

The stability of a radially stretching disc beneath a uniformly rotating fluid.

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Abstract

The steady radial stretching of a disc beneath a rigidly rotating flow with constant angular velocity is considered. The steady base flow is determined numerically for both a stretching and a shrinking disc. The convective instability properties of the flow are examined using temporal stability analysis of the governing Rayleigh equation, and typically for small to moderate radial wavenumbers, the range of azimuthal wavenumbers, β , over which the flow is unstable increases for both a stretched and shrinking disc, compared to the unstretched case. The inviscid absolute instability properties of the resulting base flows are also examined using spatio-temporal stability analysis. For suitably large stretching rates, the flow is absolutely unstable in only a small range of positive β . For small stretching rates there exists a second region of absolute instability for a range of negative β values. In this region the ‘effective’ two-dimensional base flow, comprised of a linear combination of the radial and azimuthal velocity profiles which enter the Rayleigh equation calculation, has a critical point (unlike for $\beta > 0$) which can dominate the absolute instability growth rate contribution compared to the shear layer component. A similar behaviour is found to occur for a radially shrinking disc, except these profiles have a strong shear layer structure and hence are more unstable than the stretching disc profiles. We thus find for a suitably large shrinking rate the absolute instability contribution from the critical point becomes sub-dominant to the shear layer contribution.

1 Introduction

There have been many theoretical, numerical and experimental studies which have examined the flow of a rotating fluid above an infinite disc. The first theoretical investigation of this flow setup was due to Bödewadt (1940). Bödewadt calculated the steady boundary layer flow over an infinite stationary disc in the form of a similarity solution, such that the flow is an exact solution of the Navier-Stokes equations. The steady flow itself is characterized by a radial pressure gradient which balances the centrifugal forces. This leads to a flow where fluid is pulled radially inward from infinity at the surface of the disc and ejected upward as it travels toward the centre of the disc. This type of flow exemplifies a crossflow instability in a canonical example. This flow is relevant to industrial devices such as rotor-stator systems (Itoh et al., 1990) and cavity elements of turbine engines (Owen, 1988). The Bödewadt flow is not the only exact solution to the Navier-Stokes equations for rotating fluid flows which has received much attention. von Kármán (1921) examined the steady flow produced by a rotating disc of infinite extent below a stationary fluid, while Ekman (1905) examined a related flow where the disc and the outer flow are both rotating with almost equal angular velocities.

The content of this paper focuses on the Bödewadt flow, except with the added feature that the infinite disc is able to stretch, and we examine how the stability of the boundary layer is modified by stretching the disc beneath the rigidly rotating flow. We document the steady base flow shear stresses and velocity profiles as a function of the stretching (or shrinking) rate, as well as investigate the convective and absolute instability properties (henceforth denoted by CI and AI) of the resulting flows in the limit of large Reynolds number, where the Reynolds number is defined using the angular velocity of the rotating flow. Hence we investigate how disc stretching or shrinking can control instability growth in the boundary layer, and thus control transition to turbulence.

Savaş (1983, 1987) examined the stability of Bödewadt flow by conducting experiments on the spin-down flow of a cylindrical cavity. Savaş suggests that until secondary instabilities start to grow, this experimental setup is a suitable means to study the stability of Bödewadt flow. It was found that both stationary spiral modes (often referred to as type I instability) and unsteady circular instability modes appear. The circular modes propagate towards the centre of the disc before dying out. Lopez & Weidman (1996) confirmed the experimental results of Savaş (1983, 1987) via direct axisymmetric numerical simulations and further experiments. By artificially allowing the side wall of the cylinder to continue rotating even after the end walls stopped in their simulations, Lopez & Weidman found that the presence of the inwardly propagating circular modes could persist for longer times than when the side walls were stopped with the end walls. From this they concluded that the circular instability mode generation could persist indefinitely in the Bödewadt problem. The experimental results have also been confirmed via theoretical studies (Fernandez-Feria, 2002; Mackerrell, 2005)

The inwardly propagating circular waves observed in the above experiments are of particular interest because, for the von Kármán boundary layer, Lingwood (1995) discovered an AI consisting of a coalescence of an inviscidly unstable spiral mode and an inwardly propagating spatially decaying

mode, similar to the circular modes described above. Based on this finding, Lingwood (1997) conducted a linear stability analysis of the Bödewadt, Ekman, and von Kármán boundary layer flows, investigating the AI properties of each flow. A flow is absolutely unstable if, when impulsively forced, the response to the transient disturbance grows in time at the location where the forcing is applied. If the disturbance grows in time along a characteristic with non-zero speed but decays at the point at which it was forced, then the flow is convectively unstable (Huerre & Monkewitz, 1990). Following the work of Briggs (1964) the AI characteristics of a flow can be determined via a spatio-temporal stability analysis in which both the wavenumber and frequency of the flow are allowed to become complex. This is in contrast with a temporal or spatial analysis where only the frequency or wavenumber, respectively, is allowed to become complex (Schmid & Henningson, 2001). The AI growth rate is calculated by searching for special saddle points in the complex wavenumber plane, through which the inverse Fourier transform contour can pass. For each of the three above cited flows, Lingwood (1997) calculated neutral stability contours, at finite Reynolds numbers, for the AI of the flows by transforming the governing linear partial differential equations into six first order ordinary differential equations. Using a parallel flow assumption she also conducted inviscid AI calculations and found that the Bödewadt flow, in particular, was absolutely unstable for a range of both positive and negative azimuthal wavenumbers of the instantaneous forcing.

In the present study we re-examine the inviscid CI and AI properties of the Bödewadt flow as well as for the new flows which incorporate a stretching or shrinking disc. In particular for the Bödewadt flow problem, we consider the AI of perturbations with azimuthal wavenumbers $\beta < -0.3$ which are not present in Lingwood (1997). Here we find that the growth rate of the AI reduces from its $\beta > 0$ maximum value, but for $\beta \lesssim -0.3$ the growth rate becomes dominated by an additional saddle point which produces a secondary maximum in the growth rate curve. It is shown that this additional saddle point has an ‘effective’ two-dimensional velocity profile containing a critical point away from the surface of the disc. At this azimuthal wavenumber, the contribution to the AI from this critical point is larger than that of the shear layer interaction with the disc surface, and hence its growth rate is larger. The same significant contribution from critical points on flow stability was shown for the von Kármán flow by Healey (2006) in the long wavenumber limit. In the current work it is also shown that stretching the disc beneath the rotating flow tends to reduce the effect (both maximum growth rate and existence range of azimuthal wavenumbers) of the AI, while shrinking the disc enhances the AI.

The current paper is laid out as follows. In §2 we derive the steady base flow equations, as well as the governing ordinary differential equations for the inviscid linear stability analysis in the parallel flow limit. In §3 we present numerical solutions to the base flow equations and examine the asymptotic behaviour in a region far from the surface of the disc. Convective and absolute instability results for the base flow profiles are given in §4 with a discussion and concluding remarks given in

2 Formulation

The problem we consider is a disc radially stretching at constant strain rate a in the dimensional (\hat{x}, \hat{y}) -plane, below a rigidly rotating fluid, which in the large \hat{z} far-field is rotating steadily with constant angular velocity Ω . A schematic diagram of the physical setup is given in figure 1. The

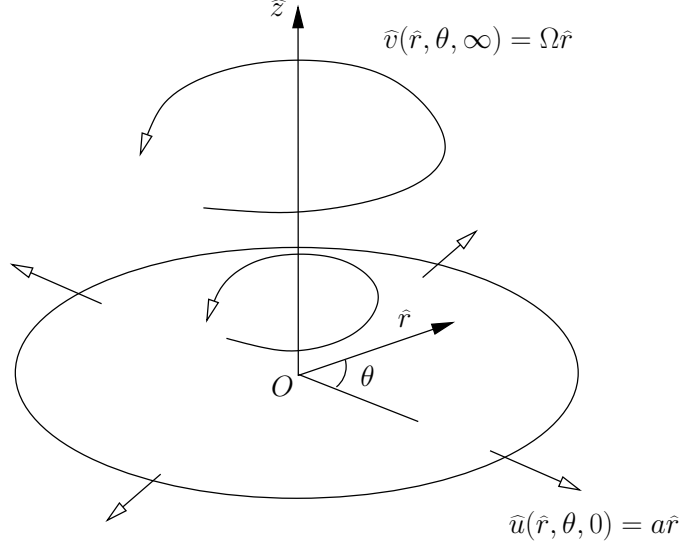


Figure 1: A schematic diagram of the radially stretching disc below a rotating fluid.

flow is governed in the inertial non-rotating reference frame by the incompressible Navier-Stokes equations

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} = -\frac{1}{\rho} \nabla \hat{p} + \nu \nabla^2 \hat{\mathbf{u}}, \quad \nabla \cdot \hat{\mathbf{u}} = 0,$$

where $\hat{\mathbf{u}}$ is the dimensional fluid velocity, ρ is the fluid density and ν is the kinematic viscosity. We use cylindrical polar coordinates $(\hat{r}, \theta, \hat{z})$ to formulate the problem with corresponding velocity components $\hat{\mathbf{u}} = (\hat{u}, \hat{v}, \hat{w})$. The boundary condition on the surface of the disc located at $z = 0$, oriented in the (x, y) -plane, is $\hat{\mathbf{u}} = (a\hat{r}, 0, 0)$ and in the far-field $\hat{z} \rightarrow \infty$, $\hat{\mathbf{u}} \rightarrow (0, \Omega\hat{r}, \hat{w}_\infty)$, where \hat{w}_∞ is not prescribed but is found as part of the solution. The form of this boundary condition suggests seeking a solution for the base flow which is axisymmetric.

We non-dimensionalize the problem by introducing the common scales: $\sqrt{\nu/\Omega}$ for lengths, $\sqrt{\nu\Omega}$ for velocities, $1/\Omega$ for time and $\rho\nu\Omega$ for pressure. Thus the dimensionless incompressible, Navier-Stokes equations in the inertial reference frame are

$$u_t + uu_r + \frac{vu_\theta}{r} + wu_z - \frac{v^2}{r} = -p_r + u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz} - \frac{u}{r^2} - \frac{2v\theta}{r^2}, \quad (2.1a)$$

$$v_t + uv_r + \frac{vv_\theta}{r} + wv_z + \frac{uv}{r} = -\frac{p_\theta}{r} + v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2} + v_{zz} - \frac{v}{r^2} + \frac{2u\theta}{r^2}, \quad (2.1b)$$

$$w_t + uw_r + \frac{vw_\theta}{r} + ww_z = -p_z + w_{rr} + \frac{w_r}{r} + \frac{w_{\theta\theta}}{r^2} + w_{zz}, \quad (2.1c)$$

$$u_r + \frac{u}{r} + \frac{v_\theta}{r} + w_z = 0, \quad (2.1d)$$

where (u, v, w) are the dimensionless velocity components, p the dimensionless pressure and (r, θ, z) are the dimensionless cylindrical polar coordinates. The subscripts in (2.1) denote partial derivatives. In these non-dimensional variables the disc boundary conditions are

$$u(r, \theta, 0) = \sigma r, \quad v(r, \theta, 0) = 0, \quad w(r, \theta, 0) = 0, \quad (2.2a)$$

and the far field conditions are

$$u(r, \theta, \infty) = 0, \quad v(r, \theta, \infty) = r, \quad w(r, \theta, \infty) = w_\infty, \quad (2.2b)$$

where w_∞ is to be determined. The parameter $\sigma = a/\Omega$ defines a one parameter family of possible solutions.

2.1 The base flow

The flow can be separated into an axisymmetric steady base flow, and a more general unsteady part whose amplitude is characterized by a small parameter $\delta \ll 1$. Thus inserting

$$u = r f'(z) + \delta \bar{u}(r, \theta, z, t), \quad (2.3a)$$

$$v = r g(z) + \delta \bar{v}(r, \theta, z, t), \quad (2.3b)$$

$$w = -2f(z) + \delta \bar{w}(r, \theta, z, t), \quad (2.3c)$$

$$p = P(r, z) + \delta \bar{p}(r, \theta, z, t), \quad (2.3d)$$

into (2.1) and (2.2) and equating terms at $O(1)$ leads to the coupled pair of nonlinear ordinary differential equations

$$f''' + 2f f'' - f'^2 + g^2 - 1 = 0, \quad (2.4a)$$

$$g'' + 2(fg' - f'g) = 0, \quad (2.4b)$$

to be solved together with the boundary conditions

$$f(0) = 0, \quad f'(0) = \sigma, \quad f'(\infty) = 0 \quad (2.5a)$$

$$g(0) = 0, \quad g(\infty) = 1. \quad (2.5b)$$

In (2.3c) the form of the base flow $f(z)$, $g(z)$ is found by the need to satisfy (2.1d) and the primes in (2.4) denote ordinary derivatives with respect to z . In deriving the above system of ordinary differential equations it is also possible to compute the base pressure field

$$p(r, z) = p_0 + 2\sigma^2 - \frac{1}{2}r^2 - 2(f' + f^2), \quad (2.6)$$

where p_0 is the constant pressure at $(r, z) = (0, 0)$. The governing ordinary differential equations (2.4) and the pressure field (2.6) can be shown to agree with (3.11)-(3.16) of Lingwood (1997) when the Rossby number $Ro \equiv 1$. Note from (2.3c) that $w_\infty = -2f(\infty)$.

The radial and azimuthal dimensional shear stresses on the disc surface are given by

$$\tau_r = \rho\nu\Omega r f''(0), \quad \tau_\theta = \rho\nu\Omega r g'(0).$$

Numerical results for the base flow equations are investigated in 3.

2.2 Linearized disturbance equations

Substituting (2.3) into (2.1) and equating terms of $O(\delta)$ furnishes the linearized disturbance equations

$$D\bar{u} + rf''\bar{w} + f'\bar{u} - 2g\bar{v} = -\bar{p}_r + L\bar{u} - \frac{\bar{u}}{r^2} - \frac{2\bar{v}_\theta}{r^2}, \quad (2.7a)$$

$$D\bar{v} + rg'\bar{w} + f'\bar{v} + 2g\bar{u} = -\frac{\bar{p}_\theta}{r} + L\bar{v} - \frac{\bar{v}}{r^2} + \frac{2\bar{u}_\theta}{r^2}, \quad (2.7b)$$

$$D\bar{w} - 2f'\bar{w} = -\bar{p}_z + L\bar{w}, \quad (2.7c)$$

$$\bar{u}_r + \frac{\bar{u}}{r} + \frac{\bar{v}_\theta}{r} + \bar{w}_z = 0, \quad (2.7d)$$

where

$$D \equiv \frac{\partial}{\partial t} - rf'' \frac{\partial}{\partial r} + g \frac{\partial}{\partial \theta} - 2f \frac{\partial}{\partial z}, \quad \text{and} \quad L \equiv \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

These linear partial differential equations have coefficients which depend upon r and z but not on θ or t , and therefore we can take Fourier and Laplace transforms respectively. However, since the resulting disturbance equations remain as partial differential equations in r and z , they can only be reduced to ordinary differential equations at leading order by considering a point on the disc far from the flow axis of rotation where the boundary layer profile is assumed to be approximately parallel. For the stability analysis presented in this paper we shall work in this limiting regime.

If we let \hat{R} be the dimensional radial position at which we perform a stability analysis of the flow on the disc, then we can introduce the Reynolds number Re such that

$$Re = \hat{R} \sqrt{\frac{\Omega}{\nu}}.$$

We next introduce a new scaled radial coordinate \tilde{r} given by

$$r = \tilde{r} Re,$$

where we assume $Re \gg 1$, to be far from the flow axis of rotation, and $\tilde{r} = O(1)$ in this region. In this region we neglect the weak non-parallel effects of the boundary layer on the disc and hypothesize that the disturbance quantities assume the waveform given by

$$\bar{u}(\tilde{r}, \theta, z, t) = u(z) \exp [iRe (\alpha\tilde{r} + \bar{\beta}\theta - \omega t)], \quad (2.8a)$$

$$\bar{v}(\tilde{r}, \theta, z, t) = v(z) \exp [iRe (\alpha\tilde{r} + \bar{\beta}\theta - \omega t)], \quad (2.8b)$$

$$\bar{w}(\tilde{r}, \theta, z, t) = w(z) \exp [iRe (\alpha\tilde{r} + \bar{\beta}\theta - \omega t)], \quad (2.8c)$$

$$\bar{p}(\tilde{r}, \theta, z, t) = Re p(z) \exp [iRe (\alpha\tilde{r} + \bar{\beta}\theta - \omega t)]. \quad (2.8d)$$

Here α and $\bar{\beta}$ are $O(1)$ scaled radial and azimuthal wavenumbers respectively, ω is the scaled angular frequency, and we have recycled the notation (u, v, w, p) with the understanding that henceforth these refer only to the perturbation quantities. The presence of the Reynolds number in the

exponential terms indicate that the resulting waves are short compared to the distance to the axis of rotation for the flow field. Thus this is the basis for the WKB approximation in (2.8).

Substituting (2.8) into the linearized disturbance equations (2.7), and taking the limit as $Re \rightarrow \infty$ leads to four ordinary differential equations for (u, v, w, p) which, on eliminating (u, v, p) , reduce to the Rayleigh equation

$$(Q - \omega) (w'' - \gamma^2 w) - Q'' w = 0, \quad (2.9a)$$

where

$$Q(z) = \alpha f'(z) + \beta g(z), \quad \gamma^2 = \alpha^2 + \beta^2, \quad (2.9b)$$

$\beta = \bar{\beta}/\tilde{r}$ and the primes denote derivatives with respect to z . Note that $Re\bar{\beta}$ is an integer, but as we are working in the large Re limit we can consider β to be a real variable. While the disturbance equation does not depend explicitly on r through the Reynolds number, there is still a radial dependence on the solutions through the quantity β . The wavenumber α and/or the frequency ω , are allowed to become complex in order to satisfy the homogenous boundary conditions

$$w(0) = 0, \quad w(\infty) = 0. \quad (2.10)$$

Hence we perform both a temporal and spatio-temporal stability analysis of (2.9) in 4, calculating α and ω such that (2.10) is satisfied.

3 Numerical solutions for the base flow

In this section we present numerical results for the solution to the one-parameter system given by (2.4) and (2.5). The system is solved using a 4th order Runge-Kutta shooting method with Newton iterations at $z = z_{\max}$ to update the values of $f''(0)$ and $g'(0)$ such that the far field boundary conditions are satisfied; see Press, *et al.* (1989). The integration domain $z \in [0, z_{\max}]$ and the integration step size are varied to ensure that the results presented here are independent of these parameters. We find $z_{\max} = 20$ is large enough to correctly capture the far field boundary conditions. The resulting values of $f''(0)$ and $g'(0)$ as a function of σ are shown in figure 2. The radial shear stress $f''(0)$ is always negative while the azimuthal shear stress $g'(0)$ is always positive for both stretching ($\sigma > 0$) and shrinking ($\sigma < 0$) discs. However, both shear stresses exhibit extrema at $\sigma \approx 0$ which corresponds to the Bödewadt problem. The asymptotic forms of the shear stresses are given by

$$f''(0) = \sigma^{3/2} [F_{0ZZ}(0) + \sigma^{-2} F_{2ZZ}(0) + \sigma^{-4} F_{4ZZ}(0)] + O(\sigma^{-9/2}), \quad (3.1a)$$

$$g'(0) = \sigma^{1/2} [G_{0Z}(0) + \sigma^{-2} G_{2Z}(0) + \sigma^{-4} G_{4Z}(0)] + O(\sigma^{-11/2}). \quad (3.1b)$$

for $\sigma > 0$ and

$$f''(0) = |\sigma|^{3/2} [\tilde{F}_{0ZZ}(0) + |\sigma|^{-1} \tilde{F}_{2ZZ}(0) + |\sigma|^{-2} \tilde{F}_{4ZZ}(0)] + O(|\sigma|^{-3/2}), \quad (3.2a)$$

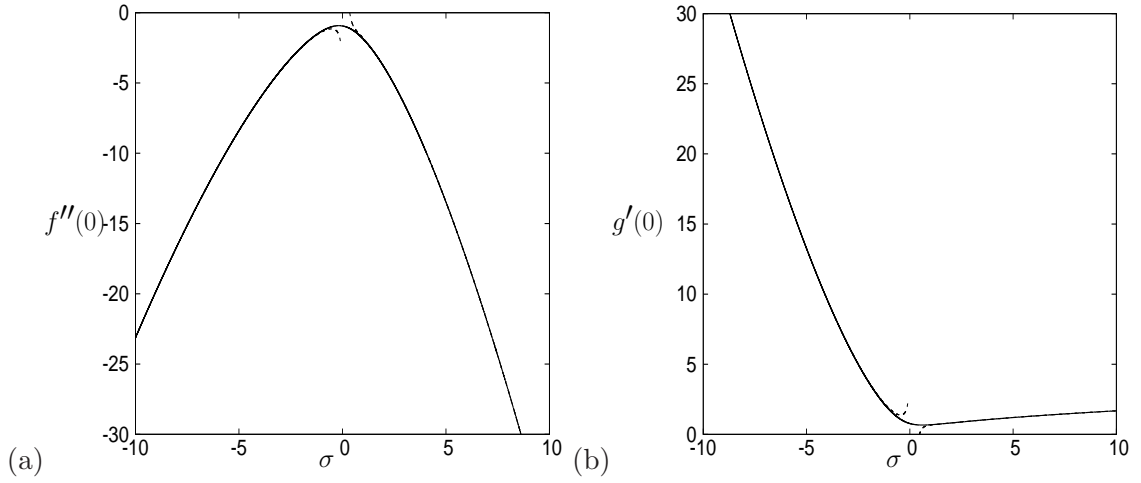


Figure 2: Plot of the numerical shear stresses (a) $f''(0)$ and (b) $g'(0)$ as a function of σ given by the solid lines and the three-term asymptotic expansions, (3.1) and (3.2), given by the dashed lines.

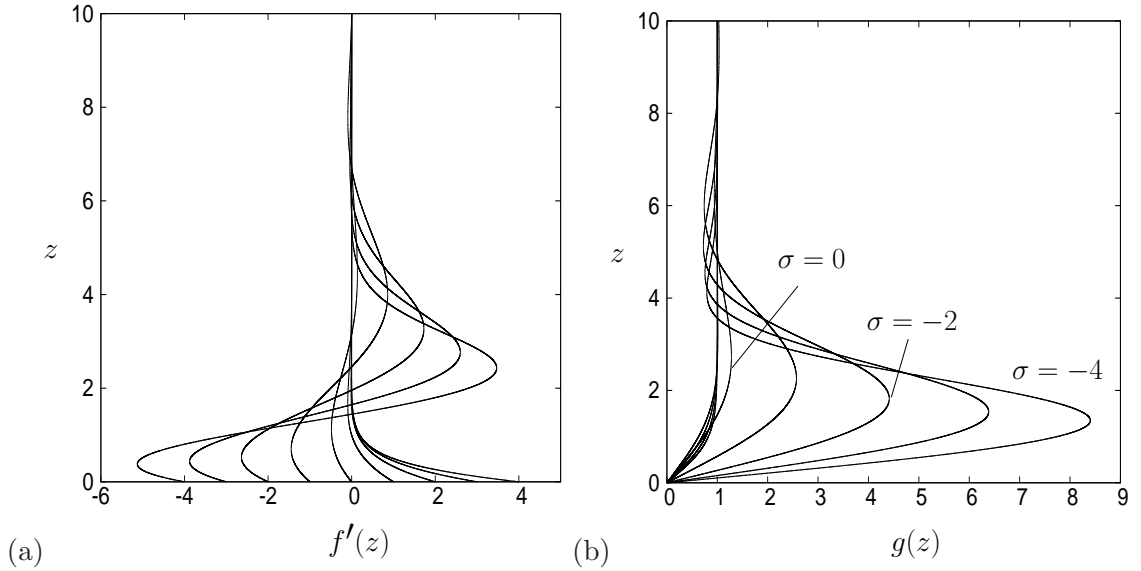


Figure 3: Plot of the velocity profiles (a) $f'(z)$ and (b) $g(z)$ for integer values of σ between -4 and 4 .

$$g'(0) = |\sigma|^{3/2} \left[\tilde{G}_{0Z}(0) + |\sigma|^{-1} \tilde{G}_{2Z}(0) + |\sigma|^{-2} \tilde{G}_{4Z}(0) \right] + O\left(|\sigma|^{-3/2}\right). \quad (3.2b)$$

for $\sigma < 0$, and are plotted as the the dashed lines in figure 2. The values of the coefficients are given along with the asymptotic analysis in Appendix A.

The corresponding radial and azimuthal velocity profiles $f'(z)$ and $g(z)$ are plotted in figure 3 for integer values of σ between -4 and 4 . For $\sigma > 0$, $g(z)$ has a simple boundary layer structure, increasing from zero at the disc to its far field value $g = 1$. There are in fact small oscillations about

$g = 1$ as $z \rightarrow \infty$, but these are not visible in the figure; see 3.1. The radial velocity $f'(z)$ behaves similarly, decreasing in magnitude from its value at the disc, σ , to $f' = 0$ as $z \rightarrow \infty$. These profiles are similar to those presented in Sahoo *et al.* (2014) who studied the Bödewadt problem with an imposed Navier slip condition at the wall, where the velocity at the wall is given as a multiple of the shear value at the wall, and thus is found as part of the solution procedure.

For $\sigma \leq 0$, on the other hand, $g(z)$ exhibits a strong wall jet structure in the boundary layer, the magnitude of which increases as σ decreases, while $f'(z)$ now includes a region of reverse flow at the disc. In both cases the oscillations as $z \rightarrow \infty$ are now obvious. In order to accurately capture these oscillations at infinity, it is necessary to compute results to a high degree of precision. For $-5 \lesssim \sigma \lesssim 1$ we find quadruple precision is required to accurately integrate out to $z = 20$. Outside this region double precision is sufficient to produce the required accuracy. Due to the reverse flow at the surface of the disc, we expect the results for $\sigma \leq 0$ to be more unstable (both convectively and absolutely) than those for $\sigma > 0$. However, the actual stability properties depend on the form of the ‘effective’ two-dimensional profile $Q(z)$ in (2.2), thus determining the stability properties is difficult without first knowing the form of $Q(z)$. Before performing the stability analysis, we first highlight the large z behaviour of the base flow, which highlights the difficulty in solving the base flow, but also allows for an alternative means for calculating the base flow, integrating from $z = \infty$ to the disc.

3.1 Oscillatory behaviour at large z

It is possible to determine the asymptotic behaviour of the velocity profile at large z by writing

$$g(z) = 1 + \delta \tilde{g}(z), \quad f(z) = f_\infty + \delta \tilde{f}(z), \quad (3.3)$$

where δ is a small parameter and $f_\infty = f(\infty)$. Substituting these expressions into (2.4) and retaining only terms at $O(\delta)$ leads to the pair of equations

$$\tilde{f}''' + 2f_\infty \tilde{f}'' + 2\tilde{g} = 0, \quad (3.4a)$$

$$\tilde{g}'' + 2f_\infty \tilde{g}' - 2\tilde{f}' = 0. \quad (3.4b)$$

Substituting (3.4b) into (3.4a) and simplifying leads to the 4th order ordinary differential equation for $\tilde{g}(z)$

$$\tilde{g}^{(iv)} + 4f_\infty \tilde{g}''' + 4f_\infty^2 \tilde{g}'' + 4\tilde{g} = 0. \quad (3.5)$$

Solving this equation, and retaining only those terms which tend to zero as $z \rightarrow \infty$, gives

$$\tilde{g}(z) = e^{\lambda_r z} (A \cos \lambda_i z + B \sin \lambda_i z), \quad (3.6a)$$

where A and B are constants which are, in principle, determined by the boundary conditions at the disc and

$$\lambda_r = -f_\infty - \frac{1}{2} \sqrt{2f_\infty^2 + 2\sqrt{f_\infty^4 + 4}}, \quad \lambda_i = \frac{2}{\sqrt{2f_\infty^2 + 2\sqrt{f_\infty^4 + 4}}}. \quad (3.6b)$$

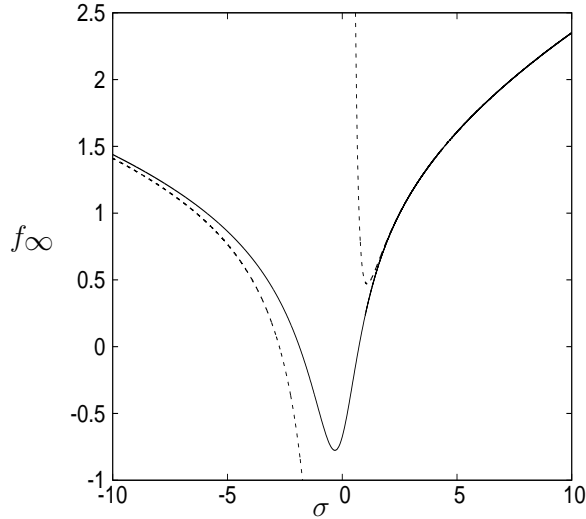


Figure 4: Plot of the constant $f_\infty = f(\infty)$ from the numerical solution as a function of σ . The dashed lines represent the large $|\sigma|$ asymptotic results from Appendix A.

It is clear from the form of $\lambda_r < 0$ and $\lambda_i > 0$ that the constant f_∞ is important in determining the large z behaviour, and is plotted in figure 4.

The exponential form of the solution in (3.6a) and the quadratic nonlinearities which appear in (2.4) suggest that the large z form can be approximated by expanding in powers of $\exp(\lambda_r z)$, essentially $\delta = 1$ in (3.3), and $\exp(\lambda_r z)$ becomes the small component of the expansion. The first few terms in the expansion for $g(z)$ are

$$g(z) \sim 1 + e^{\lambda_r z} (A \cos \lambda_i z + B \sin \lambda_i z) + e^{2\lambda_r z} (\alpha + \beta \cos \lambda_i z + \gamma \sin \lambda_i z), \quad (3.7)$$

where the forms of α , β and γ are given in Appendix B. The corresponding form of $f(z)$ can be found using (3.2b), but as it is not required for the subsequent analysis, we do not express its full form here.

The undetermined constants A and B in (3.7) are found from the numerical solution by matching the position of the stationary points, z_m of \tilde{g} , and the value of the function, $\tilde{g}_m = \tilde{g}(z_m)$, at these points. This furnishes a pair of equations at each stationary point which can be solved to give

$$A = \frac{\tilde{g}_m e^{-\lambda_r z_m}}{\lambda_i} (\lambda_i \cos \lambda_i z_m + \lambda_r \sin \lambda_i z_m), \quad B = \frac{\tilde{g}_m e^{-\lambda_r z_m}}{\lambda_i} (\lambda_i \sin \lambda_i z_m - \lambda_r \cos \lambda_i z_m). \quad (3.8)$$

Performing this calculation at each stationary point we find that the values of A and B converge as the value of z_m increases, at least to 4 or 5 significant figures, which we find to be sufficient for graphical accuracy. The large z form of $g(z)$ (3.7) is plotted together with the numerical solution for $\sigma = \{5, 0, -5\}$ in figure 5. The values of A and B for these results are given in table 1. We have found in our study that correctly determining these far field oscillations is important when undertaking the stability analysis, in the subsequent section.

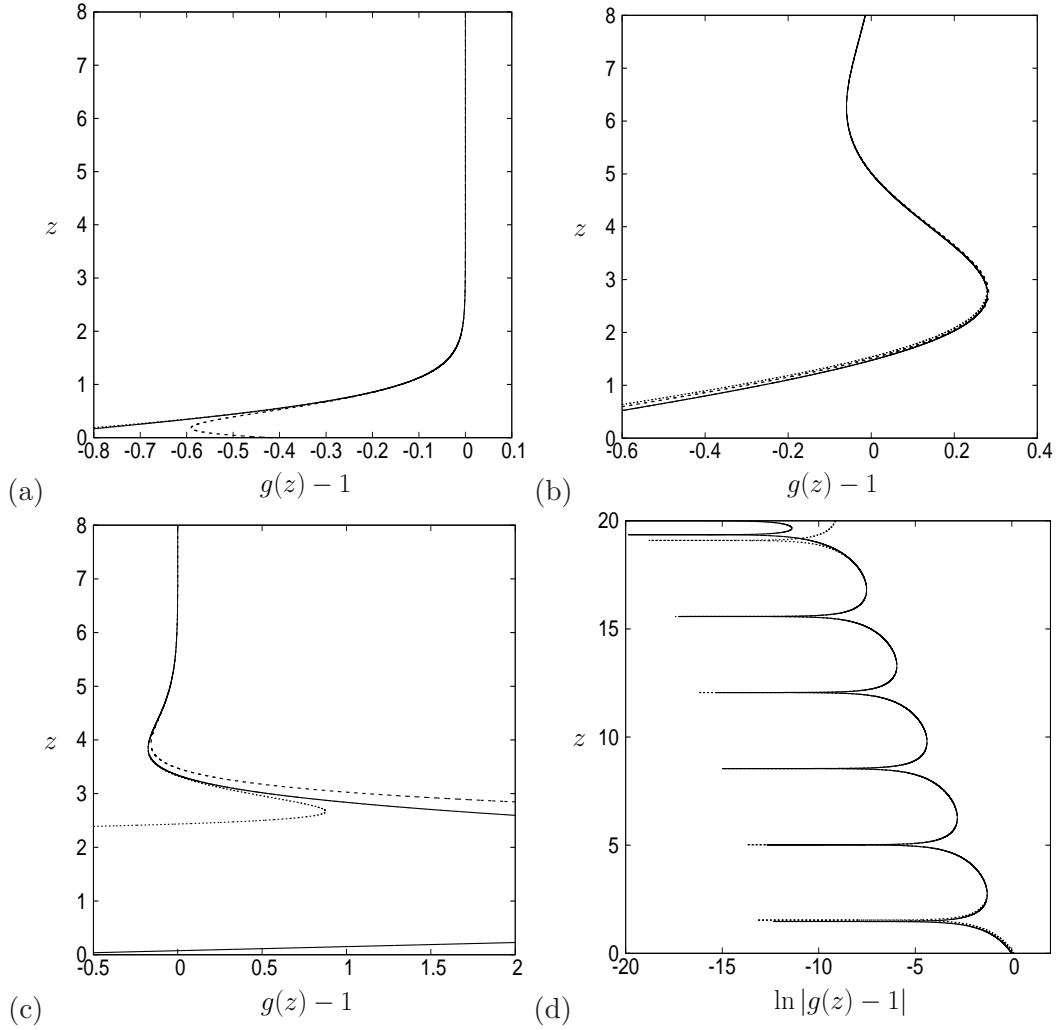


Figure 5: Plot of the azimuthal velocity $g(z) - 1$ for (a) $\sigma = 5$, (b) $\sigma = 0$ and (c) $\sigma = -5$. In each panel the numerical solution is given by the solid line and the two-term and three-term asymptotic results from (3.7) which are given by the dashed line and dotted line respectively. Panel (d) plots $\ln|g(z) - 1|$ to better show the agreement between the numeric and asymptotic solution for $\sigma = 0$. Here only the three-term asymptotic result is plotted.

4 Stability properties of base flow profiles

4.1 Convective instability

To investigate the effect of the stretching disc on the CI properties of the flow, we consider a temporal stability analysis of (2.9) for the real values of the radial wavenumber $\alpha = 0.2, 0.4$ and 0.8 . The values of $\alpha = 0.2$ and 0.4 are in the parameter region where the Bödewadt flow is highly convectively unstable, while $\alpha = 0.8$ lies at the edge of the instability region. The stability properties are found by solving the dispersion relation $\Delta(\alpha, \omega; \beta, \sigma) = 0$ where $\Delta(\alpha, \omega; \beta, \sigma) = w(\infty; \alpha, \omega; \beta, \sigma)$

σ	A	B
-5	346.0600	1377.75896
0	-1.04752	0.23777
5	-0.41458	-6.28651

Table 1: Table of values of A and B from (3.6) used in figure 5.

is found by solving (2.9) for a given base velocity profile $Q(z)$ ensuring the boundary condition (2.10) at $z = \infty$ is satisfied. This quantity equals zero when the correct value of ω is chosen for a given α . The dispersion relation is solved using the same shooting process as used for the base flow. In this section we investigate how the range of azimuthal wavenumbers, β , over which the flow is unstable ($\text{Im}(\omega) = \omega_i > 0$) varies with σ . In a CI analysis we are interested in the growth of individual waves not their interaction with each other.

In figures 6(a) and 6(b) we plot the growth rate ω_i for the CI with $\alpha = 0.2$ for $\sigma \geq 0$ and $\sigma \leq 0$ respectively. The results show for this wavenumber that, as expected, due to the increased amount of reverse flow at the surface of the disc for $\sigma \leq 0$ (see figure 3), the growth rate is larger than the $\sigma = 0$ result. In fact, as figure 7(a) shows, for $\sigma \leq 0$ the amount of shear in the base flow $Q(z)$ (here we plot $Q(z)/\alpha = f'(z) + \frac{\beta}{\alpha}g(z)$) is greater than the Bödewadt flow for all values of β/α plotted, while the converse is true for the stretched disc $\sigma > 0$.

As for the range of β values exhibiting a CI, we first observe that the stability results consist of multiple unstable modes, two of which exchange their dominance in the flow at $\beta = 0.385$ for $\sigma = 0$ which is seen by the discontinuity in gradient of ω_i in figure 6(a). In figure 7(b) we plot $|w|$ for each of these modes at $\alpha = \beta = 0.2$ for $\sigma = 0$. Here the dominant mode has a mode shape with maximum value close to the disc at $z = 2.8835$. When we compare this with the solid line for the $\alpha/\beta = 1$ result in figure 7(a) we observe this maximum occurring at the edge of the lower shear layer in $Q(z)$. The sub-dominant mode in figure 7(b) has its maximum at $z = 6.3985$ which corresponds to the edge of the second shear layer of $Q(z)$. Therefore we note that both shear layers are contributing to the stability of the flow, with the lower shear layer contribution dominating over the majority of the instability region and the upper shear layer dominating for larger β values.

So for $\sigma < 0$ the range of β values for the existence of the CI increases from the Bödewadt result, albeit after an initial decrease in size for $0 > \sigma \geq -0.5$, and the value of β at which the maximum growth rate occurs also decreases. For $\sigma > 0$ in figure 6(a) the maximum value of the growth rate moves to larger values of β but interestingly, the maximum value of ω_i decreases from $\sigma = 0$ to $\sigma = 1$, and then increases at $\sigma = 2$. The overall range of β values for which there is an instability increases as σ is increased, but note that ω_i for $\beta < 0$ reduces to small values by $\sigma = 2$; thus observing these modes at this parameter value in a set of spin-down experiments similar to those of Savaş (1983, 1987) would be difficult.

For $\alpha = 0.5$ in figures 6(c) and 6(d) a similar behaviour is observed as for the $\alpha = 0.2$ case. Here the range of β values over which an instability exists increases for both $\sigma > 0$ and $\sigma < 0$ compared

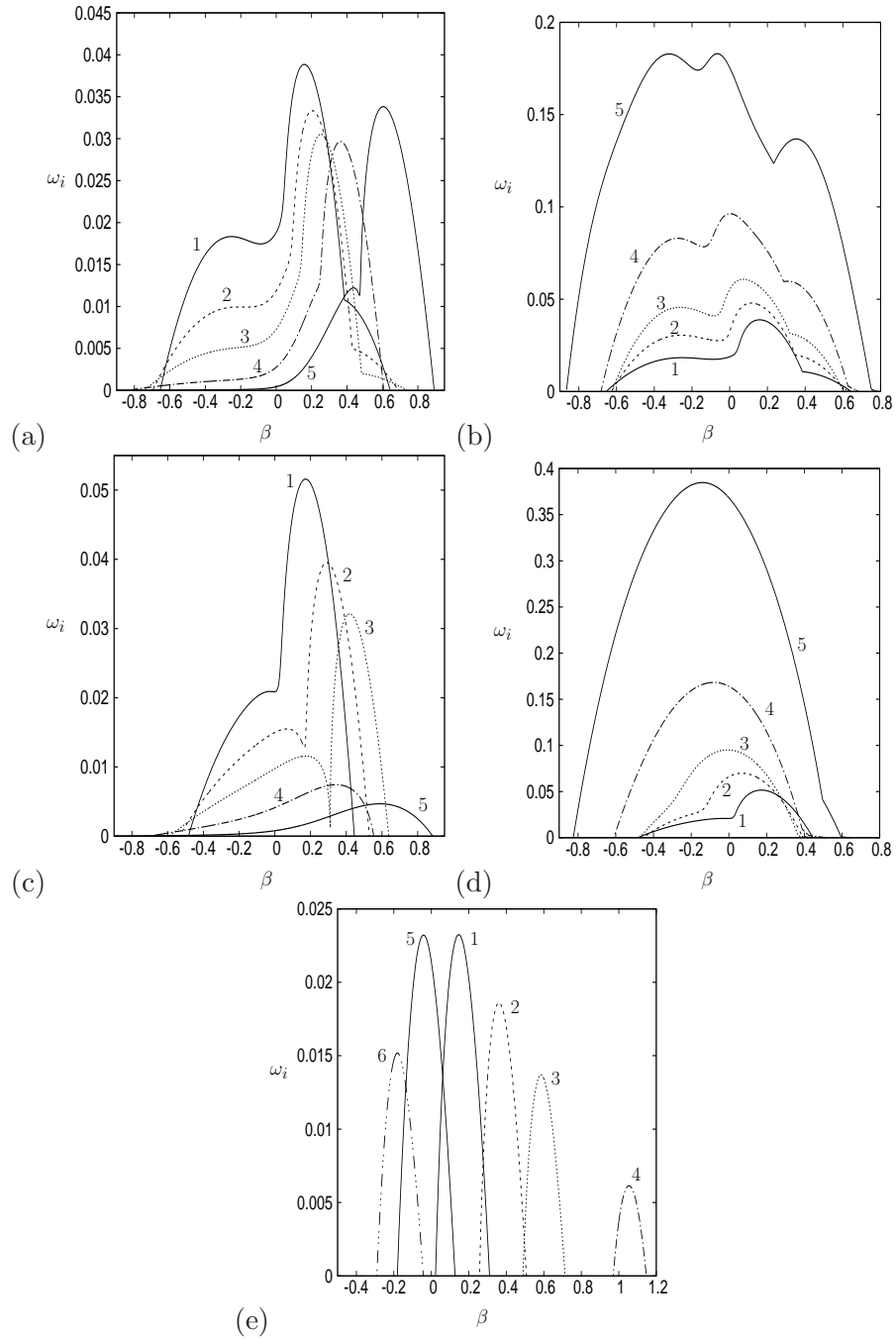


Figure 6: Plots of the temporal stability growth rates $\omega_i(\beta)$ for (a,b) $\alpha = 0.2$, (c,d) $\alpha = 0.5$ and (e) $\alpha = 0.8$. In panels (a,c) the lines $\sigma = \{0, 0.25, 0.5, 1, 2\}$ are labelled 1 to 5 respectively, in panels (b,d) the lines $\sigma = \{0, -0.25, -0.5, -1, -2\}$ are labelled 1 to 5 respectively, while in (e) the lines $\sigma = \{0, 0.25, 0.5, 1, -0.25, -0.5\}$ are labelled 1 to 6 respectively.

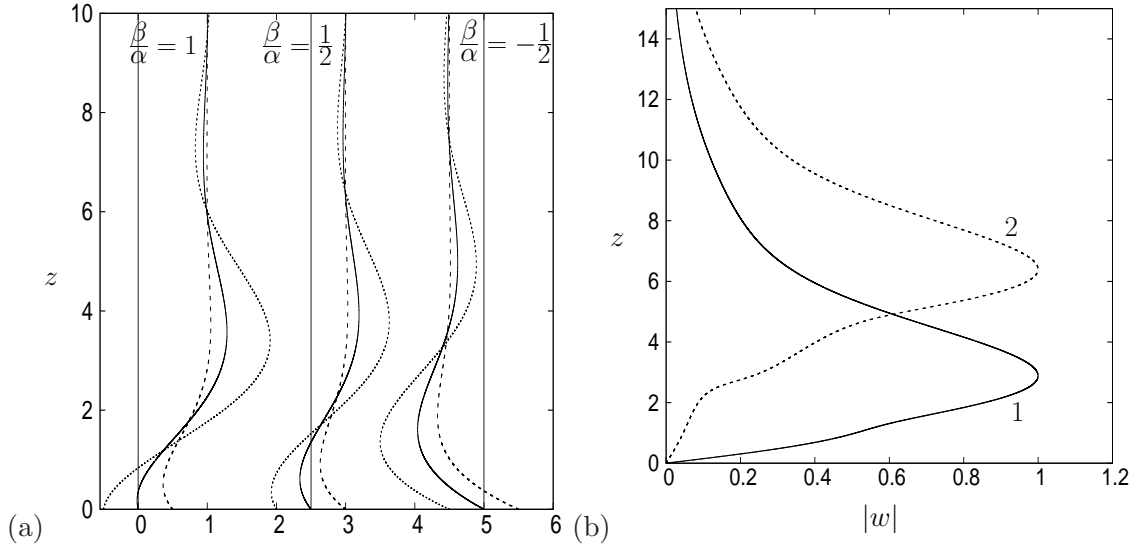


Figure 7: (a) Plot of $f'(z) + \frac{\beta}{\alpha}g(z)$, for $\frac{\beta}{\alpha} = \{1, 1/2, -1/2\}$. The three cases are separated by a constant, and in each case the solid line represents $\sigma = 0$, the long dashed line represents $\sigma = 0.5$ and the short dashed line represents $\sigma = -0.5$. (b) Plot of $|w|$, for $\alpha = \beta = 0.2$ and $\sigma = 0$. Result 1 represents the dominant eigenmode with $\omega = 0.0725 + 0.0278i$ while result 2 gives the sub-dominant eigenmode with $\omega = 0.2312 + 0.0094i$.

to the Bödewadt result. The significant difference at this wavenumber is the rate at which the maximum value of ω_i increases for $\sigma < 0$ and decreases for $\sigma > 0$ is much larger than for $\alpha = 0.2$. Also note that here the maximum value of ω_i continues to decrease at $\sigma = 2$, unlike in the $\alpha = 0.2$ case, thus showing that the form of $Q(z)$ for $\sigma = 2$ is more unstable to long waves (small α) but more stable to shorter waves (larger α).

For $\alpha = 0.8$ in figure 6(e) it is a different story. Here the size of the azimuthal wavenumber region is approximately constant for the different values of σ considered, but the position of the instability region varies. Note that here the corresponding results for $\sigma = 2, -1$ and -2 are stable.

The results in this section indicate that the region of azimuthal wavenumbers for CI typically increases for a small to moderate fixed value of α when the disc is both stretched ($\sigma > 0$) and shrunk ($\sigma < 0$). The corresponding behaviour for AI is considered in 4.2 with particular interest on how individual modes interact with one another.

4.2 Absolute instability

While the CI of a flow depends on the stability of individual waves, its AI is a consequence of the interaction of these waves as they propagate as wavepackets. The calculation of the response of the base flow to infinitesimal perturbations in a frame of reference fixed on the disturbance position for both the Bödewadt and Ekman layer flows can be found in Lingwood (1997), and for the von Kármán flow in Lingwood (1995). We apply the same fundamental approach to the base flows in this paper.

Absolute instabilities of the flow correspond to *special* saddle points of the dispersion relation $\Delta(\alpha, \omega; \beta, \sigma) = 0$ which satisfy the Briggs criteria for AI (Briggs, 1964; Brevdo & Bridges, 1996; Lingwood, 1997; Healey, 2007; Juniper, 2007; Turner et al., 2011), and the position of the saddle points are found by simultaneously solving $\Delta(\alpha, \omega; \beta, \sigma) = \Delta_\alpha(\alpha, \omega; \beta, \omega) = 0$. Note here that both α and ω are typically complex.

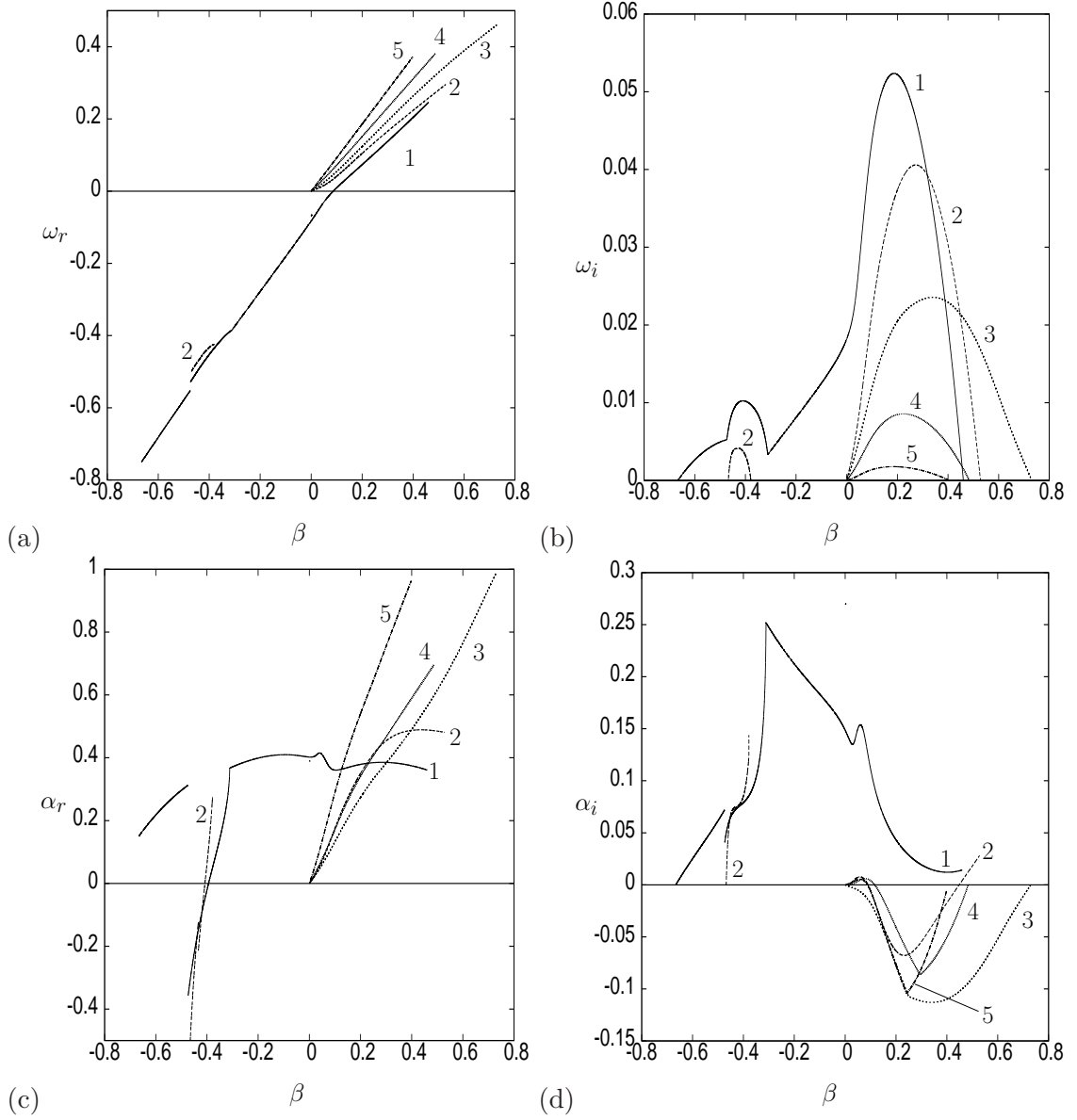


Figure 8: Plots of (a) $\omega_r(\beta)$, (b) $\omega_i(\beta)$, (c) $\alpha_r(\beta)$ and (d) $\alpha_i(\beta)$ for $\sigma = \{0, 0.25, 0.5, 1, 2\}$ labelled 1 to 5 respectively.

In figure 8 we plot (a) $\omega_r(\beta)$, (b) $\omega_i(\beta)$, (c) $\alpha_r(\beta)$ and (d) $\alpha_i(\beta)$ when $\sigma > 0$ for the dominant

saddle point for values of β where the flow is absolutely unstable. Here the stretching parameter takes the values $\sigma = \{0, 0.25, 0.5, 1, 2\}$. The result for $\sigma = 0$ is the Bödewadt result presented in figure 9(b) and 9(d) of Lingwood (1997) (denoted by the Rossby number $Ro = 1$). We note that while our results are in good agreement with those of Lingwood (1997) for $\beta > 0$, we obtain subtly different results for $\beta < 0$. However our results are qualitatively similar to those of Lingwood (1997) when her Rossby number is $Ro = 0.9$ and 0.8 in the range $-0.3 \leq \beta \leq 0$. We believe this discrepancy is due to the difficulty in accurately calculating the base flow for this value of σ , as described in §3. The fact that our results demonstrate qualitative agreement in behaviour with the $Ro = 0.9$ and 0.8 results of Lingwood (1997) gives us confidence in our numerical procedure.

The Bödewadt result has a maximum growth rate at $\beta = 0.1871$ and the AI extends over the range of azimuthal wavenumbers $\beta \in [-0.6659, 0.4605]$. At $\beta = -0.4096$ there exists a second, smaller, maximum growth rate which is caused by a second saddle point on the inversion contour becoming the dominant saddle point. As the disc stretching rate is increased to $\sigma = 0.25$ the two maximum growth rate values decrease in magnitude, and in fact the AI actually vanishes for a range of β values between these two maxima. The two maxima move to $\beta = 0.2725$ and $\beta = -0.4364$ respectively, and the flow is absolutely unstable for azimuthal wavenumbers $\beta \in [-0.4673, -0.3785] \cup [0.0001, 0.5280]$. As σ increases to 0.5 , the flow is no longer absolutely unstable for $\beta < 0$, but the positive range of β for which an AI exists is larger still. Beyond this value, the range of β values for the AI decreases and for $\sigma = 2$ the flow is only absolutely unstable for $\beta \in [0, 0.3991]$, but this instability is weak, with a maximum growth rate of $\omega_i = 1.773 \times 10^{-3}$. Note that this result is in contrast to the CI result in figure 6 which shows the range of β values for CI increases as σ is increased from $\sigma = 0$, and at $\sigma = 2$, the magnitude of ω_i for the CI is much larger.

In figure 9(a) we plot $\text{Re}(Q) = \alpha_r f' + \beta g$, which is the velocity profile which gives the AI, for $(\sigma, \beta) = \{(0, 0.2), (0, -0.2), (0, -0.4), (0.5, 0.2), (2, 0.2)\}$. These profiles show a boundary layer shear flow structure for $\sigma = 0$, with the maximum value of $\text{Re}(Q)$ close to the surface of the disc increasing and moving upward as β is reduced. As σ is increased for fixed $\beta = 0.2$ we observe that the shear layer structure of the base flow profile for an AI is reduced, and this coincides with the flow becoming less absolutely unstable. If we examine $|Q - \omega|$ for the same profiles in figure 9(b), we now observe why for $(\sigma, \beta) = (0, -0.4)$ there is an increase in the growth rate ω_i as β is reduced for $\sigma = 0$. This result (label 3 in figure 12) has a critical point at $z_c = 4.1026$, and this critical point makes an additional contribution to the AI of the flow (Healey, 2006). Healey (2006) demonstrates for the von Kármán rotating disc boundary layer, in the long-wave limit, that the contribution of a critical point (a second critical point in that case) dominates the AI properties for small β values. A similar phenomena is occurring here, except it is the appearance of a critical point in $Q(z)$ as β is reduced, for $\sigma = 0$, which is responsible for the additional increase in AI growth rate. When the critical point exists in $Q(z)$, it is likely that nonlinearity will first occur close to this critical point in the inviscid theory. The result plotted as label 5 in figure 9(b) also appears to exhibit a critical point at $z \approx 2$; however this is merely a point with a small value of $|Q - \omega|$, and not a critical point with $Q = \omega$. This is why the flow stability properties for these parameters are similar to those for other base flows with $\beta > 0$. Note, for profiles with a critical point, we modify our numerical

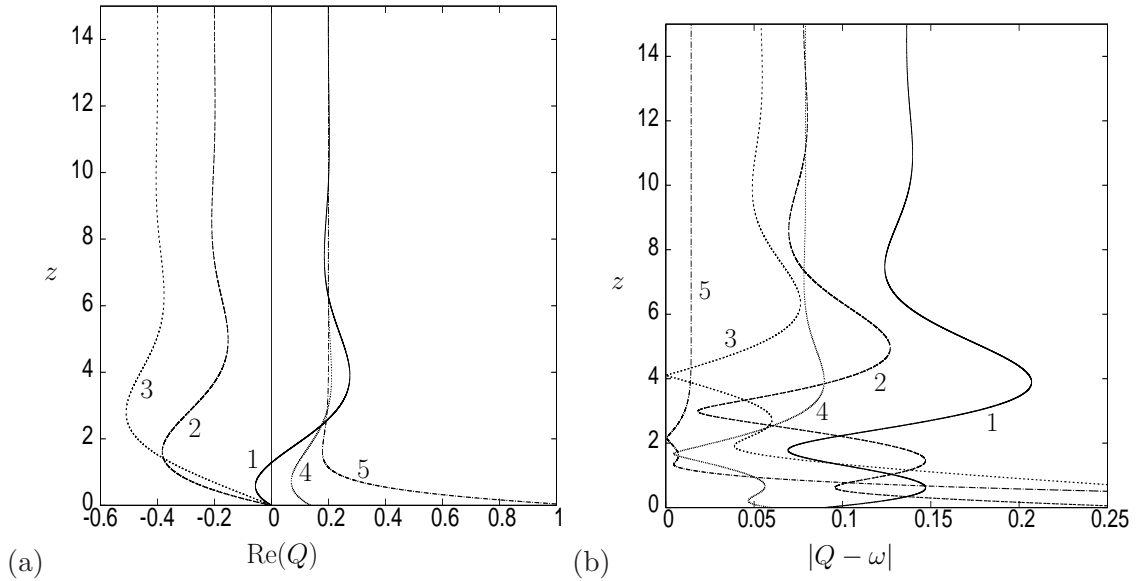


Figure 9: Plot of (a) $\text{Re}(Q(z))$, (b) $|Q(z) - \omega|$ labelled 1 to 5 respectively for $(\sigma, \beta) = \{(0, 0.2), (0, -0.2), (0, -0.4), (0.5, 0.2), (2, 0.2)\}$.

scheme to integrate up to the critical layer and then jump to the other side without modifying w or dw/dz . We can do this because we consider the critical layer to be nonlinear and not viscous (Benney & Bergeron, 1969).

The growth rate curves in figure 10(b) demonstrate that as σ is decreased from zero the growth rate ω_i typically increases for all values of β as was the case for the CI, except again at the edges of the stability domain, for $\sigma \in [-0.5, 0]$. For $\sigma = -0.5$ this instability range is $\beta \in [-0.6219, 0.4328]$. The maximum value of the growth rate increases as σ decreases and when $\sigma = -0.5$, $\omega_i = 0.09238$ which is almost twice the maximum value when $\sigma = 0$.

The base velocity profiles $\text{Re}(Q)$ in figure 11(a) show that the AI in this parameter regime is dominated by the size of the shear layer within the boundary layer. For $(\sigma, \beta) = (-0.5, -0.4)$ (label 3) in figure 11(b) we again observe a critical point at $z_c = 3.6078$ which dominates the AI properties of the flow, over the shear layer magnitude. However, as σ is reduced further the critical point moves towards the disc until its contribution to the AI of the flow becomes sub-dominant to the contribution from the shear layer. This is seen to have occurred by $\sigma = -2$ in figure 11(b) for $\beta = -0.4$.

5 Discussion and conclusions

In this paper we examined the flow generated by the linear radial stretching of a disc below a rotating fluid with constant angular velocity Ω and kinematic viscosity ν . We studied both the velocity profiles of the steady base flow, as well as the inviscid CI and AI properties of this flow to infinitesimal perturbations via linear stability theory. The steady base flow was nondimensionalised

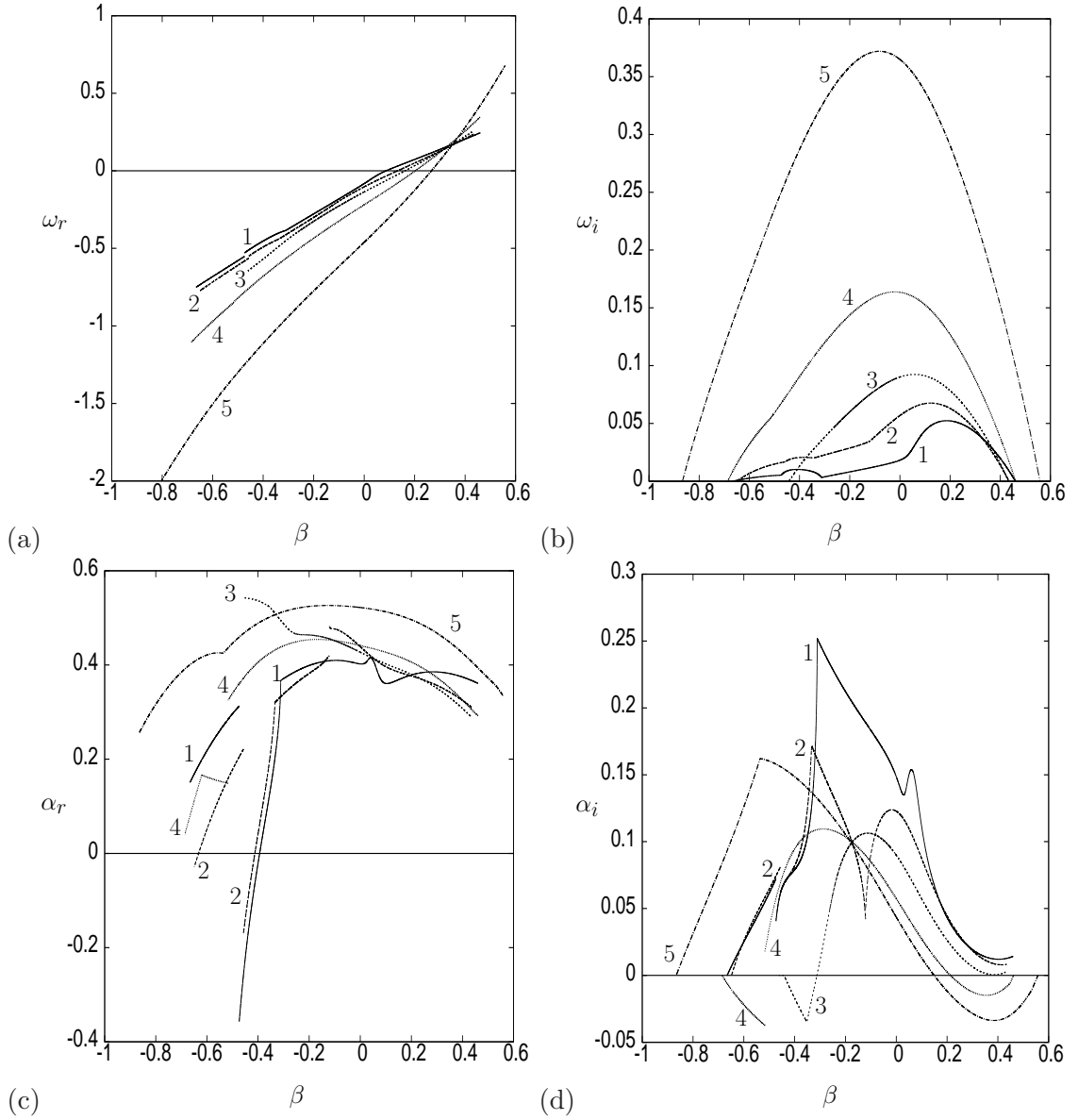


Figure 10: Plots of (a) $\omega_r(\beta)$, (b) $\omega_i(\beta)$, (c) $\alpha_r(\beta)$ and (d) $\alpha_i(\beta)$ labelled 1 to 5 respectively for $\sigma = \{0, -0.25, -0.5, -1, -2\}$.

and solely characterised by a single nondimensional parameter $\sigma = a/\Omega$, where a is the radial rate of strain of the disc. Numerical results of the base flow equations showed that the radial component of the shear stress on the disc is always directed towards the origin, while the azimuthal component of the disc shear stress is directed in the positive θ -direction.

For $\sigma \gtrsim 1$ the radial velocity component consisted of a velocity profile which obtained its maximum value at the surface of the disc and decreased to zero outside the boundary layer, while the azimuthal component was zero at the disc and increased to a unit value outside the boundary

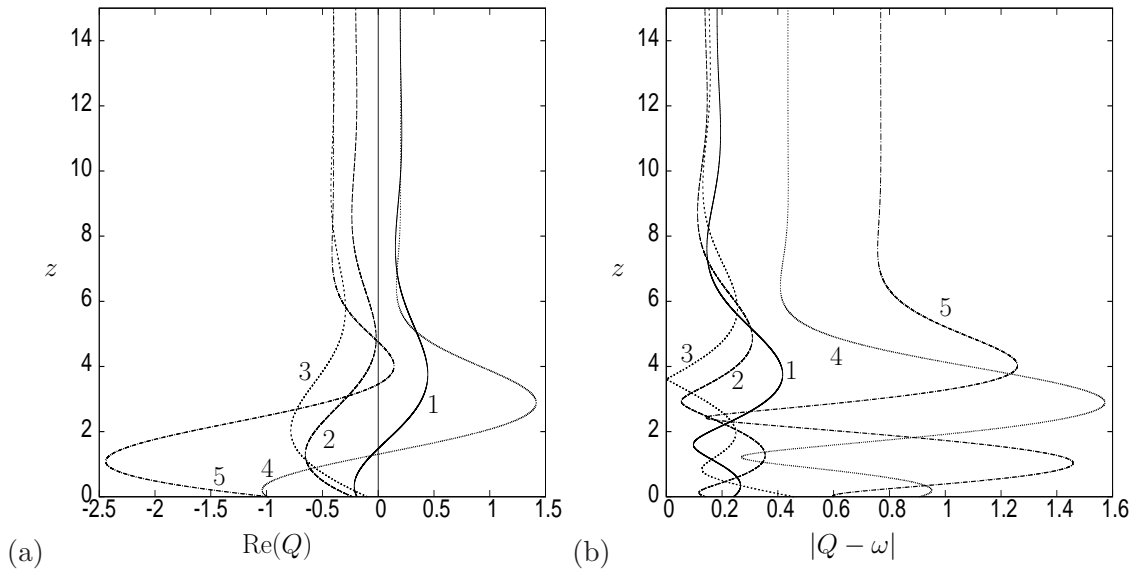


Figure 11: Plots of (a) $\text{Re}(Q(z))$, (b) $|Q(z) - \omega|$ labelled 1 to 5 respectively for $(\sigma, \beta) = \{(-0.5, 0.2), (-0.5, -0.2), (-0.5, -0.4), (-2, 0.2), (-2, -0.4)\}$.

layer. For $\sigma \lesssim -1$ on the other hand, the radial velocity profile consisted of a shear layer profile with a region of reverse flow at the disc, while the radial component had a wall jet structure with its maximum velocity confined to the boundary layer. For $-1 \lesssim \sigma \lesssim 1$ the flow transitioned between these two states. In particular, the case $\sigma = 0$ is the problem studied by Bödewadt (1940). For all values of σ the flow was shown to have oscillations outside the boundary layer which decay as $z \rightarrow \infty$. These oscillations are small for $\sigma > 0$ and more visible for $\sigma \leq 0$, and it was found that accurately calculating these oscillations was important in the subsequent stability analysis.

By considering infinitesimal perturbations of the base flows we examined the inviscid stability properties of the flow, in particular AI properties, to investigate the potential of the stretching disc to enhance or inhibit transition. We considered the stability of the flow at large radii, such that the flow is assumed to act parallel to the disc, so that non-parallel contributions to the stability calculation are neglected. For $\sigma = 2$ the flow was found to be absolutely stable except for a small region of azimuthal wavenumbers $\beta \in [0, 0.3991]$, where the magnitude of the AI growth rate was determined by the magnitude of the shear layer in the ‘effective’ two-dimensional velocity profile $Q(z) = \alpha f'(z) + \beta g(z)$. As σ was reduced the growth rate increased, and eventually a second instability region was found to exist for $\beta < 0$ (see figure 8(b)) where the magnitude of the growth rate was determined by the contribution of a critical point which occurs in $Q(z)$. Note a critical point does not occur for $\beta > 0$. As σ is decreased further the AI due to the shear layer grows in both magnitude and in the range of β values for which it exists, and ultimately dominates the whole instability range when $\sigma = -2$. For this value of σ the AI velocity profiles with a critical point still exist, and produce an AI, but they are sub-dominant to the shear layer contribution. The CI results are in accord with the AI results for $\sigma \leq 0$, but for $\sigma \geq 0$ the existence region for

the CI increases with increasing σ , at least up to $\sigma = 2$ considered here.

Acknowledgement

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A Asymptotic analysis for $|\sigma| \gg 1$

In this appendix we consider the asymptotic form of the base flow for large disc stretching and shrinking rates $|\sigma|$. We consider the two asymptotic regimes of a rapid stretching disc $\sigma \gg 1$ and a rapidly shrinking disc $-\sigma \gg 1$.

A.1 The rapidly stretching disc: $\sigma \gg 1$

The velocity profiles in figure 3 for $\sigma > 0$ suggests for large positive σ there exists a boundary layer close to the disc. The thickness of this boundary layer is $O(\sigma^{-1/2})$, outside this layer the outer solution is trivially

$$g(z) = 1, \quad f(z) = f_\infty. \quad (\text{A.1})$$

The inner solution is found by introducing a new stretched boundary layer variable Z , such that $z = |\sigma|^{-1/2}Z$, and the scaled inner functions $F(Z)$ and $G(Z)$

$$f = \sigma^{1/2}F(Z), \quad g = G(Z). \quad (\text{A.2})$$

Introducing these scaled variables into (2.4) leads to

$$F_{ZZZ} + 2FF_{ZZ} - F_Z^2 + \sigma^{-2}(G^2 - 1) = 0, \quad F(0) = 0, \quad F_Z(0) = 1, \quad F_Z(\infty) = 0, \quad (\text{A.3a})$$

$$G_{ZZ} + 2(FG_Z - F_ZG) = 0, \quad G(0) = 0, \quad G(\infty) = 1. \quad (\text{A.3b})$$

We seek an asymptotic solution to these equations where we expand $F(Z)$ and $G(Z)$ in powers of σ^{-2}

$$\begin{aligned} F(Z) &= F_0(Z) + \sigma^{-2}F_2(Z) + \sigma^{-4}F_4(Z) + O(\sigma^{-6}), \\ G(Z) &= G_0(Z) + \sigma^{-2}G_2(Z) + \sigma^{-4}G_4(Z) + O(\sigma^{-6}). \end{aligned}$$

Inserting these expansions into (3.8) and equating powers of σ^{-2} leads to the following hierarchy of system equations: at $O(1)$

$$\begin{aligned} F_{0ZZZ} + 2F_0F_{0ZZ} - F_{0Z}^2 &= 0, \quad F_0(0) = 0, \quad F_{0Z}(0) = 1, \quad F_{0Z}(\infty) = 0, \\ G_{0ZZ} + 2(F_0G_{0Z} - F_{0Z}G_0) &= 0, \quad G_0(0) = 0, \quad G_0(\infty) = 1; \end{aligned}$$

then at $O(\sigma^{-2})$

$$\begin{aligned} F_{2ZZZ} + 2(F_0F_{2ZZ} + F_2F_{0ZZ}) - 2F_{0Z}F_{2Z} + G_0^2 - 1 &= 0, \\ F_2(0) = 0, \quad F_{2Z}(0) = 0, \quad F_{2Z}(\infty) &= 0, \\ G_{2ZZ} + 2(F_0G_{2Z} + F_2G_{0Z} - F_{0Z}G_2 - F_{2Z}G_0) &= 0, \\ G_2(0) = 0, \quad G_2(\infty) &= 0; \end{aligned}$$

and at $O(\sigma^{-4})$

$$\begin{aligned} F_{4ZZZ} + 2(F_0F_{4ZZ} + F_2F_{2ZZ} + F_4F_{0ZZ}) - 2F_{0Z}F_{4Z} - F_{2Z}^2 + 2G_0G_2 &= 0, \\ F_4(0) = 0, \quad F_{4Z}(0) = 0, \quad F_{4Z}(\infty) &= 0, \\ G_{4ZZ} + 2(F_0G_{4Z} + F_2G_{2Z} + F_4G_{0Z} - F_{0Z}G_4 - F_{2Z}G_2 - F_{4Z}G_0) &= 0, \\ G_4(0) = 0, \quad G_4(\infty) &= 0. \end{aligned}$$

These systems of equations are solved consecutively using the same numerical scheme as for the full system.

The resulting asymptotic shear stress values are found to be

$$\begin{aligned} F_{0ZZ}(0) &= -1.173720738913, & G_{0Z}(0) &= 0.528652929118, \\ F_{2ZZ}(0) &= -0.772897415441, & G_{2Z}(0) &= 0.203722589589, \\ F_{4ZZ}(0) &= 0.114592927248, & G_{4Z}(0) &= -0.071141123212. \end{aligned}$$

A.2 The rapidly shrinking disc: $-\sigma \gg 1$

For $\sigma < 0$ the boundary layer thickness is again $O(|\sigma|^{1/2})$, and the outer solution is also (A.1). Using the same boundary layer variable Z as in Appendix A.1 but the scaled inner functions $\tilde{F}(Z)$ and $\tilde{G}(Z)$

$$f = |\sigma|^{1/2}\tilde{F}(Z), \quad g = |\sigma|\tilde{G}(Z).$$

in (2.4) leads to the system

$$\tilde{F}_{ZZZ} + 2\tilde{F}\tilde{F}_{ZZ} - \tilde{F}_Z^2 + \tilde{G}^2 - |\sigma|^{-2} = 0, \quad \tilde{F}(0) = 0, \quad \tilde{F}_Z(0) = -1, \quad \tilde{F}_Z(\infty) = 0, \quad (\text{A.4a})$$

$$\tilde{G}_{ZZ} + 2(\tilde{F}\tilde{G}_Z - \tilde{F}_Z\tilde{G}) = 0, \quad \tilde{G}(0) = 0, \quad \tilde{G}(\infty) = |\sigma|^{-1}. \quad (\text{A.4b})$$

The $|\sigma|^{-1}$ term in the boundary condition for \tilde{G} suggests an asymptotic solution in the form

$$\begin{aligned} \tilde{F}(Z) &= \tilde{F}_0(Z) + |\sigma|^{-1}\tilde{F}_1(Z) + |\sigma|^{-2}\tilde{F}_2(Z) + O(|\sigma|^{-3}), \\ \tilde{G}(Z) &= \tilde{G}_0(Z) + |\sigma|^{-1}\tilde{G}_1(Z) + |\sigma|^{-2}\tilde{G}_2(Z) + O(|\sigma|^{-3}). \end{aligned}$$

Inserting these expansions into (3.10) and equating powers of $|\sigma|^{-1}$ leads to the following hierarchy of system equations: at $O(1)$

$$\begin{aligned}\tilde{F}_{0ZZZ} + 2\tilde{F}_0\tilde{F}_{0ZZ} - \tilde{F}_{0Z}^2 + \tilde{G}_0^2 &= 0 & \tilde{F}_0(0) &= 0, & \tilde{F}_{0Z}(0) &= -1, & \tilde{F}_{0Z}(\infty) &= 0, \\ \tilde{G}_{0ZZ} + 2(\tilde{F}_0\tilde{G}_{0Z} - \tilde{F}_{0Z}\tilde{G}_0) &= 0 & \tilde{G}_0(0) &= 0, & \tilde{G}_0(\infty) &= 0;\end{aligned}$$

then at $O(|\sigma|^{-1})$

$$\begin{aligned}\tilde{F}_{1ZZZ} + 2\left(\tilde{F}_0\tilde{F}_{1ZZ} + \tilde{F}_1\tilde{F}_{0ZZ}\right) - 2\tilde{F}_{0Z}\tilde{F}_{1Z} + 2\tilde{G}_0\tilde{G}_1 &= 0 \\ \tilde{F}_1(0) &= 0, & \tilde{F}_{1Z}(0) &= 0, & \tilde{F}_{1Z}(\infty) &= 0, \\ \tilde{G}_{1ZZ} + 2\left(\tilde{F}_0\tilde{G}_{1Z} + \tilde{F}_1\tilde{G}_{0Z} - \tilde{F}_{0Z}\tilde{G}_1 - \tilde{F}_{1Z}\tilde{G}_0\right) &= 0 \\ \tilde{G}_1(0) &= 0, & \tilde{G}_1(\infty) &= 1;\end{aligned}$$

and at $O(|\sigma|^{-2})$

$$\begin{aligned}\tilde{F}_{2ZZZ} + 2\left(\tilde{F}_0\tilde{F}_{2ZZ} + \tilde{F}_1\tilde{F}_{1ZZ} + \tilde{F}_2\tilde{F}_{0ZZ}\right) - 2\tilde{F}_{0Z}\tilde{F}_{2Z} - \tilde{F}_{1Z}^2 + 2\tilde{G}_0\tilde{G}_2 + \tilde{G}_1^2 - 1 &= 0, \\ \tilde{F}_2(0) &= 0, & \tilde{F}_{2Z}(0) &= 0, & \tilde{F}_{2Z}(\infty) &= 0, \\ \tilde{G}_{2ZZ} + 2\left(\tilde{F}_0\tilde{G}_{2Z} + \tilde{F}_1\tilde{G}_{1Z} + \tilde{F}_2\tilde{G}_{0Z} - \tilde{F}_{0Z}\tilde{G}_2 - \tilde{F}_{1Z}\tilde{G}_1 - \tilde{F}_{2Z}\tilde{G}_0\right) &= 0, \\ \tilde{G}_2(0) &= 0, & \tilde{G}_2(\infty) &= 0.\end{aligned}$$

Numerically solving these systems give the asymptotic shear stress values as

$$\begin{aligned}\tilde{F}_{0ZZ}(0) &= -0.725131902079, & \tilde{G}_{0Z}(0) &= 1.157973962096, \\ \tilde{F}_{1ZZ}(0) &= -0.000000000088, & \tilde{G}_{1Z}(0) &= 0.000000000018, \\ \tilde{F}_{2ZZ}(0) &= -0.633741006320, & \tilde{G}_{2Z}(0) &= 0.690472088227.\end{aligned}$$

B Coefficients of the large z asymptotic expansion

In the large- z asymptotic form of $g(z)$ in (3.7) the coefficients α , β and γ are given by

$$\begin{aligned}\alpha &= \frac{\zeta_1 + \zeta_2}{8(8f_\infty\lambda_r^3 + 4\lambda_r^4 + 4f_\infty^2\lambda_r^2 + 1)}, \\ \beta &= \frac{1}{8(\pi_1^2 + \pi_2^2)} (\pi_1(\zeta_1 - \zeta_2) + \pi_2\zeta_3), \\ \gamma &= \frac{1}{8(\pi_1^2 + \pi_2^2)} (-\pi_2(\zeta_1 - \zeta_2) + \pi_1\zeta_3),\end{aligned}$$

where

$$\begin{aligned}
\pi_1 &= 8f_\infty\lambda_r^3 - 24f_\infty\lambda_i^2\lambda_r + 4\lambda_i^4 - 24\lambda_i^2\lambda_r^2 + 4\lambda_r^4 - 4f_\infty^2\lambda_i^2 + 4f_\infty^2\lambda_r^2 + 1, \\
\pi_2 &= -8\lambda_i(f_\infty + 2\lambda_r)(f_\infty\lambda_r - \lambda_i^2 + \lambda_r^2), \\
\zeta_1 &= \frac{1}{2} (8f_\infty^2\lambda_i^2 - 4f_\infty^2\lambda_r^2 + 4f_\infty\lambda_i^2\lambda_r - 4f_\infty\lambda_r^3 + \lambda_i^4 - 4\lambda_i^2\lambda_r^2 - \lambda_r^4 - 4) A^2 \\
&\quad - \lambda_i^2 (4f_\infty\lambda_r - \lambda_i^2 + 5\lambda_r^2 - 2f_\infty^2) B^2 \\
&\quad - 2\lambda_i (f_\infty + \lambda_r) (2f_\infty\lambda_r - \lambda_i^2 + \lambda_r^2) AB, \\
\zeta_2 &= -\lambda_i^2 (4f_\infty\lambda_r - \lambda_i^2 + 5\lambda_r^2 - 2f_\infty^2) A^2 \\
&\quad + \frac{1}{2} (8f_\infty^2\lambda_i^2 - 4f_\infty^2\lambda_r^2 + 4f_\infty\lambda_i^2\lambda_r - 4f_\infty\lambda_r^3 + \lambda_i^4 - 4\lambda_i^2\lambda_r^2 - \lambda_r^4 - 4) B^2 \\
&\quad + 2\lambda_i (f_\infty + \lambda_r) (2f_\infty\lambda_r - \lambda_i^2 + \lambda_r^2) AB, \\
\zeta_3 &= 2\lambda_i (f_\infty + \lambda_r) (2f_\infty\lambda_r - \lambda_i^2 + \lambda_r^2) (A^2 - B^2) \\
&\quad + (4f_\infty^2\lambda_i^2 - 4f_\infty^2\lambda_r^2 + 12f_\infty\lambda_i^2\lambda_r - 4f_\infty\lambda_r^3 - \lambda_i^4 + 6\lambda_i^2\lambda_r^2 - \lambda_r^4 - 4) AB.
\end{aligned}$$

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