FAR DOWNSTREAM ANALYSIS FOR THE BLASIUS BOUNDARY-LAYER STABILITY PROBLEM

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[Received 20 February 2006. Revise 23 April 2007]

Summary

In this paper, we examine the large Reynolds number (Re) asymptotic structure of the wave number in the Orr–Sommerfeld region for the Blasius boundary layer on a semi-infinite flat plate given by Goldstein (1983, *J. Fluid Mech.*, **127**, 59–81). We show that the inclusion of the term which contains the leading-order non-parallel effects, at $O(\mathrm{Re}^{-1/2})$, leads to a non-uniform expansion. By considering the far downstream form of each term in the asymptotic expansion, we derive a length scale at which the non-uniformity appears, and compare this position with the position seen in plots of the wave number.

1. Introduction

When a body is placed in a parallel mean flow, which contains a small-amplitude unsteady perturbation, the interaction of this perturbation with the boundary layer at areas of 'receptivity' produces a collection of eigenmodes (**1**). These areas of receptivity occur in regions where the non-parallel effects of the mean flow are important, such as at the leading edge of a body (**2**), at an element of surface roughness (**3**) or at regions of marginal separation (**4**). As these eigenmodes move downstream of the receptivity region, they match, in the matched asymptotic expansion sense, to the Tollmien–Schlichting (T-S) modes in the nearly parallel Orr–Sommerfeld region. All these T-S modes experience exponential decay as they move along the body, except one, which eventually grows downstream of the receptivity area, and hence the growth rate calculation for this T-S wave is important in the prediction of transition. Typically, growth rate calculations have used Orr–Sommerfeld theory, although this method does not include the slow growth in the boundary-layer thickness. Other numerical studies have incorporated these non-parallel effects, although they are not rigourous in an asymptotic sense (**5**, **6**).

Goldstein (**2**) made a breakthrough in the receptivity/stability problem when he derived the asymptotic form of the wave number/growth rate and mode shape in the Orr–Sommerfeld region on a semi-infinite flat plate, and showed that the T-S modes in this region match to the Lam–Rott asymptotic eigenmodes (**7**, **8**) from the leading-edge region of the plate. Goldstein (**2**) provided the asymptotic expansion for the wave number in the Orr–Sommerfeld region up to and including the $O(\epsilon^3 \ln \epsilon)$ term, where $\epsilon = \mathrm{Re}^{-1/6}$. However, in an earlier NASA report, Goldstein calculated the $O(\epsilon^3)$ term of the wave number, which turns out to be important, as it includes the non-parallel effect of the boundary layer (**9**). Goldstein (**2**) concentrated his analysis on the growing T-S wave;

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Q. Jl Mech. Appl. Math, Vol. 60. No. 3
Advance Access publication 11 July 2007. doi:10.1093/qjmam/hbm011

however, his formulation of the Orr–Sommerfeld problem also incorporated the other exponentially decaying T-S modes, which were studied in more depth by Hultgren (**10**).

The advantage of the asymptotic expansion over the existing numerical procedures is that they provide a link between the receptivity which occurs at the leading edge of the plate and the amplitude of the T-S wave downstream. Hence, the complete amplitude of the T-S wave is known, and there are no unknown constants to fix, such as the initial amplitude of the T-S wave as it enters the Orr–Sommerfeld region, unlike in previous numerical studies (**6**). Turner and Hammerton (**11**) used this connection between the leading-edge Lam–Rott modes and the T-S modes to numerically calculate the wave number of the T-S wave by the use of the parabolized stability equations (PSE) (**12**). The advantage of the PSE over full DNS is that the numerical procedure is quicker as the most dangerous upstream propagating eigenmode has been eliminated (**13**). However, Turner and Hammerton (**11**) noted, when comparing their results to the results of Goldstein (**2**), that the inclusion of the non-parallel $O(\epsilon^3)$ term made the asymptotics appear to be non-uniform far downstream. Turner and Hammerton also demonstrated that the inclusion of this $O(\epsilon^3)$ term is essential for the matching of the Lam–Rott asymptotic eigenmodes to the T-S modes in the Orr–Sommerfeld region at values of $\epsilon \gtrsim 0{\cdot}05$. This statement of non-uniformity was never investigated in their paper, and it is addressed here.

In section 2, we formulate the governing equation for the wave number and include the equation for the $O(\epsilon^3)$ term along with the form of the undetermined constants not given in (**9**). We also show that the $O(\epsilon^3)$ equation can be simplified by the explicit evaluation of most of the integrals. In section 3, we consider the form of the small ϵ asymptotic expansion for the wave number, when we include more terms in the expansion, and show that the non-uniform behaviour occurs when the $O(\epsilon^3)$ term is included. We then produce the large downstream asymptotic form of each of the terms from the small ϵ asymptotic expansion, and show that the asymptotics do indeed become non-uniform with the inclusion of the $O(\epsilon^3)$ term, and we give a streamwise position at which this occurs.

2. Formulation

We consider a small two-dimensional harmonic disturbance of frequency ω, acting on the Blasius boundary-layer flow on a semi-infinite flat plate. The free stream has density ρ, and streamwise velocity U_∞; therefore, the corresponding length, time, velocity and pressure scales we consider are $\omega^{-1}U_\infty$, ω^{-1}, U_∞ and ρU_∞^2, respectively. We introduce non-dimensional coordinates $(x, \eta = y/(\epsilon^3(2x)^{1/2}))$, which are in the streamwise and normal directions to the plate, respectively. Non-dimensionalizing the vorticity–stream function form of the Navier–Stokes equation with respect to these scales and linearizing about the Blasius boundary-layer mean flow $(\Psi = \epsilon^3(2x)^{1/2} f(\eta) + \psi(x_1, \eta)e^{it})$ give an equation for the perturbation stream function, ψ, as

$$-i\tilde{\nabla}^2\psi + x^{\frac{1}{2}}\left[\frac{\partial(x^{-1}\tilde{\nabla}^2\psi, x^{1/2} f)}{\partial(x,\eta)} + \frac{\partial(x^{-1/2} f'', \psi)}{\partial(x,\eta)}\right] = \tilde{\nabla}^2\left(\frac{1}{2x}\tilde{\nabla}^2\psi\right) \quad (\eta, x > 0), \quad (2.1)$$

where

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial\eta^2} + 2\epsilon^6 x\frac{\partial^2}{\partial x^2} + \epsilon^6\frac{\partial}{\partial x},$$

$$\epsilon^6 = \mathrm{Re}^{-1} = F = \frac{\nu\omega}{U_\infty^2},$$

and ν is the kinematic viscosity of the fluid. The Reynolds number Re is based on the acoustic length scale U_∞/ω and $F = \omega\nu/U_\infty^2$ is the dimensionless frequency, commonly used in stability calculations. The function $f(\eta)$ is the usual Blasius function that satisfies

$$f''' + ff'' = 0,$$

with boundary conditions $f(0) = f'(0) = 0$ and $f' \longrightarrow 1$ as $\eta \longrightarrow \infty$. In (2.1), correction terms, which remain uniformly small in the region we consider, have been dropped. The parameter ϵ^6 is the inverse of the Reynolds number, which is assumed to be large; hence, $\epsilon \ll 1$. We utilize this fact later when forming our asymptotic expansions.

Following the work of Goldstein (**2**), we seek a solution for the perturbation stream function, ψ, in (2.1) in the form of travelling waves

$$\psi = \epsilon^{-(2\tau_j+1)} A(x_1)\gamma(x_1,\eta)\exp\left(\frac{i}{\epsilon}\int_0^x \hat{\kappa}_j(x_1,\epsilon)dx\right), \tag{2.2}$$

where $x_1 = \epsilon^2 x$ is a slow streamwise coordinate, $A(x_1)$ is a slowly varying function to be determined by the analysis, $\gamma(x_1, \eta)$ is a mode shape and $\hat{\kappa}_j(x_1, \epsilon)$ is the wave number of the jth mode, which has an associated constant τ_j. The constant τ_j is found by solving a solvability condition for the receptivity problem in the leading-edge region, $x = O(1)$ (**2**). The form of this constant was simplified by Hammerton and Kerschen (**14**) and given by

$$\tau_j = -\frac{889 - 16\rho_j^3}{1260}.$$

Here ρ_j are the roots of $\mathrm{Ai}'(-\rho_j) = 0$, where Ai' is the derivative of the Airy function. In this paper, we concentrate solely on the first root of this problem ($\rho_1 = 1{\cdot}0188$), which corresponds to the unstable Tollmien–Schlichting (T-S) wave and displays streamwise growth downstream of the lower branch point. The other modes for this problem are important close to the leading edge of the body; however, once we pass the lower branch neutral stability point, the amplitudes of these modes decay exponentially; hence, we do not consider them here (**10**). In expression (2.2), it is assumed that $\hat{\kappa}_1$ and x_1 are $O(1)$ in the region of the lower branch point, whereas $x_1 = O(\epsilon^{-2})$ and $\hat{\kappa}_1 = O(\epsilon^{1/2})$ at the upper branch point (**10**).

Substituting (2.2) into (2.1) and applying the parallel flow assumption lead to the Orr–Sommerfeld problem; see (**2**, section 4), where the wave number, phase velocity and Reynolds number are now given by

$$\alpha = \epsilon\bar{\alpha} = \epsilon(2x_1)^{1/2}\hat{\kappa}_1,$$

$$c = \epsilon\bar{c} = \epsilon\hat{\kappa}_1^{-1},$$

$$R = \epsilon^{-4}(2x_1)^{1/2},$$

respectively. By matching the asymptotic solution in the main deck to the outer inviscid and inner viscous layers of the resulting equation, Goldstein (**2**) showed that the equation for the determination for $\kappa \equiv \hat{\kappa}_1$ is

$$\tilde{x}_1^{3/2} + (\epsilon e^{i\pi/4}\zeta_0^{3/2})\tilde{x}_1\left(2 - \frac{\tilde{x}_1^{3/2}J_1}{i\zeta_0^3}\right) + (\epsilon e^{i\pi/4}\zeta_0^{3/2})^2\tilde{x}_1^{1/2}\left(1 + \frac{2\tilde{x}_1^{3/2}J_2}{i\zeta_0^3} - \frac{\tilde{x}_1^3 J_3}{\zeta_0^6}\right)$$
$$- \frac{e^{i\pi/4}(\tilde{x}_1\zeta_0)^{3/2}\epsilon^3}{2U_0'^2}\ln\left(\frac{\epsilon e^{i\pi/4}\zeta_0^{3/2}}{\tilde{x}_1^{1/2}U_0'}\right) = H(\zeta_0) \equiv \frac{e^{5i\pi/2}\zeta_0^2\mathrm{Ai}'(\zeta_0)}{\int_{\infty_1}^{\zeta_0}\mathrm{Ai}(\zeta)d\zeta}, \tag{2.3}$$

where

$$\begin{aligned}
\tilde{x}_1 &\equiv \frac{2x_1}{U_0'^2},\\
\zeta_0 &= e^{-5i\pi/6}\left(\frac{\tilde{x}_1^{1/2}}{\kappa}\right)^{2/3},\\
J_1 &\equiv U_0'\int_0^\infty\left(U^2 - \frac{1}{U^2} + \frac{1}{U_0'^2\eta^2}\right)d\eta,\\
J_2 &= -U_0'\int_0^\infty\left(\frac{1}{U^3} - \frac{2}{U^2} + U - \frac{1}{(U_0'\eta)^3} + \frac{2}{(U_0'\eta)^2}\right)d\eta,\\
J_3 &= J_1^2 - 2U_0'^2\int_0^\infty U^2\int_\eta^\infty\left(U^2 - \frac{1}{U^2}\right)d\eta\, d\eta,\\
U &= f'(\eta),
\end{aligned} \tag{2.4}$$

and the subscript 1 on ∞ is used to indicate that the path of integration tends to infinity in the sector $-\frac{1}{3}\pi < \arg(\zeta) < \frac{1}{3}\pi$. The constants J_1, J_2 and J_3 take on the values 0·92809, −2·09322 and 1·28777, respectively, and $U_0' = f''(0) = 0{\cdot}46960$. Hultgren (**10**) offered an alternative form of (2.3) which is numerically more accurate, especially near the upper branch point (**15**). However, as we are only interested in forming an asymptotic expansion for the wave number for small ϵ, the form of (2.3) is acceptable.

The error in (2.3) is of $O(\epsilon^3)$; hence, an asymptotic expansion for κ using this equation would be valid up to $O(\epsilon^3\ln\epsilon)$. However, the neglected non-parallel terms enter the problem at $O(\epsilon^3)$; therefore, in order to construct an accurate asymptotic expansion for the wave number κ, including the non-parallel effects, we require the equation for the slowly varying amplitude function $A(x_1)$. This equation is found by matching the inviscid Rayleigh solution to the viscous wall layer solution at $O(\epsilon^4)$. This analysis is carried out in (**9**, Appendix C), so we just quote the result here. The equation for $d\ln A/dx_1$ is given by

$$2\bar{\alpha}\frac{d\ln A}{dx_1} + \bar{\alpha}_{x_1} - \frac{\bar{\alpha}}{2x_1} + \frac{\bar{\alpha}}{\bar{c}}\sum_{n=0}^{3}\tilde{A}_n\bar{c}^n\bar{\alpha}^{(3-n)}$$
$$= \pi U_0'\mathrm{Bi}'(\zeta_0)\int_0^\infty\left(\bar{H}_1\frac{d\ln A}{dx_1} + \bar{H}_2\right)\mathrm{Ai}(\zeta)d\bar{\eta} - \frac{\bar{\alpha}\zeta_0}{\bar{c}}\int_0^\infty\left(Wi\left(\bar{H}_1\frac{d\ln A}{dx_1} + \bar{H}_2\right.\right.$$
$$\left.\left. - \frac{U_0'}{2x_1} + \frac{iU_0'}{2\bar{c}}\bar{\eta}^2(a_1 + U_0'\bar{\eta}) + \frac{\bar{c}(a_1+\bar{c})}{2iU_0'}\right) + \frac{\bar{c}(a_1+\bar{c})}{2iU_0'\zeta}\right)d\bar{\eta}, \tag{2.5}$$

where

$$
\begin{aligned}
&Wi(\zeta) = -\pi \left(\mathrm{Ai}(\zeta) \int_{\zeta_0}^{\zeta} \mathrm{Bi}(\xi) d\xi - \mathrm{Bi}(\zeta) \int_{\infty_1}^{\zeta} \mathrm{Ai}(\xi) d\xi \right), \\
&\bar{H}_1 \equiv \bar{D}(\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0), \\
&\bar{H}_2 \equiv \bar{D} \left(\frac{\partial}{\partial x_1}(\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0) + \frac{1}{4x_1}\bar{D}(\bar{\eta}^2 \bar{D}\bar{\gamma}_0) - \frac{iU_0'}{3!\bar{c}}\bar{\eta}^3 \left(\bar{\gamma}_0 - \frac{1}{4}\bar{\eta}\bar{D}\bar{\gamma}_0 \right) \right), \\
&\zeta = \zeta_0 \left(1 - \frac{U_0'\bar{\eta}}{\bar{c}} \right), \\
&a_1 = \bar{c} \left(-1 + \frac{\mathrm{Ai}'(\zeta_0)}{\zeta_0 \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds} \right).
\end{aligned} \tag{2.6}
$$

The derivation of the $O(1)$ constants, $\tilde{A}_n$, from this matching procedure at $O(\epsilon^4)$, not given in (**9**), can be found in (**16**). These constants take the form

$$
\begin{aligned}
\tilde{A}_0 &= iU_0' \left(\frac{2J_1 J_3}{U_0'^4} - \frac{J_1^3}{U_0'^4} + J_6 \right), \\
\tilde{A}_1 &= iU_0' \left(\frac{4J_1 J_2}{U_0'^3} + \frac{6J_1^2}{U_0^3} - J_5 - \frac{4J_3}{U_0'^3} \right), \\
\tilde{A}_2 &= iU_0' \left(J_4 - \frac{8J_2}{U_0'^2} - \frac{10J_1}{U_0'^2} - \frac{5}{24U_0'^4} \right), \\
\tilde{A}_3 &= \frac{i}{2U_0'^2},
\end{aligned}
$$

where

$$
\begin{aligned}
J_4 &= -\frac{1}{U_0'} \int_0^{\infty} \left(\frac{3}{U^4} - 1 + 8U - 10U^2 - \frac{3}{(U_0'\eta)^4} - \frac{\delta}{2U_0'^3\eta} \right) d\eta, \\
J_5 &= \frac{4}{U_0'} \left(\int_0^{\infty} (U^2 - U) \int_{\eta}^{\infty} \left(U^2 - \frac{1}{U^2} \right) d\eta\, d\eta \right. \\
&\quad \left. + \int_0^{\infty} U^2 \int_{\eta}^{\infty} \left(2U^2 - \frac{1}{U^3} - U \right) d\eta\, d\eta \right), \\
J_6 &= \frac{1}{U_0'} \left(\int_0^{\infty} U^2 \left(4 \int_{\eta}^{\infty} U^2 \int_{\eta}^{\infty} \left(U^2 - \frac{1}{U^2} \right) d\eta\, d\eta \right. \right. \\
&\quad \left. \left. + \left(\int_{\eta}^{\infty} \left(U^2 - \frac{1}{U^2} \right) d\eta \right)^2 d\eta \right) \right),
\end{aligned}
$$

and $\delta = 1$ for $0 < \eta < 1$ and $\delta = 0$ for $\eta > 1$. The constants $\tilde{A}_0$ to $\tilde{A}_3$ have the values $30{\cdot}25292i$, $-42{\cdot}05954i$, $6{\cdot}28404i$ and $2{\cdot}26733i$, respectively. Equation (2.5) is solved by first solving (2.3) for ζ_0 and κ at each streamwise step, and then using these to solve for $d\ln(A)/dx_1$.

We found that most of the integrals in (2.5) can either be written in closed form or simplified. Thus, we can write (2.5) as

$$
\begin{aligned}
(2\bar{\alpha} - \omega_1 L_1)\frac{d\ln(A)}{dx_1} = &-\bar{\alpha}_{x_1} + \frac{\bar{\alpha}}{2x_1} - \frac{\bar{\alpha}}{\bar{c}}\sum_{n=0}^{3}\tilde{A}_n \bar{c}^n \bar{\alpha}^{(3-n)} \\
&+ \omega_1\left(\frac{\zeta_{0x_1}}{\zeta_0} - \frac{\bar{c}_{x_1}}{\bar{c}} - \frac{\mathrm{Ai}(\zeta_0)\zeta_{0x_1}}{\int_\infty^{\zeta_0}\mathrm{Ai}(s)ds} - \frac{1}{x_1}\right)L_1 + \frac{\omega_1\zeta_{0x_1}}{\zeta_0}L_2 \\
&+ \left(\omega_2 - \frac{\omega_1}{4x_1} - \frac{\omega_1\bar{c}_{x_1}}{\bar{c}}\right)L_4 - \frac{\omega_1}{2x_1}L_3 - \omega_2 L_6 - \omega_2\zeta_0 L_5 + \frac{\omega_2}{12}L_9 \\
&- \frac{\pi\bar{c}\mathrm{Bi}'(\zeta_0)}{\zeta_0}\left(\omega_4 I_8 + (\omega_4\zeta_0 - \omega_3)I_7 + \frac{U_0'}{2x_1}\int_\infty^{\zeta_0}\mathrm{Ai}(s)ds\right) \\
&+ \frac{\bar{c}\bar{\alpha}(a_1+\bar{c})}{2iU_0'^2}\int_{\zeta_0}^{\infty}\left(Wi + \frac{1}{\zeta}\right)d\zeta, \qquad (2.7)
\end{aligned}
$$

where

$$
\begin{aligned}
L_j &= \frac{\bar{\alpha}}{U_0'}K_j - \frac{\pi\bar{c}\mathrm{Bi}'(\zeta_0)}{\zeta_0}I_j, \\
\omega_1 &= \frac{U_0'}{\int_\infty^{\zeta_0}\mathrm{Ai}(\zeta)d\zeta}, \\
\omega_2 &= \frac{i\bar{c}^2}{2\zeta_0^3 U_0' \int_\infty^{\zeta_0}\mathrm{Ai}(s)ds}, \\
\omega_3 &= \omega_2\mathrm{Ai}'(\zeta_0), \\
\omega_4 &= \omega_2\int_\infty^{\zeta_0}\mathrm{Ai}(s)ds,
\end{aligned}
$$

and I_1 to I_9 and K_1 to K_9 are given in the Appendix as functions of ζ_0.

Equation (2.3) can be solved numerically for κ, using a root search in the complex plane, and then this value could be substituted into (2.7) to find $d\ln(A)/dx_1$. Although this method retains all the $O(\epsilon^3)$ terms, it also retains some terms of $O(\epsilon^4)$ and smaller, whereas other terms of $O(\epsilon^4)$ from the function $A(x_1)$ are missing, and these terms are important because they contain the non-parallel behaviour. Therefore, in section 3, we construct an asymptotic expansion for the wave number, κ, in powers of ϵ. The asymptotic expansion eliminates the above problem because it allows us to retain all the terms up to a given order in our solution or none at all depending on where we decide to truncate our asymptotic solution.

3. Results and large $\tilde{x}_1$ asymptotics

In this section, we form an asymptotic solution for the wave number, κ, in terms of the small parameter ϵ, which includes the leading-order non-parallel effects, which enters the problem at $O(\epsilon^3)$.

To form this expansion, we note that since $H(\zeta_0)$ is an analytic function of ζ_0, it is clear that the total wave number, κ_{Tot}, has an asymptotic expansion of the form

$$\kappa_{\mathrm{Tot}} = \kappa_0 + \epsilon\kappa_1 + \epsilon^2\kappa_2 + \epsilon^3 \ln\epsilon\,\kappa_3 + \epsilon^3\left(\kappa_4 - i\frac{d\ln(A)}{dx_1}\right) + O(\epsilon^4\ln\epsilon), \tag{3.1}$$

where κ_0 to κ_4 are found from (2.3).

Inserting $\kappa = \kappa_0 + \epsilon\kappa_1 + \epsilon^2\kappa_2 + \epsilon^3\ln\epsilon\,\kappa_3 + \epsilon^3\kappa_4 + O(\epsilon^4\ln\epsilon)$ into (2.3) and (2.4), expanding the function $H(\zeta_0)$ about

$$\zeta_{00} = e^{-5\pi i/6}\left(\frac{\tilde{x}_1^{1/2}}{\kappa_0}\right)^{2/3}, \tag{3.2}$$

and equating powers of ϵ, give

$$H(\zeta_{00}) = \tilde{x}_1^{3/2}, \tag{3.3}$$

$$\frac{\kappa_1}{\kappa_0} = -\frac{3}{2}e^{i\pi/4}\zeta_{00}^{1/2}\tilde{x}_1\left(2 - \frac{\tilde{x}_1^{3/2}J_1}{i\zeta_{00}^3}\right)\Big/ H'(\zeta_{00}), \tag{3.4}$$

$$\frac{\kappa_2}{\kappa_0} = -\frac{1}{3}\left(\frac{1}{2} - \frac{H''(\zeta_{00})\zeta_{00}}{H'(\zeta_{00})}\right)\left(\frac{\kappa_1}{\kappa_0}\right)^2 + 3e^{-i\pi/4}\left(\frac{\tilde{x}_1}{\zeta_{00}}\right)^{5/2} J_1\left(\frac{\kappa_1}{\kappa_0}\right)\Big/ H'(\zeta_{00})$$
$$-\frac{3}{2}i\zeta_{00}^2\tilde{x}_1^{1/2}\left(1 + \frac{2\tilde{x}_1^{3/2}J_2}{i\zeta_{00}^3} - \frac{\tilde{x}_1^3 J_3}{i\zeta_{00}^6}\right)\Big/ H'(\zeta_{00}), \tag{3.5}$$

$$\frac{\kappa_3}{\kappa_0} = \frac{3}{4U_0'^2}e^{i\pi/4}\zeta_{00}^{1/2}\tilde{x}_1^{3/2}\big/ H'(\zeta_{00}). \tag{3.6}$$

The function $H(\zeta_{00})$ is defined by (2.3) and the primes on H denote derivatives with respect to ζ_{00} (**2**). The form of κ_4 is not given in (**2**), and we found it to be

$$\frac{\kappa_4}{\kappa_0} = \frac{1}{27}\left(47 - 15\frac{\zeta_{00}H''(\zeta_{00})}{H'(\zeta_{00})} - 2\frac{\zeta_{00}^2H'''(\zeta_{00})}{H'(\zeta_{00})}\right)\left(\frac{\kappa_1}{\kappa_0}\right)^3 + \frac{2}{3}\left(1 + \frac{\zeta_{00}H''(\zeta_{00})}{H'(\zeta_{00})}\right)\frac{\kappa_1\kappa_2}{\kappa_0^2}$$
$$-\frac{3}{2}e^{-i\pi/4}\left(\frac{\tilde{x}_1}{\zeta_{00}}\right)^{5/2} J_1\left(\left(\frac{\kappa_1}{\kappa_0}\right)^2 - 2\frac{\kappa_2}{\kappa_0}\right)\Big/ H'(\zeta_{00})$$
$$+3i\zeta_{00}^2\tilde{x}_1^{1/2}\left(1 + \frac{\tilde{x}_1^3 J_3}{\zeta_{00}^6}\right)\left(\frac{\kappa_1}{\kappa_0}\right)\Big/ H'(\zeta_{00}) + \frac{\kappa_3}{\kappa_0}\ln\left(\frac{e^{i\pi/4}\zeta_{00}^{3/2}}{\tilde{x}_1^{1/2}U_0'}\right). \tag{3.7}$$

To solve these equations, we first solve (3.3) for ζ_{00}. This is done by performing a complex eigenvalue search, where the initial eigenvalue is given as $\zeta_{00} = -\rho_1 = -1{\cdot}0188$ at $\tilde{x}_1 = 0$. At the next streamwise step, a box is placed around the previous eigenvalue in the complex plane, and a search for the next eigenvalue is made within this box. Also, at each step the value for ζ_{00} is used in (3.2) to solve for κ_0, and then both of these are used to solve for κ_1, κ_2, κ_3 and κ_4 in turn. To find the $O(\epsilon^3)$

correction term to κ_{Tot} from (2.7), we require only the leading-order term of $d\ln(A)/dx_1$. Thus, we solve (2.7) by substituting in the leading-order terms of $\zeta_0 = \zeta_{00}$ and $\kappa = \kappa_0$ to make (3.1) a true asymptotic expansion. The asymptotic expansion for κ_{Tot} at different levels of approximation are plotted in Fig. 1.

In Fig. 1, we plot the asymptotic results for the wave number, κ_{Tot}, as a function of $\tilde{x}_1$ for $\epsilon = F^{1/6} = 0{\cdot}1$. For this value of ϵ, the neutral stability point ($\text{Im}(\kappa_{\text{Tot}}) = 0$) occurs at $\tilde{x}_1 \approx 4$, which corresponds to a downstream Reynolds number $R_x = U_\infty x^*/\nu = U_0'^2\tilde{x}_1/(2\epsilon^8) \approx 4{\cdot}4 \times 10^7$, where x^* is a dimensional distance downstream. The wave number, κ_{Tot}, in Fig. 1 appears to be uniform up to and including the $O(\epsilon^3\ln\epsilon)$ term, as the inclusion of each extra term only changes the wave number by a small amount. However, when we include the $O(\epsilon^3)$ term, we see that the asymptotic expansion appears to become non-uniform far downstream. The addition of the $O(\epsilon^3)$ term also appears to change the form of κ_{Tot} significantly for small $\tilde{x}_1$, which can be seen near the first maximum of $\text{Im}(\kappa)$, close to $\tilde{x}_1 = 0{\cdot}5$, in Fig. 1(b). However, this behaviour is required so that the wave number matches on to the large downstream asymptotic form of the first Lam–Rott eigenmode from the leading-edge region (**2**). Goldstein (**2**) showed that, for this matching to take place, the function $A(x_1)$, in the limit as $x_1 \longrightarrow 0$, must behave like

$$\frac{d\ln(A)}{dx_1} \sim \frac{1+2\tau_1}{2x_1},$$

where $\tau_1 = -0{\cdot}6921$. This property can been seen to hold numerically, in Fig. 2, which plots $d\ln(A)/dx_1$ as a function of $\tilde{x}_1$, along with $(1+2\tau_1)/2x_1$. It is clear that the imaginary part of $d\ln(A)/dx_1$ tends to zero for small $\tilde{x}_1$, and before $\tilde{x}_1 \approx 0{\cdot}5$, the real part of $d\ln(A)/dx_1$ is indistinguishable from $(1+2\tau_1)/2x_1$. Therefore, the wave number has the correct behaviour for small $\tilde{x}_1$, so we need only to concern ourselves with the large $\tilde{x}_1$ behaviour of the wave number.

Figure 3 shows the real parts of κ_0 to κ_4 as a function of $\tilde{x}_1$. We see that the modulus of each term, at $\tilde{x}_1 = 100$, increases as we move from κ_0 to κ_4. The form of the imaginary parts in Fig. 4 is

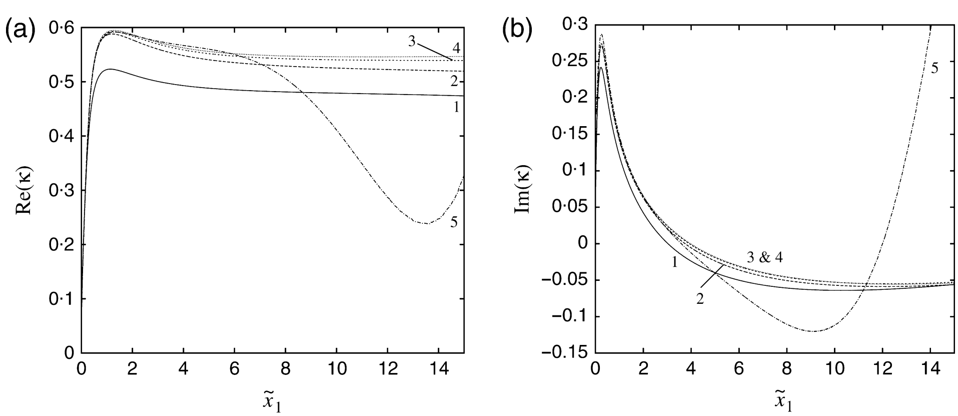

Fig. 1 Plot of (a) the real part and (b) the imaginary part of the wave number κ_{Tot} as a function of $\tilde{x}_1$, up to $1 - O(1)$, $2 - O(\epsilon)$, $3 - O(\epsilon^2)$, $4 - O(\epsilon^3\ln\epsilon)$ and $5 - O(\epsilon^3)$, where $\epsilon = F^{1/6} = 0{\cdot}1$ in this case. Note that $R_x = U_0'^2\tilde{x}_1/(2\epsilon^8)$

slightly different because κ_0, κ_1 and κ_3 all decay with increasing $\tilde{x}_1$; however, κ_2 and κ_4 both grow with increasing $\tilde{x}_1$. The asymptotic expansion for κ_{Tot} will remain valid, as long as $|\epsilon\kappa_1| < |\kappa_0|$ and $|\epsilon^2\kappa_2| < |\epsilon\kappa_1|$, etc. From Figs 3 and 4, it is not clear if the non-uniformity seen in Fig. 1 is due to any of the terms κ_0 to κ_4 becoming non-uniform, so we consider the large $\tilde{x}_1$ asymptotic form of κ_0 to κ_4 to determine if they lead to the breakdown of (3.1).

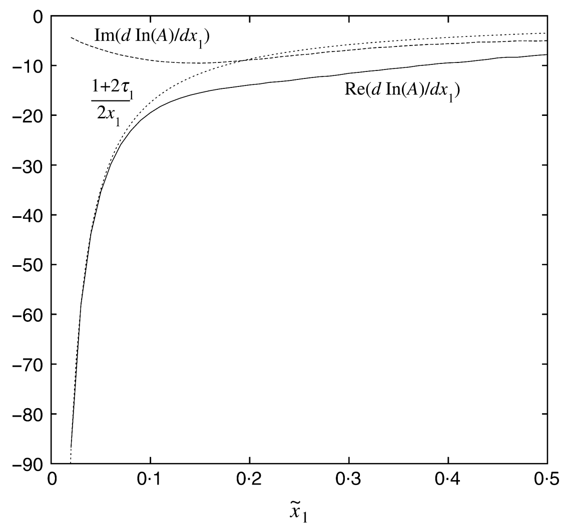

Fig. 2 Plot of the real and imaginary part of $d\ln(A)/dx_1$, showing the matching onto $(1+2\tau_1)/2x_1$ for small x_1

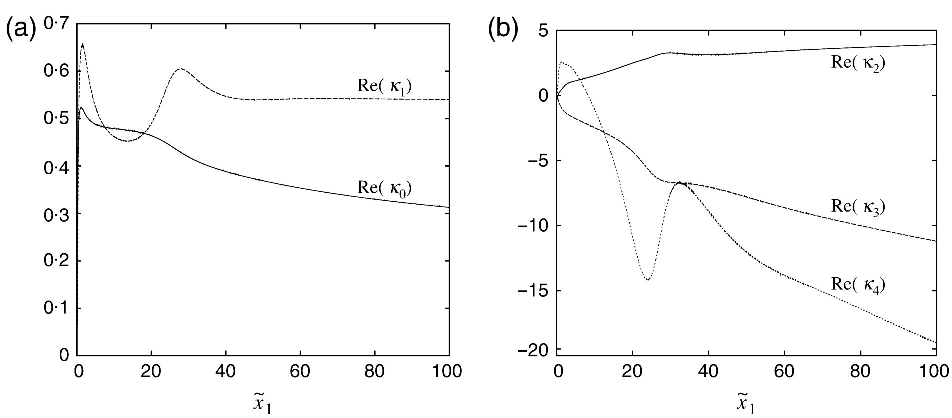

Fig. 3 Plot of the real parts of (a) κ_0 and κ_1 and (b) κ_2, κ_3 and κ_4, as a function of $\tilde{x}_1$

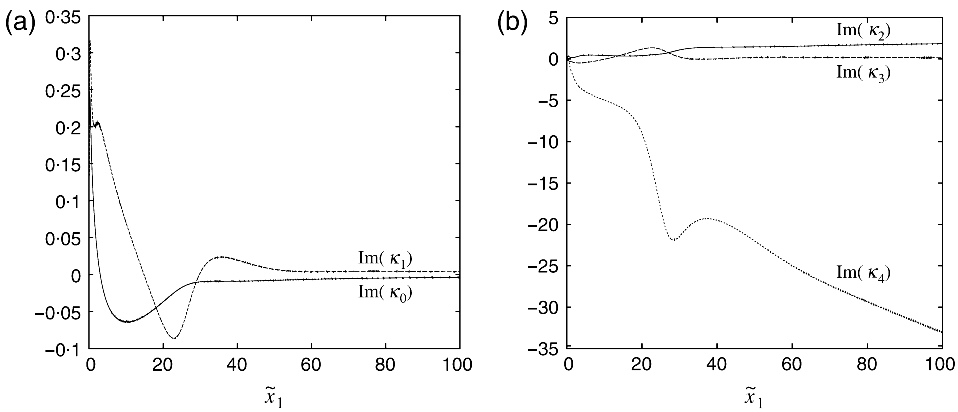

Fig. 4 Plot of the imaginary parts of (a) κ_0 and κ_1 and (b) κ_2, κ_3 and κ_4, as a function of $\tilde{x}_1$

3.1 *Large $\tilde{x}_1$ asymptotics of* (3.1)

To find out when the asymptotic form of (3.1) breaks down, we consider the large $\tilde{x}_1$ form of (3.3) to (3.7). To find the large $\tilde{x}_1$ form of κ_0, we first expand the function $H(\zeta_{00})$ in (3.3) for large ζ_{00}. To do this, we require the large ζ_{00} asymptotic forms of $\mathrm{Ai}'(\zeta_{00})$ and $\int_\infty^{\zeta_{00}} \mathrm{Ai}(\zeta)d\zeta$, which from Abramowitz and Stegun (**17**) are given by

$$\mathrm{Ai}'(\zeta_{00}) \sim -\frac{\zeta_{00}^{1/4}}{2\pi^{1/2}}\left(1 + \frac{7}{48}\zeta_{00}^{-3/2} - \frac{455}{4608}\zeta_{00}^{-3} + \frac{95095}{663552}\zeta_{00}^{-9/2} - \frac{40415375}{127401984}\zeta_{00}^{-6}\right)\exp\left(-\frac{2}{3}\zeta_{00}^{3/2}\right), \tag{3.8}$$

$$\int_\infty^{\zeta_{00}} \mathrm{Ai}(\zeta)d\zeta \sim -\frac{1}{2\pi^{1/2}}\left(\zeta_{00}^{-3/4} - \frac{41}{48}\zeta_{00}^{-9/4} + \frac{9241}{4608}\zeta_{00}^{-15/4} - \frac{5075225}{663552}\zeta_{00}^{-21/2} + \frac{5153008945}{127401984}\zeta_{00}^{-27/4}\right)\exp\left(-\frac{2}{3}\zeta_{00}^{3/2}\right), \tag{3.9}$$

which have error terms of $O(\zeta_{00}^{-29/4}\exp(-\frac{2}{3}\zeta_{00}^{3/2}))$ and $O(\zeta_{00}^{-33/4}\exp(-\frac{2}{3}\zeta_{00}^{3/2}))$, respectively.

Substituting these into (3.3), we find

$$\tilde{x}_1^{3/2} = e^{5\pi i/2}\left(\zeta_{00}^3 + \zeta_{00}^{3/2} - \frac{5}{4} + \frac{151}{32}\zeta_{00}^{-3/2} + O(\zeta_{00}^{-3})\right). \tag{3.10}$$

Just considering the first term on the right-hand side of (3.10), we find that to leading order

$$\zeta_{00} = e^{-5\pi i/6}\tilde{x}_1^{1/2};$$

hence, we can find the higher-order terms of ζ_{00} by looking for a solution of the form $\zeta_{00} = e^{-5\pi i/6}\tilde{x}_1^{1/2} + \hat{\zeta_{00}}$. On inserting this into (3.10), we find that

$$\zeta_{00} = e^{-5\pi i/6}\tilde{x}_1^{1/2} - \frac{1}{3}e^{5\pi i/12}\tilde{x}_1^{-1/4} + \frac{17}{36}e^{5\pi i/6}\tilde{x}_1^{-1} + O(\tilde{x}_1^{-7/4}). \tag{3.11}$$

Hence, we note that the real part of $\zeta_{00}^{3/2}$, which appears in the exponential term of (3.8) and (3.9), is negative, and so in this region $\mathrm{Ai}'(\zeta_{00})$ and $\int_{\infty}^{\zeta_{00}} \mathrm{Ai}(\zeta)d\zeta$ are both exponentially growing functions.

We can now use (3.11) to find the large $\tilde{x}_1$ form of the wave number terms, which are

$$\begin{aligned}
\kappa_0 &= \tilde{x}_1^{-1/4} + \frac{1}{2}e^{5\pi i/4}\tilde{x}_1^{-1} + O(\tilde{x}_1^{-7/4}),\\
\kappa_1 &= \frac{1}{2}(2 - J_1) - \frac{1}{2}e^{\pi i/4}\left(1 - \frac{3}{2}J_1\right)\tilde{x}_1^{-3/4} + O(\tilde{x}_1^{-3/2}),\\
\kappa_2 &= \frac{1}{2}\left[\left(1 - \frac{1}{2}J_1\right)\left(1 - \frac{5}{2}J_1\right) - (1 + 2J_2 - iJ_3)\right]\tilde{x}_1^{1/4}\\
&\quad + \frac{1}{2}e^{\pi i/4}\left[2(J_2 - iJ_3) + J_1(4 - 3J_1)\right]\tilde{x}_1^{-1/2} + O(\tilde{x}_1^{-5/4}),\\
\kappa_3 &= -\frac{1}{4U_0'^2}\tilde{x}_1^{1/2} + \frac{1}{8U_0'^2}e^{\pi i/4}\tilde{x}_1^{-1/4} + O(\tilde{x}_1^{-1}),\\
\kappa_4 &= -\frac{1}{16U_0'^2}\tilde{x}_1^{1/2}\ln(\tilde{x}_1) + \left[\frac{40}{27} - \frac{67}{18}J_1 + \frac{37}{9}J_1^2 - \frac{32}{27}J_1^3 + iJ_3(1 - J_1) + \frac{1}{2}J_3(J_1 - 2)\right.\\
&\quad \left. + 2J_2(J_1 - 1) - \frac{1}{4U_0'^2}\ln\left(-\frac{1}{U_0'}\right)\right]\tilde{x}_1^{1/2} + \frac{e^{\pi i/4}}{32U_0'^2}\tilde{x}_1^{-1/4}\ln(\tilde{x}_1)\\
&\quad + e^{\pi i/4}\left[-\frac{20}{27} + \frac{29}{6}J_1 - \frac{1345}{144}J_1^2 + \frac{3395}{864}J_1^3 - \frac{7}{4}iJ_3\left(1 - \frac{23}{14}J_1\right) + 2J_3\left(1 - \frac{3}{4}J_1\right)\right.\\
&\quad \left. + \frac{3}{2}J_2\left(1 - \frac{5}{2}J_1\right) + \frac{1}{8U_0'^2}\left(\ln\left(-\frac{1}{U_0'}\right) - 1\right)\right]\tilde{x}_1^{-1/4} + O(\tilde{x}_1^{-1}\ln(\tilde{x}_1)),
\end{aligned}$$

where J_1, J_2 and J_3 are the numerical coefficients given in section 2.

Figure 5 shows a plot of the numerical and large $\tilde{x}_1$ asymptotic forms of both κ_0 and κ_1. We see that the asymptotics and the numerics are in good agreement for $\tilde{x}_1$ larger than 50, and this result holds true for κ_2, κ_3 and κ_4. The asymptotic form of the wave number breaks down when $|\kappa_0| = |\epsilon\kappa_1|$; hence, by comparing the leading-order terms of the large $\tilde{x}_1$ asymptotics for κ_0 and κ_1, we note that the asymptotic form of the wave number (3.1) breaks down when

$$\tilde{x}_1^{-1/4} = \epsilon,$$

that is, when $\tilde{x}_1 = O(\epsilon^{-4})$. For the value of $\epsilon = 0.1$ chosen in Fig. 1, this breakdown point would be when $\tilde{x}_1 = O(10\,000)$, which is well outside the range of values considered here. Thus, the non-uniformity seen in Fig. 1 must be due to the function $A(x_1)$.

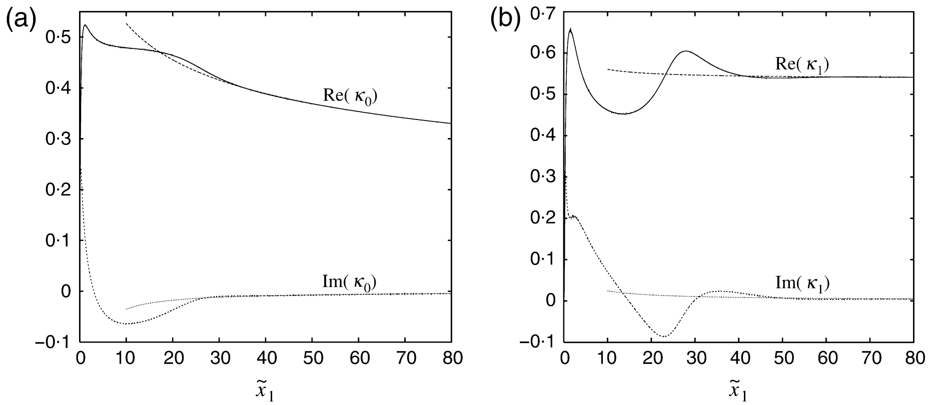

Fig. 5 Plot of the numerical and large $\tilde{x}_1$ form of (a) κ_0 and (b) κ_1, as a function of $\tilde{x}_1$

3.2 *Large $\tilde{x}_1$ asymptotics of $d\ln(A)/dx_1$*

To find the large $\tilde{x}_1$ asymptotic form of (2.7), we note that we again require the large ζ_{00} asymptotic forms of both $\mathrm{Ai}'(\zeta_{00})$ and $\int_\infty^{\zeta_{00}} \mathrm{Ai}(\zeta)d\zeta$ obtained above, and also the large ζ_{00} asymptotic form of $\mathrm{Ai}(\zeta_{00})$ and $\mathrm{Bi}(\zeta_{00})$. From (3.11), we can see that for large $\tilde{x}_1$, ζ_{00} lies in the sector $|\arg(-\zeta_{00})| < \frac{2}{3}\pi$; hence, by comparing the large ζ_{00} forms of $\mathrm{Ai}(\zeta_{00})$ and $\mathrm{Bi}(\zeta_{00})$ in Abramowitz and Stegun (**17**) we deduce that in the sector we are considering

$$\mathrm{Bi}(\zeta_{00}) = -i\,\mathrm{Ai}(\zeta_{00}) + R(\zeta_{00}), \tag{3.12}$$

where $R(\zeta_{00})$ is the remainder term of $\mathrm{Bi}(\zeta_{00})$, and its large ζ_{00} asymptotic form is given by

$$R(\zeta_{00}) = \frac{\zeta_{00}^{-1/4}}{\pi^{1/2}}\left(1 + \frac{5}{48}\zeta_{00}^{-3/2} + \frac{385}{4608}\zeta_{00}^{-3} + \frac{85085}{663552}\zeta_{00}^{-9/2} + O(\zeta_{00}^{-6})\right)\exp\left(\frac{2}{3}\zeta_{00}^{3/2}\right). \tag{3.13}$$

Note also that in this sector $\exp\left(-\frac{2}{3}\zeta_{00}^{3/2}\right)$ is exponentially large, while $\exp\left(\frac{2}{3}\zeta_{00}^{3/2}\right)$ is exponentially small.

Using the fact that (3.12) holds in this sector, we can simplify the form of the L_j terms which appear in (2.7). These terms can be written in their large ζ_{00} form as

$$L_j \sim -\bar{c}i\pi\int_\infty^{\zeta_{00}} \mathrm{Ai}(\zeta)d\zeta\,\tilde{F}_j(\zeta_{00})\left\{\frac{\bar{\alpha}}{\bar{c}U_0'} - \frac{\mathrm{Ai}'(\zeta_{00})}{\zeta_{00}\int_\infty^{\zeta_{00}}\mathrm{Ai}(s)ds}\right\} + O\left(\exp\left(-\frac{2}{3}\zeta_{00}^{3/2}\right)\right), \tag{3.14}$$

where the function $\tilde{F}_j(\zeta_{00})$ is the evaluation of I_j, with ζ_0 replaced by ζ_{00}, which is $O\left(\exp\left(-\frac{2}{3}\zeta_{00}^{3/2}\right)^2\right)$. In equation (2.7), all the L_j terms are multiplied by ω_1 or ω_2 which are both $O\left(\exp\left(\frac{2}{3}\zeta_{00}^{3/2}\right)\right)$; thus, the leading-order term of $\omega_k L_j$ ($k = 1$ or 2) is exponentially large of $O\left(\exp\left(-\frac{2}{3}\zeta_{00}^{3/2}\right)^2\right)$, while the correction terms are $O(1)$. Now using (3.3), we can write

$$\frac{\mathrm{Ai}'(\zeta_{00})}{\zeta_{00}\int_\infty^{\zeta_{00}}\mathrm{Ai}(s)ds} = e^{-5\pi i/2}\frac{\tilde{x}_1^{3/2}}{\zeta_{00}^3},$$

which on inserting (3.11), and using the fact that

$$\bar{\alpha} = (2x_1)^{1/2}\kappa = U_0'\tilde{x}_1^{1/2}\kappa_0,$$
$$\bar{c} = \frac{1}{\kappa} = \frac{1}{\kappa_0},$$

gives

$$\frac{\mathrm{Ai}'(\zeta_{00})}{\zeta_{00}\int_{\infty}^{\zeta_{00}}\mathrm{Ai}(s)ds} = \frac{\bar{\alpha}}{U_0'\bar{c}}.$$

Therefore, comparing this with (3.3), we see that the term in braces in (3.14) is zero; hence, all the combinations $\omega_k L_j$ ($k = 1$ or 2) have no exponential growth and are at most $O(1)$.

The left-hand side of (2.7), $2\bar{\alpha} - \omega_1 L_1$, now has no exponential growth, and to find the leading-order term, we consider the error term in (3.14) multiplied by ω_1. In this case, $j = 1$ and the correction term to $\omega_1 L_1$ from (3.14) is

$$\frac{\bar{\alpha}}{2} + \frac{2\bar{\alpha}}{3}\frac{\zeta_{00}\mathrm{Ai}(\zeta_{00})}{\int_{\infty}^{\zeta_{00}}\mathrm{Ai}(s)ds} - \frac{\pi U_0'\bar{c}\tilde{F}_1(\zeta_{00})R'(\zeta_{00})}{\zeta_{00}\int_{\infty}^{\zeta_{00}}\mathrm{Ai}(s)ds}$$
$$+\bar{\alpha}\pi\left(-\frac{1}{6}(\mathrm{Ai}'(\zeta_{00})R(\zeta_{00}) + \mathrm{Ai}(\zeta_{00})R'(\zeta_{00})) - \frac{2}{3}\zeta_{00}\mathrm{Ai}'(\zeta_{00})R'(\zeta_{00}) + \frac{2}{3}\zeta_{00}^2\mathrm{Ai}(\zeta_{00})R(\zeta_{00})\right),$$

where $R(\zeta_{00})$ is given in (3.13) and in this case

$$\tilde{F}_1(\zeta_{00}) = \frac{1}{3}(2\zeta_{00}^2\mathrm{Ai}(\zeta_{00})^2 - 2\zeta_{00}\mathrm{Ai}'(\zeta_{00})^2 - \mathrm{Ai}(\zeta_{00})\mathrm{Ai}'(\zeta_{00})).$$

Thus, it is straightforward to show that the leading-order term of $(2\bar{\alpha} - \omega_1 L_1)$ is just $2\bar{\alpha}$. However, on the right-hand side of (2.7), there is still some exponential growth, and in fact the leading-order term is

$$\frac{\bar{c}\bar{\alpha}(a_1 + \bar{c})}{2iU_0'^2}\int_{\zeta_{00}}^{\infty}\left(Wi(\zeta) + \frac{1}{\zeta}\right)d\zeta.$$

Using the definition of a_1 in (2.6), with ζ_0 replaced by ζ_{00} we see that $a_1 + \bar{c} = \bar{\alpha}/U_0'$. Hence, the equation for $d\ln(A)/dx_1$ becomes

$$\frac{d\ln(A)}{dx_1} = \frac{\bar{\alpha}\bar{c}}{4iU_0'^3}\int_{\zeta_{00}}^{\infty}\left(Wi(\zeta) + \frac{1}{\zeta}\right)d\zeta. \tag{3.15}$$

The above integral can be approximated for large ζ_{00} by using the fact that $Wi'' - \zeta Wi = 1$; thus, by integrating by parts we find

$$I = \int_{\zeta_{00}}^{\infty}\left(Wi(\zeta) + \frac{1}{\zeta}\right)d\zeta = -Wi'(\zeta_{00})\left(\frac{1}{\zeta_{00}} + \frac{2}{\zeta_{00}^4}\right) - Wi(\zeta_{00})\left(\frac{1}{\zeta_{00}^2} + \frac{8}{\zeta_{00}^5}\right)$$
$$- \frac{2}{3\zeta_{00}^3} + 40\int_{\zeta_{00}}^{\infty}\frac{Wi(\zeta)}{\zeta^6}d\zeta, \tag{3.16}$$

where

$$Wi(\zeta_{00}) = \pi\mathrm{Bi}(\zeta_{00})\int_{\infty_1}^{\zeta_{00}}\mathrm{Ai}(\xi)d\xi \quad \text{and} \quad Wi'(\zeta_{00}) = \pi\mathrm{Bi}'(\zeta_{00})\int_{\infty_1}^{\zeta_{00}}\mathrm{Ai}(\xi)d\xi. \tag{3.17}$$

Substituting (3.11) into (3.16) and (3.17) above, we find that, in terms of $\tilde{x}_1$, the first two asymptotic terms of the integral I can be given as

$$I \sim ie^{2/3}\left(\frac{1}{4}e^{5\pi i/4}\tilde{x}_1^{-3/4} - \frac{29}{96}e^{5\pi i/2}\tilde{x}_1^{-3/2} + O(\tilde{x}_1^{-9/4})\right)\exp\left(-\frac{4}{3}e^{-5\pi i/4}\tilde{x}_1^{3/4}\right). \tag{3.18}$$

Figure 6 plots the numerical solution of the integral I along with the solution given by the first five terms of the asymptotic expansion given in (3.16). We see that even for these moderate values of $\tilde{x}_1$, the asymptotics and numerics are in reasonable agreement, in both shape and magnitude. The agreement between the asymptotics and the numerics for larger values of $\tilde{x}_1$ is illustrated in Fig. 7, where (a) is the real part and (b) is the imaginary part of the integral I. In Fig. 7, curve 1 represents the full numerical integration, curve 2 is the leading-order term from the asymptotic expansion (3.16) $(-Wi'(\zeta_{00})/\zeta_{00})$, curve 3 is the first five terms of the asymptotic expansion given in (3.16) and curve 4 is the leading-order term of I from (3.18). Although the two leading-order expansions do not agree as well as the five-term higher-order expansion, their shape and magnitude show that these are valid approximations to the full numerical result at leading order.

Using the expansion (3.18) in (3.15), we find that the first two terms in the large $\tilde{x}_1$ expansion of $d\ln(A)/dx_1$ are

$$\frac{d\ln(A)}{dx_1} = -\frac{\tilde{x}_1^{1/2}e^{2/3}}{4U_0'^2}\left(-\frac{1}{4}e^{5\pi i/4}\tilde{x}_1^{-3/4} + \frac{29}{96}e^{5\pi i/2}\tilde{x}_1^{-3/2}\right)\exp\left(-\frac{4}{3}e^{-5\pi i/4}\tilde{x}_1^{3/4}\right),$$

which to leading order is

$$\frac{d\ln(A)}{dx_1} = \frac{e^{2/3}e^{5\pi i/4}}{16U_0'^2}\tilde{x}_1^{-1/4}\exp\left(-\frac{4}{3}e^{-5\pi i/4}\tilde{x}_1^{3/4}\right).$$

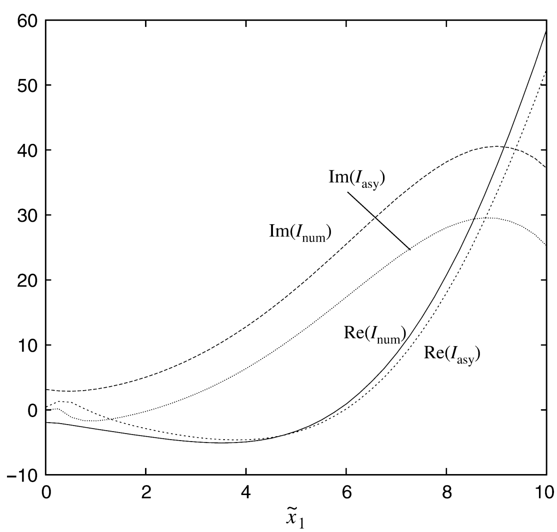

Fig. 6 Plot of the real and imaginary parts of the integral I as a function of $\tilde{x}_1$, for both the numerical solution and the asymptotic solution given by the first five terms of (3.16)

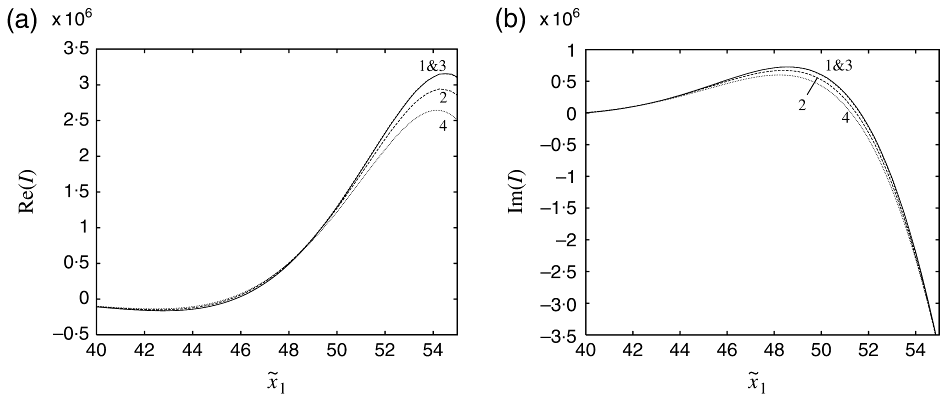

Fig. 7 Plot of (a) the real part and (b) the imaginary part of the integral I as a function of $\tilde{x}_1$, where 1 is the numerical solution, 2 is the leading-order term of (3.16), 3 is the first five terms of (3.16) and 4 is the leading-order term of (3.18)

Therefore, comparing the leading-order asymptotic term of κ_0 with the one above, we find that the asymptotic expansion breaks down when

$$\tilde{x}_1^{-1/4} = \epsilon^3 \tilde{x}_1^{-1/4} \exp\left(-\frac{4}{3} e^{-5\pi i/4} \tilde{x}_1^{3/4}\right),$$

which leads to a breakdown when

$$\tilde{x}_1 = O((-\ln \epsilon)^{4/3}).$$

For the value of $\epsilon = F^{1/6} = 0{\cdot}1$ used in Fig. 1, this would lead to a breakdown around $\tilde{x}_1 = 3{\cdot}0$ ($R_x \approx 3{\cdot}3 \times 10^7$), which is in reasonable agreement with the observations in Fig. 1.

4. Conclusions

We have shown that for the Blasius boundary layer on a semi-infinite flat plate, the large Reynolds number asymptotic expansion for the wave number, κ_{Tot}, given by Goldstein (**2**) becomes non-uniform downstream, due to the non-parallel flow effects. In the limit as $\epsilon \to 0$, the lower branch neutral stability point ($(\kappa_{\mathrm{Tot}}) = 0$) occurs at $\tilde{x}_1 = 3{\cdot}03$, and the non-uniformity occurs far downstream of this, where it is unimportant. However, for practical values of ϵ, this non-uniformity moves towards the lower branch point, and in fact when $\epsilon = 0{\cdot}1$, the non-uniformity occurs before the lower branch point. This shows that any T-S wave amplitudes calculated via these asymptotics downstream of the leading edge would become less accurate with the inclusion of the important $O(\epsilon^3)$ term. However, the numerical methods of Gaster (**5**), Saris and Nayfeh (**6**) and Bertolotti *et al.* (**12**) do not show this non-uniformity and are in good agreement with the numerical and experimental results. These numerical schemes include terms which are $O(1)$ and $O(R_x^{-1/2} = 2^{1/2}\epsilon^4/U_0'\tilde{x}_1^{1/2})$. The $O(R_x^{-1/2})$ terms when changed into our variables correspond to being $O(\epsilon^3)$, and as the neglected terms in the above numerical schemes are $O(R_x^{-1})$, or $O(\epsilon^6)$, we note that these schemes include more terms than our asymptotic expansion. Also, these schemes handle all the $O(R_x^{-1/2})$ terms together, which corresponds to combining terms of order ϵ^3, $\epsilon^4 \ln \epsilon$, ϵ^4, $\epsilon^5 \ln \epsilon$, ϵ^5 and $\epsilon^6 \ln \epsilon$ into one single equation. It appears that this approach either removes the non-uniformity or handles it in such a way as to not be an issue. Thus, the inclusion of these extra terms could help to deal with the non-uniformity; however, the asymptotic evaluation of these terms is far from trivial.

The value of $\epsilon = 0{\cdot}1$ ($F = 1 \times 10^{-6}$) used in this paper has illustrated the importance of the $O(\epsilon^3)$ term in the asymptotic expansion; however, in experiments, a typical value for ϵ is between 0·18 and 0·25 ($34 \times 10^{-6} < F < 244 \times 10^{-6}$); hence, the inclusion of the O(ϵ^3) term becomes even more significant for these cases. Thus, to accurately calculate the T-S wave amplitude downstream of the leading-edge region, we require a numerical method in the Orr–Sommerfeld region similar to the method of Turner and Hammerton (**11**). A numerical method in the Orr–Sommerfeld region would also be needed if the asymptotics were extended to bodies with non-zero pressure gradients, for the same reason, although the asymptotics up to $O(\epsilon^3 \ln \epsilon)$ would provide reasonable results, at least for small ϵ.

Acknowledgement

I am very grateful to Dr P. W. Hammerton for his encouraging comments. This work was partially supported by the EPSRC (EP/D032202/1).

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APPENDIX

Evaluation of integrals

This Appendix displays the form of the integrals in section 2, as well as their evaluation or simplification.

Some of the integrals in I_1 to I_9 involve the integration of $\mathrm{Ai}(\zeta)^2$, which can be integrated by noting that $w = \mathrm{Ai}(\zeta)^2$ satisfies the differential equation

$$w''' - 4\zeta w' - 2w = 0.$$

Rearranging this to give $2w = w''' - 4\zeta w'$ and integrating with respect to ζ gives

$$\int w\, d\zeta = -\frac{1}{2}(w'' - 4\zeta w).$$

Thus, the integrals I_1 to I_9 can be evaluated by using the above expression and using integration by parts:

$$I_1 = \int_{\zeta_0}^{\infty} (\zeta - \zeta_0)\mathrm{Ai}(\zeta)^2\, d\zeta = \frac{1}{3}(2\zeta_0^2\mathrm{Ai}(\zeta_0)^2 - 2\zeta_0\mathrm{Ai}'(\zeta_0)^2 - \mathrm{Ai}(\zeta_0)\mathrm{Ai}'(\zeta_0)),$$

$$I_2 = \int_{\zeta_0}^{\infty} \zeta(\zeta - \zeta_0)\mathrm{Ai}(\zeta)\mathrm{Ai}'(\zeta)d\zeta = \frac{1}{6}(2\mathrm{Ai}(\zeta_0)\mathrm{Ai}'(\zeta_0) - \zeta_0^2\mathrm{Ai}(\zeta_0)^2 + \zeta_0\mathrm{Ai}'(\zeta_0)^2),$$

$$I_3 = \int_{\zeta_0}^{\infty} \mathrm{Ai}(\zeta)\int_{\infty}^{\zeta} \mathrm{Ai}(s)ds\, d\zeta = -\frac{1}{2}\left(\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds\right)^2,$$

$$I_4 = \int_{\zeta_0}^{\infty} (\zeta - \zeta_0)^2\mathrm{Ai}'(\zeta)\mathrm{Ai}(\zeta)d\zeta$$

$$= -I_1 = -\frac{1}{3}(2\zeta_0^2\mathrm{Ai}(\zeta_0)^2 - 2\zeta_0\mathrm{Ai}'(\zeta_0)^2 - \mathrm{Ai}(\zeta_0)\mathrm{Ai}'(\zeta_0)),$$

$$I_5 = \int_{\zeta_0}^{\infty} (\zeta - \zeta_0)^2\mathrm{Ai}(\zeta)\int_{\infty}^{\zeta} \mathrm{Ai}(s)ds\, d\zeta$$

$$= (\mathrm{Ai}(\zeta_0) + \zeta_0\mathrm{Ai}'(\zeta_0))\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds - \frac{\zeta_0^2}{2}\left(\int_{\infty}^{\zeta_0} Ai(s)ds\right)^2 + \frac{3}{2}\mathrm{Ai}'(\zeta_0)^2 - 2\zeta_0\mathrm{Ai}(\zeta_0)^2,$$

$$I_6 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^3 \mathrm{Ai}(\zeta) \int_{\infty}^{\zeta} \mathrm{Ai}(s) ds\, d\zeta = -(\zeta_0^2 \mathrm{Ai}'(\zeta_0) + \zeta_0 \mathrm{Ai}(\zeta_0)) \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds$$

$$+ \left(\frac{\zeta_0^3}{2} - 1\right) \left(\int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds\right)^2 - \frac{7}{2}\zeta_0 \mathrm{Ai}'(\zeta_0)^2 + 4\zeta_0^2 \mathrm{Ai}(\zeta_0)^2 - \mathrm{Ai}(\zeta_0)\mathrm{Ai}'(\zeta_0),$$

$$I_7 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^2 \mathrm{Ai}(\zeta) d\zeta = \zeta_0 \mathrm{Ai}'(\zeta_0) + \mathrm{Ai}(\zeta_0) - \zeta_0^2 \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds,$$

$$I_8 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^3 \mathrm{Ai}(\zeta) d\zeta = (\zeta_0^3 - 2) \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds - \zeta_0^2 \mathrm{Ai}'(\zeta_0) - \zeta_0 \mathrm{Ai}(\zeta_0),$$

$$I_9 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^4 \mathrm{Ai}(\zeta)^2 d\zeta = \frac{1}{9}\left(\left(\frac{128}{35}\zeta_0^4 - \frac{80}{7}\zeta_0\right) \mathrm{Ai}'(\zeta_0)^2 + \left(\frac{64}{5}\zeta_0^2 - \frac{128}{35}\zeta_0^5\right) \mathrm{Ai}(\zeta_0)^2\right.$$

$$\left. + \left(\frac{64}{35}\zeta_0^3 - 4\right) \mathrm{Ai}(\zeta_0)\mathrm{Ai}'(\zeta_0)\right).$$

The integrals K_1 to K_9 are evaluated in a similar way to the integrals above. However, in this case, some of the integrals involve having to integrate $\mathrm{Ai}(\zeta)Wi(\zeta)$, where $Wi(\zeta)$ is defined in (2.6). In this case, the function $v = \mathrm{Ai}(\zeta)Wi(\zeta)$ satisfies the differential equation

$$v''' - 4\zeta v' - 2v = 3\mathrm{Ai}'.$$

Again, this is rearranged to give $2v = v''' - 4\zeta v' - 3\mathrm{Ai}'$, which on integration with respect to ζ leads to the expression

$$\int v\, d\zeta = -\frac{1}{2}(v'' - 4\zeta v - 3\mathrm{Ai}).$$

Thus, using this and integration by parts lead to the evaluation of the integrals below:

$$K_1 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0) \mathrm{Ai}(\zeta) Wi(\zeta) d\zeta$$

$$= \frac{1}{3}\left(\frac{3}{2} - \frac{\pi}{2}(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}(\zeta_0) + \mathrm{Ai}(\zeta_0)\mathrm{Bi}'(\zeta_0)) - 2\pi\zeta_0(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}'(\zeta_0) - \zeta_0 \mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0))\right)$$

$$\times \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds + \frac{2}{3}\zeta_0 \mathrm{Ai}(\zeta_0),$$

$$K_2 = \int_{\zeta_0}^{\infty} \zeta(\zeta-\zeta_0) \mathrm{Ai}'(\zeta) Wi(\zeta) d\zeta$$

$$= \frac{1}{6}\pi(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}(\zeta_0) + \mathrm{Ai}(\zeta_0)\mathrm{Bi}'(\zeta_0)) \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds + \frac{1}{6}\pi\zeta_0 \mathrm{Ai}'(\zeta_0)\mathrm{Bi}'(\zeta_0) \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds$$

$$- \frac{1}{6}\pi\zeta_0^2 \mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0) \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds - \frac{5}{6}\int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds - \frac{1}{12}\zeta_0^3 \int_{\infty}^{\zeta_0} \mathrm{Ai}(s) ds$$

$$- \frac{1}{12}\zeta_0 \mathrm{Ai}(\zeta_0) + \frac{1}{12}\zeta_0^2 \mathrm{Ai}'(\zeta_0),$$

$$K_3 = \int_{\zeta_0}^{\infty} Wi(\zeta) \int_{\infty}^{\zeta} \mathrm{Ai}(s)ds\, d\zeta = \frac{3}{2}\pi \int_{\zeta_0}^{\infty} \mathrm{Bi}(\zeta)\left(\int_{\infty}^{\zeta} \mathrm{Ai}(s)ds\right)^2 d\zeta,$$

$$K_4 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^2 \mathrm{Ai}'(\zeta) Wi(\zeta) d\zeta$$

$$= \frac{1}{3}\Big(\frac{\pi}{2}(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}(\zeta_0) + \mathrm{Ai}(\zeta_0)\mathrm{Bi}'(\zeta_0)) + 2\pi\zeta_0(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}'(\zeta_0) - \zeta_0\mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0))\Big)$$

$$\times \int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds - \frac{5}{6}\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds + \frac{1}{6}\zeta_0^3 \int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds - \frac{5}{6}\zeta_0\mathrm{Ai}(\zeta_0) - \frac{1}{6}\zeta_0^2\mathrm{Ai}'(\zeta_0),$$

$$K_5 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^2 Wi(\zeta) \int_{\infty}^{\zeta} \mathrm{Ai}(s)ds\, d\zeta = \pi(\mathrm{Bi}(\zeta_0) + \zeta_0\mathrm{Bi}'(\zeta_0))\left(\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds\right)^2$$

$$- 2\zeta_0\pi\,\mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0)\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds + \frac{3}{2}\pi\,\mathrm{Ai}'(\zeta_0)\mathrm{Bi}'(\zeta_0)\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds - \frac{9}{4}\zeta_0^2 \int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds$$

$$+ \frac{9}{4}\zeta_0\mathrm{Ai}'(\zeta_0) - \frac{3}{4}\mathrm{Ai}(\zeta_0) + \frac{3}{2}\pi\zeta_0^2 \int_{\zeta_0}^{\infty} \mathrm{Bi}(\zeta)\left(\int_{\infty}^{\zeta} \mathrm{Ai}(s)ds\right)^2 d\zeta,$$

$$K_6 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^3 Wi(\zeta) \int_{\infty}^{\zeta} \mathrm{Ai}(s)ds\, d\zeta = -\pi\zeta_0(\mathrm{Bi}(\zeta_0) + \zeta_0\mathrm{Bi}'(\zeta_0))\left(\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds\right)^2$$

$$- \frac{1}{2}\pi(\mathrm{Ai}(\zeta_0)\mathrm{Bi}'(\zeta_0) + \mathrm{Ai}'(\zeta_0)\mathrm{Bi}(\zeta_0))\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds + 4\pi\zeta_0^2\mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0)\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds$$

$$- \frac{7}{2}\pi\zeta_0\mathrm{Ai}'(\zeta_0)\mathrm{Bi}'(\zeta_0)\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds + \frac{11}{4}\zeta_0^3\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds + \frac{1}{2}\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds - \frac{11}{4}\zeta_0^2\mathrm{Ai}'(\zeta_0)$$

$$+ \frac{9}{4}\zeta_0\mathrm{Ai}(\zeta_0) + 3\pi\left(1 - \frac{\zeta_0^3}{2}\right)\int_{\zeta_0}^{\infty} \mathrm{Bi}(\zeta)\left(\int_{\infty}^{\zeta} \mathrm{Ai}(s)ds\right)^2 d\zeta,$$

$$K_7 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^2 \mathrm{Ai}(\zeta) Wi(\zeta) d\zeta = \frac{1}{5}\pi\,\mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0)\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds$$

$$+ \frac{4}{15}\zeta_0\Big(\frac{\pi}{2}(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}(\zeta_0) + \mathrm{Ai}(\zeta_0)\mathrm{Bi}'(\zeta_0)) + 2\pi\zeta_0(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}'(\zeta_0) - \zeta_0\mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0))\Big)$$

$$\times \int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds - \frac{8}{15}\zeta_0^2\mathrm{Ai}(\zeta_0) + \frac{3}{5}\mathrm{Ai}'(\zeta_0) - \zeta_0\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds,$$

$$K_8 = \int_{\zeta_0}^{\infty} (\zeta-\zeta_0)^3 \mathrm{Ai}(\zeta) Wi(\zeta) d\zeta = \frac{3}{7}\pi(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}'(\zeta_0) - \zeta_0\mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0))\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds$$

$$- \frac{6}{7}\zeta_0 K_7 - \frac{15}{14}\mathrm{Ai}(\zeta_0) - \frac{9}{14}\zeta_0\mathrm{Ai}'(\zeta_0) + \frac{9}{14}\zeta_0^2\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds,$$

$$K_9 = \int_{\zeta_0}^{\infty} (\zeta - \zeta_0)^4 \mathrm{Ai}(\zeta) Wi(\zeta) d\zeta$$

$$= -\frac{4}{9}\left(\frac{\pi}{2}(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}(\zeta_0) + \mathrm{Ai}(\zeta_0)\mathrm{Bi}'(\zeta_0)) + 2\pi\zeta_0(\mathrm{Ai}'(\zeta_0)\mathrm{Bi}'(\zeta_0) - \zeta_0\mathrm{Ai}(\zeta_0)\mathrm{Bi}(\zeta_0))\right)$$

$$\times \int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds - \frac{8}{9}\zeta_0 K_8 + \frac{14}{9}\zeta_0\mathrm{Ai}(\zeta_0) + \frac{2}{3}\zeta_0^2\mathrm{Ai}'(\zeta_0) + 2\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds - \frac{2}{3}\zeta_0^3 \int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds.$$

The integral K_3 (and also K_5 and K_6) could not be evaluated explicitly; however, they were simplified, as shown above. Thus, when we are in the region where $\mathrm{Bi}(\zeta) \sim -i\mathrm{Ai}(\zeta)$ to leading order, as in (3.12), then the leading-order term of K_3 is given by

$$K_3 \sim \frac{i\pi}{2}\left(\int_{\infty}^{\zeta_0} \mathrm{Ai}(s)ds\right)^3,$$

which means that property (3.14) still holds for L_3, L_5 and L_6.