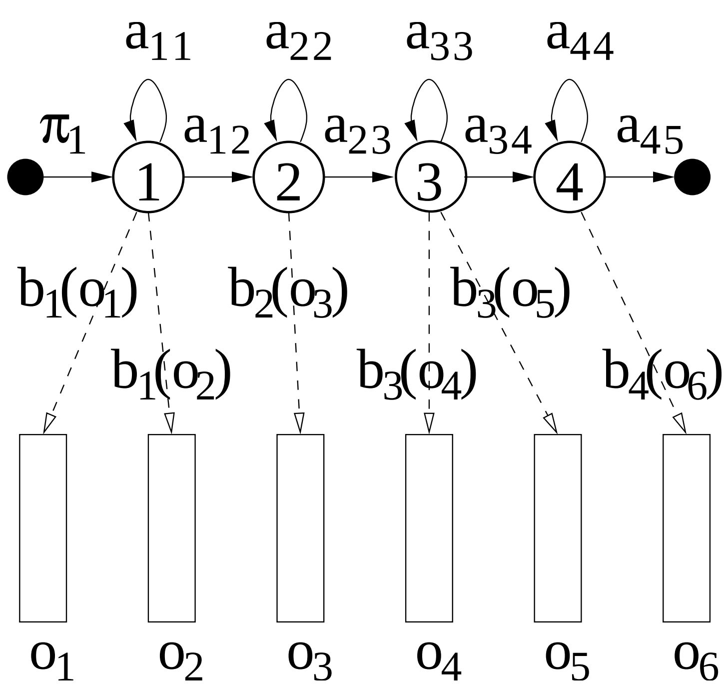
producing a sequence
$X = \{1, 1, 2, 3, 3, 4\}$.

Hidden Markov Model, λ

(a) Initial-state probabilities,
$\pi = \{\pi_i\} = \{P(x_1 = i)\}$ for $1 \le i \le N$;

(b) State-transition probabilities,
$A = \{a_{ij}\} = \{P(x_t = j | x_{t-1} = i)\}$ for $1 \le i, j \le N$;

(c) Discrete output probabilities,
$B = \{b_i(k)\} = \{P(o_t = k | x_t = i)\}$ for $1 \le i \le N$ and $1 \le k \le K$.



producing observations with a state sequence
$X = \{1, 1, 2, 3, 3, 4\}$.

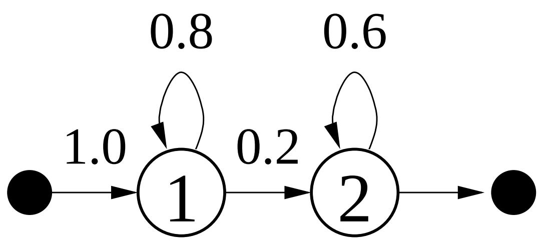
Three problems for HMMs

1. Compute likelihoods $P(\mathcal{O}|\lambda)$;

2. Find best state sequence X^*;

3. Re-estimate model parameters $\Lambda = \{\lambda\}$.

Likelihood for MM state sequence



Transition probabilities:

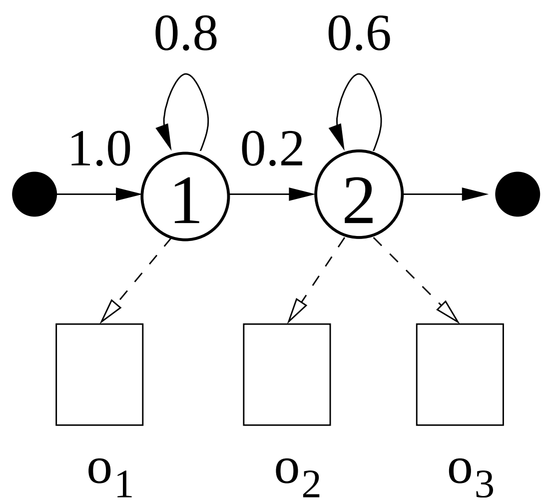
$$\pi = \{\pi_i\} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \text{and}$$

$$A = \left\{a_{ij}\right\} = \begin{bmatrix} 0.8 & 0.2 \\ 0 & 0.6 \end{bmatrix}.$$

Probability of a certain state sequence, $X = \{1, 2, 2\}$:

$$\begin{aligned} P(X|\mathcal{M}) &= \pi_1 \, a_{12} \, a_{22} \\ &= 1 \cdot \\ &= \end{aligned} \tag{1}$$

Likelihood for HMM state sequence



Output probabilities: $B = \begin{bmatrix} b_1(k) \\ b_2(k) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0 & 0.9 & 0.1 \end{bmatrix}$.

Probability with a certain state sequence, $X = \{1, 2, 2\}$:

$$\begin{aligned} P(\mathcal{O}, X|\lambda) &= P(\mathcal{O}|X, \lambda)P(X|\lambda) \\ &= \pi_1 b_1(o_1)\, a_{12} b_2(o_2)\, a_{22} b_2(o_3) \\ &= 1 \cdot \\ &\approx \end{aligned} \tag{2}$$

Problem 1: Computing $P(\mathcal{O}|\lambda)$

Joint probability of the observations and state sequence, for a given model λ:

$$\begin{aligned} P(\mathcal{O}, X|\lambda) &= P(\mathcal{O}|X, \lambda)P(X|\lambda) \\ &= \pi_1 b_1(o_1)\, a_{11} b_1(o_2)\, a_{12} b_2(o_3) \ldots \end{aligned} \tag{3}$$

To get the total probability of the observations, we must sum across all possible state sequences:

$$P(\mathcal{O}|\lambda) = \sum_X P(\mathcal{O}|X, \lambda)P(X|\lambda). \tag{4}$$

Forward procedure

Consider $\alpha_t(i) = P(o_1, o_2, \ldots, o_t, x_t = i|\lambda)$:

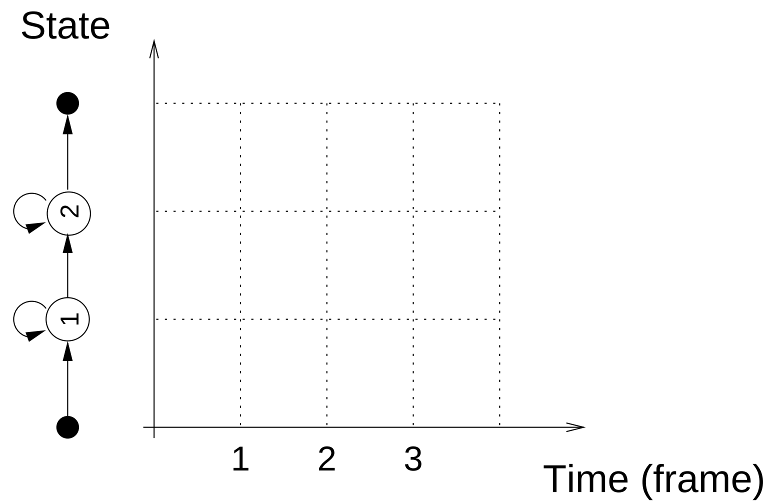
1. Initially,
   $\alpha_1(i) = \pi_i \, b_i(o_1)$, for $1 \leq i \leq N$;

2. For $t = 2, 3, \ldots, T$,
   $\alpha_t(j) = \left[\sum_{i=1}^{N} \alpha_{t-1}(i) \, a_{ij}\right] b_j(o_t)$, for $1 \leq j \leq N$; (5)

3. Finally,
   $P(\mathcal{O}|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$.

Thus, we can solve Problem 1 efficiently by recursion.

Worked example of the forward procedure



Backward procedure

Define $\beta_t(i) = P(o_{t+1}, o_{t+2}, \ldots, o_T | x_t = i, \lambda)$:

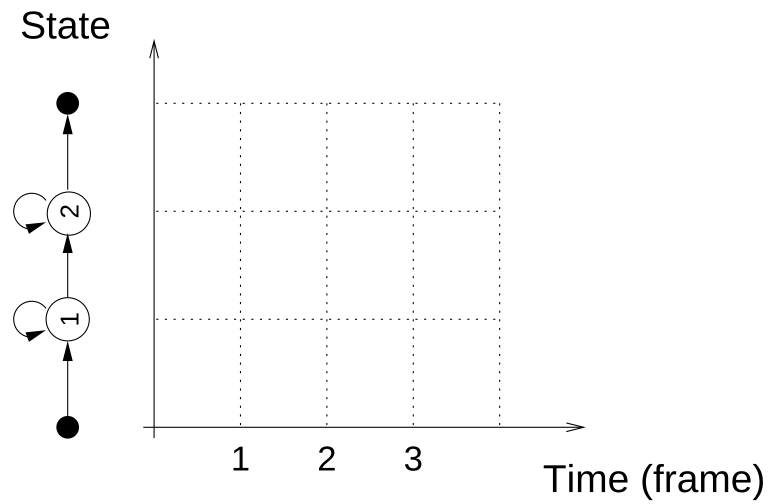
1. Initially,
   $\beta_T(i) = 1,$ for $1 \le i \le N$;

2. For $t = T-1, T-2, \ldots, 1,$
   $\beta_t(i) = \sum_{j=1}^{N} a_{ij}\, b_j(o_{t+1})\, \beta_{t+1}(j),$ for $1 \le i \le N$; (6)

3. Finally,
   $P(\mathcal{O}|\lambda) = \sum_{i=1}^{N} \pi_i\, b_i(o_1)\, \beta_1(i).$

We now have another efficient way of computing $P(\mathcal{O}|\lambda)$.

Worked example of the backward procedure



Problem 2: best state sequence

Given observations $\mathcal{O} = \{o_1, \ldots, o_T\}$, find the state sequence $X = \{x_1, \ldots, x_T\}$ with greatest likelihood:

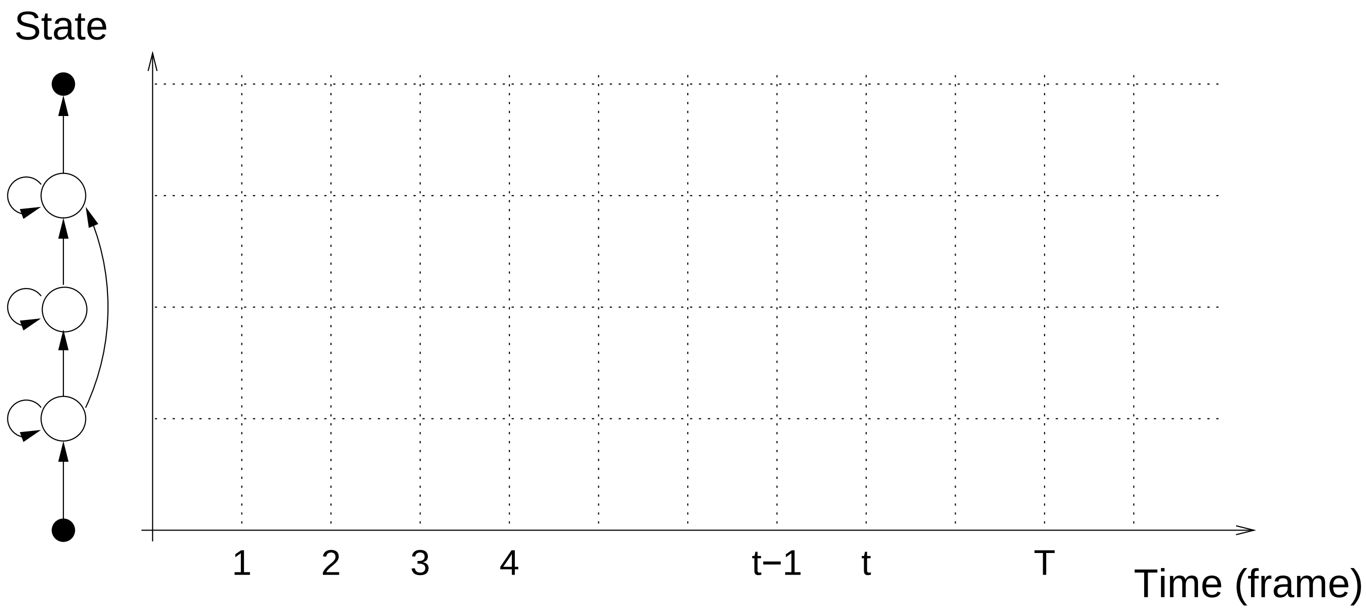
$$\begin{aligned} X^* &= \arg\max_X P(\mathcal{O}, X|\lambda) \\ &= \arg\max_X \Delta(X) \end{aligned} \qquad (7)$$

where

$$\Delta(X) = \pi_{x_1} b_{x_1}(o_1). \prod_{t=2}^{T} a_{x_{t-1}x_t} b_{x_t}(o_t). \qquad (8)$$

The *Viterbi algorithm* is an inductive algorithm that allows us to find the optimal state sequence X^* efficiently.

Step 1

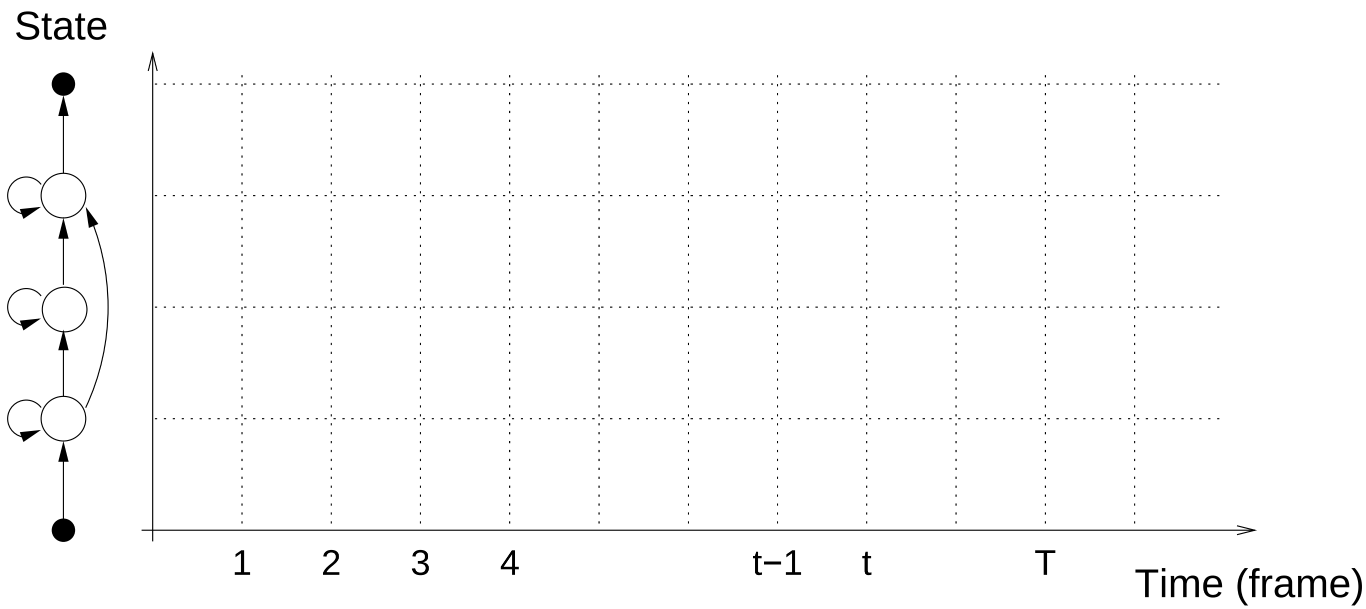


1. Initially,

$$\delta_1(i) = \pi_i b_i(o_1)$$

$$\psi_1(i) = 0 \qquad \text{for } 1 \leq i \leq N;$$

Step 2

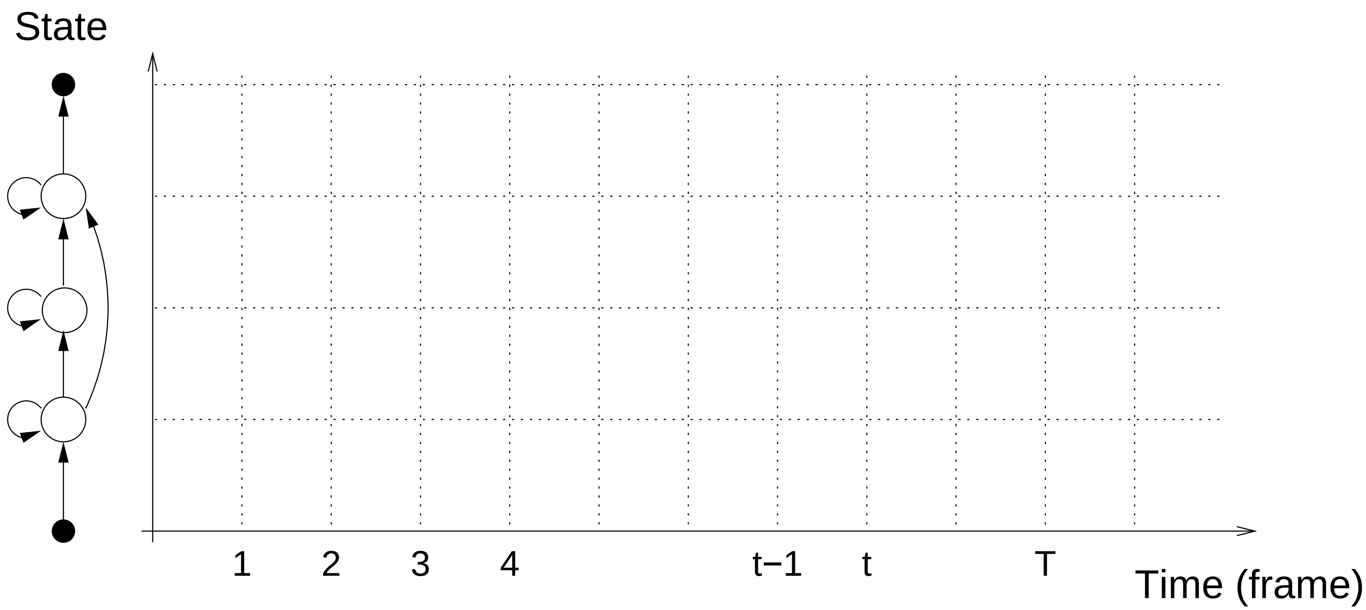


2. For $t = 2, \ldots, T$,

$$\delta_t(j) = \max_i \left[\delta_{t-1}(i) a_{ij}\right] b_j(o_t)$$

$$\psi_t(j) = \arg\max_i \left[\delta_{t-1}(i) a_{ij}\right] \qquad \text{for } 1 \le j \le N;$$

Steps 3 and 4



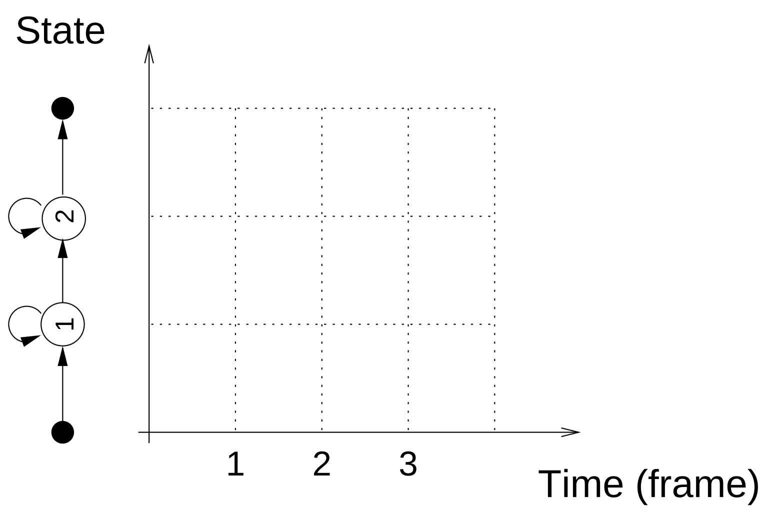
3. Finally,

$$\Delta^* = \max_i [\delta_T(i)]$$
$$x_T^* = \arg\max_i [\delta_T(i)];$$

4. Trace back, for $t = T-1, T-2, \ldots, 1$,

$$x_t^* = \psi_{t+1}\left(x_{t+1}^*\right), \quad \text{and} \quad X^* = \{x_1^*, x_2^*, \ldots, x_T^*\}. \qquad (9)$$

Viterbi by numbers



Reformulating the optimisation

Recall the likelihood calculation,

$$\begin{aligned} P(\mathcal{O}, X|\lambda) &= P(\mathcal{O}|X, \lambda) P(X|\lambda) \\ &= \pi_1 b_1(o_1)\, a_{11} b_1(o_2)\, a_{12} b_2(o_3) \ldots \end{aligned}$$

Now, taking the negative logarithm of eq. 8 gives

$$Q(X) = -\left[\ln\left(\pi_{x_1} b_{x_1}(o_1)\right) + \sum_{t=2}^{T} \ln\left(a_{x_{t-1}x_t} b_{x_t}(o_t)\right)\right]. \tag{10}$$

Hence, eq. 7 becomes

$$X^* = \arg\min_X Q(X). \tag{11}$$

Summary of the Viterbi algorithm

1. Initially,

$$q_1(i) = -\ln \pi_i - \ln(b_i(o_1))$$
$$\psi_1(i) = 0 \qquad \text{for } 1 \leq i \leq N;$$

2. For $t = 2, \ldots, T$,

$$q_t(j) = \min_i \left[q_{t-1}(i) - \ln a_{ij}\right] - \ln\left(b_j(o_t)\right)$$
$$\psi_t(j) = \arg\min_i \left[q_{t-1}(i) - \ln a_{ij}\right] \qquad \text{for } 1 \leq j \leq N;$$

3. Finally,

$$Q^* = \min_i [q_T(i)]$$
$$x_T^* = \arg\min_i [q_T(i)];$$

4. Trace back, for $t = T-1, T-2, \ldots, 1$,

$$x_t^* = \psi_{t+1}\left(x_{t+1}^*\right), \quad \text{and} \quad X^* = \{x_1^*, x_2^*, \ldots, x_T^*\}. \tag{12}$$

Today's summary

- Recap. of MMs and HMMs
- Computing likelihoods, $P(\mathcal{O}|\lambda)$
  - for Markov models
  - for Hidden Markov models
- Finding the best state sequence, X^*
  - Viterbi algorithm
  - Trellis diagrams

Next time

- Setting the parameters in the models $\Lambda = \{\lambda\}$
  - Baum-Welch re-estimation
  - Forward-backward algorithm
  - Continuous output pdfs

Homework

Implement the Viterbi algorithm for your own problem, formulating it in terms of hidden states and observations:

- parameters determined empirically or heuristically
- parameters estimated by least squares (e.g., μ, Σ)