Details of the proof of equivalence: the “cosine” versus “vertical eigenfunction” representations

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1 Introduction

The boundary-value problem for the linear horizontally-forced sloshing problem can be solved using two different classes of eigenfunction expansions. The first will be referred to as the "cosine" expansion since the organizing centre is a cosine series in the $x-$direction (the horizontal direction), and the second is called the “vertical eigenfunction expansion” since the organizing centre is a class of $y-$direction (the vertical direction) eigenfunctions. These two methods lead to very different forms of solution, and their equivalence is not obvious. This technical report gives the details of a proof of equivalence. The strategy of the proof was suggested to the authors by McIver [7].

2 Governing equations

The basic boundary-value problem, for the function $\widehat{\phi}(x,y)$, under consideration is

$$\widehat{\phi}_{xx} + \widehat{\phi}_{yy} = 0\,, \quad 0 < y < h_0\,, \quad 0 < x < L\,, \tag{2.1}$$

with boundary conditions

$$\begin{aligned} \widehat{\phi}_y &= 0 \quad \text{at} \quad y = 0\,, \\ \widehat{\phi}_y &= K\widehat{\phi} \quad \text{at} \quad y = h_0\,, \quad K = \tfrac{\omega^2}{g}\,, \\ \widehat{\phi}_x &= 1 \quad \text{at} \quad x = 0, L\,, \end{aligned} \tag{2.2}$$

where K is considered here to be a given positive constant, and the boundary conditions at $x = 0, L$ have been simplified for this report. The aim is to find the particular solution due to the inhomogeneous boundary condition at $x = 1$. This boundary-value problem arises in the problem of horizontally-forced sloshing (cf. Graham & Rodriquez [4] and §2.2.2 of Linton & McIver [6]), in the linear analysis of *tuned sloshing dampers* (cf. Ikeda & Nakagawa [5]) and *tuned liquid dampers* (TLDs) (cf. Frandsen [3]), and in the analysis of dynamic-coupling between vehicle motion and fluid sloshing in Cooker’s experiment (cf. Alemi Ardakani et al. [2]).

3 The "cosine expansion"

Transform $\widehat{\phi}$ so that the inhomogeneous boundary conditions at $x = 0, L$ are moved to $y = h_0$. Then a cosine expansion in the $x-$direction can be used. Let

$$\widehat{\phi}(x,y) = \left(x - \tfrac{1}{2}L\right) + \widehat{\Phi}(x,y)\,. \tag{3.1}$$

The function $\widehat{\Phi}(x,y)$ then satisfies Laplace's equation and the following boundary conditions

$$\begin{aligned} \widehat{\Phi}_y &= \tfrac{\omega^2}{g}\widehat{\Phi} + K\left(x - \tfrac{1}{2}L\right)\,, \quad y = h_0 \\ \widehat{\Phi}_y &= 0\,, \quad y = 0 \\ \widehat{\Phi}_x &= 0\,, \quad x = 0, L\,. \end{aligned} \tag{3.2}$$

The function x has the following cosine expansion

$$x - \frac{L}{2} = \sum_{n=0}^{\infty} p_n \cos(\alpha_n x)\,, \quad \text{with} \quad p_n = -\frac{4}{L\alpha_n^2}\,, \quad \alpha_n = (2n+1)\frac{\pi}{L}\,. \tag{3.3}$$

Considering only the solution due to the inhomogeneous term in the boundary condition at $y = h_0$, a proposed infinite cosine expansion is

$$\widehat{\Phi}(x,y) = \sum_{n=0}^{\infty} A_n(y)\, \cos(\alpha_n x)\,.$$

This function satisfies the boundary conditions at $x = 0$ and $x = L$. To satisfy Laplace's equation and the boundary condition at $y = 0$, $A_n(y)$ are required to be proportional to $\cosh(\alpha_n y)$, giving

$$\widehat{\Phi}(x,y) = \sum_{n=0}^{\infty} a_n \frac{\cosh(\alpha_n y)}{\cosh(\alpha_n h_0)} \cos(\alpha_n x)\,. \tag{3.4}$$

The coefficients a_n are then determined by substituting into the boundary condition at $y = h_0$ and using (3.3),

$$a_n = \frac{K}{K_n - K}\, p_n\,, \quad \text{with} \quad K_n := \alpha_n \tanh(\alpha_n h_0)\,. \tag{3.5}$$

Substituting (3.4) with (3.5) and the cosine expansion (3.3) gives the required solution for $\widehat{\phi}$,

$$\widehat{\phi}(x,y) = \sum_{n=0}^{\infty} p_n\, f_n(y)\, \cos(\alpha_n x)\,, \tag{3.6}$$

with

$$f_n(y) = 1 + \left(\frac{K}{K_n - K}\right) \frac{\cosh(\alpha_n y)}{\cosh(\alpha_n h_0)}\,. \tag{3.7}$$

This form of solution was first proposed in [4] and then later used in [3] for a linear analysis of TLDs.

3.1 A property of the functions $f_n(y)$

In the proof of equivalence, the functions $f_n(y)$ will need to be expanded in another series of functions – the vertical eigenfunctions. For this transformation, a useful property of these functions is

$$f_n'(h_0) - Kf_n(h_0) = 0\,. \tag{3.8}$$

This follows since

$$f_n'(y) - Kf_n(y) = \alpha_n \left(\frac{K}{K_n - K}\right) \frac{\sinh(\alpha_n y)}{\cosh(\alpha_n h_0)} - K - K\left(\frac{K}{K_n - K}\right) \frac{\cosh(\alpha_n y)}{\cosh(\alpha_n h_0)}\,,$$

and so

$$\begin{aligned} f_n'(h_0) - Kf_n(h_0) &= \alpha_n \left(\frac{K}{K_n - K}\right) \tanh(\alpha_n h_0) - K - K\left(\frac{K}{K_n - K}\right) \\ &= \left(\frac{KK_n}{K_n - K}\right) - K - K\left(\frac{K}{K_n - K}\right) = 0\,. \end{aligned}$$

4 The vertical eigenfunction expansion

An alternative is to use the vertical direction as an organizing centre. There is a complete set of eigenfunctions $\{\psi_0(y), \psi_1(y), \ldots\}$ satisfying a boundary value problem in y. The required details of these eigenfunctions following [6] are recorded in Appendix A.

In terms of the infinite set of vertical eigenfunctions, the general solution of the boundary value problem (2.1)-(2.2) is

$$\widehat{\phi}(x, y) = \sum_{n=0}^{\infty} A_n(x)\psi_n(y)\,, \tag{4.1}$$

where the vertical eigenfunctions $\psi_n(y)$ satisfy the BVP (A-1) in Appendix A.

Laplace's equation and the properties of the ψ_n-eigenfunctions give

$$\begin{aligned} A_0(x) &= A_0^{(1)} \cos k_0 x + A_0^{(2)} \sin k_0 x \\ A_n(x) &= A_n^{(1)} \cosh k_n x + A_n^{(2)} \sinh k_n x\,, \quad n = 1, 2, \ldots\,. \end{aligned} \tag{4.2}$$

Impose the boundary condition at $x = 0$:

$$\sum_{n=0}^{\infty} A_n'(0)\psi_n(y) = 1\,.$$

Now use the fact that $1 = \sum_{n=0}^{\infty} c_n \psi_n(y)$ as shown in Appendix B with c_n defined in (B-2), to give

$$A_n'(0) = A_n'(L) = c_n\,, \quad n = 0, 1, \cdots\,. \tag{4.3}$$

Substitution and the assumption

$$\sin\left(\tfrac{1}{2} k_0 L\right) \neq 0\,,$$

(neglecting the $1:1$ resonance [2] in this report) gives

$$A_0(x) = \frac{c_0}{k_0}\bigg(\sin(k_0 x) - \tan\left(\tfrac{1}{2}\,k_0 L\right)\cos(k_0 x)\bigg)\,, \qquad (4.4)$$

and for $n \geq 1$,

$$A_n(x) = \frac{c_n}{k_n}\left(\sinh(k_n x) - \tanh(\tfrac{1}{2}\,k_n L)\,\cosh(k_n x)\right)\,. \qquad (4.5)$$

Hence the solution in terms of the vertical eigenfunctions is

$$\begin{aligned}\widehat{\phi}(x,y) \;&=\; \frac{c_0}{k_0}\bigg(\sin(k_0 x) - \tan\left(\tfrac{1}{2}k_0 L\right)\cos(k_0 x)\bigg)\psi_0(y) \\ &\quad + \sum_{n=1}^{\infty}\frac{c_n}{k_n}\left(\sinh(k_n x) - \tanh(\tfrac{1}{2}k_n L)\,\cosh(k_n x)\right)\psi_n(y)\,.\end{aligned}$$

This solution should equal the solution (3.6) obtained using the cosine expansion. To show this the strategy is to expand each $A_n(x)$ in a cosine series and expand each $f_n(y)$ in terms of the vertical eigenfunctions.

5 Cosine expansion of $A_n(x)$ for $n = 0, 1, 2, \ldots$

The first step is to expand (4.4) and (4.5) in cosine series. For this the two key integrals are

$$\int_0^L \left[\sin(k_0 x) - \tan(\tfrac{1}{2}\,k_0 L)\cos(k_0 x)\right]\cos(\alpha_n x)\,\mathrm{d}x = -\frac{2k_0}{\alpha_n^2 - k_0^2}\,, \qquad (5.1)$$

and

$$\int_0^L \left[\sinh(k_m x) - \tan(\tfrac{1}{2}\,k_m L)\cos(k_m x)\right]\cos(\alpha_n x)\,\mathrm{d}x = -\frac{2k_m}{k_m^2 + \alpha_n^2}\,. \qquad (5.2)$$

These two integrals can be verified directly using elementary calculus.

Expand $A_0(x)$ in a cosine series and use (5.1)

$$A_0(x) := \frac{c_0}{k_0}\left(\sin(k_0 x) - \tan\left(\tfrac{1}{2}\,k_0 L\right)\cos(k_0 x)\right) = \sum_{m=0}^{\infty} u_m\,\cos(\alpha_m x)\,, \qquad (5.3)$$

with

$$u_m = \frac{4c_0}{L}\frac{1}{k_0^2 - \alpha_m^2} = c_0 p_m \frac{\alpha_m^2}{\alpha_m^2 - k_0^2}\,. \qquad (5.4)$$

Similarly, using (5.2),

$$\frac{c_n}{k_n}\left(\sinh(k_n x) - \tanh\left(\tfrac{1}{2}\,k_n L\right)\cosh(k_n x)\right) = \sum_{m=0}^{\infty} U_m^{(n)}\,\cos(\alpha_m x)\,, \qquad (5.5)$$

with

$$U_m^{(n)} = -\frac{4c_n}{L}\frac{1}{k_n^2 + \alpha_m^2} = c_n p_m \frac{\alpha_m^2}{k_n^2 + \alpha_m^2}\,. \qquad (5.6)$$

Hence the vertical eigenfunction expansion can be expressed in terms of a cosine series,

$$\widehat{\phi}(x,y) = \sum_{m=0}^{\infty} u_m \cos(\alpha_m x)\psi_0(y) + \sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty} U_m^{(n)} \cos(\alpha_m x)\right]\psi_n(y)\,.$$

or

$$\widehat{\phi}(x,y) = \sum_{m=0}^{\infty}\left[u_m\psi_0(y) + \sum_{n=0}^{\infty} U_m^{(n)}\psi_n(y)\right]\cos(\alpha_m x)\,. \tag{5.7}$$

Comparison of (5.7) with (3.6) suggests that the term in brackets in (5.7) should be related to the sequence of functions $f_n(y)$.

6 Vertical eigenfunction expansion of $f_n(y)$

The most difficult part of the proof of equivalence is the expansion of the functions in (3.7) in terms of vertical eigenfunctions

$$f_m(y) = 1 + \left(\frac{K}{K_m - K}\right)\frac{\cosh(\alpha_m y)}{\cosh(\alpha_m h_0)} = \sum_{n=0}^{\infty} F_n^{(m)}\,\psi_n(y)\,. \tag{6.1}$$

There are two key facts needed. The functions $f_m(y)$ should be defined so that $f_m'(h_0) = Kf_m(h_0)$ (and this is the case as shown in (3.8)). Secondly the infinite set of identities (A-5) need to be used.

6.1 Computing $F_0^{(m)}$

In this subsection it is proved that

$$F_0^{(m)} = c_0\,\frac{\alpha_m^2}{\alpha_m^2 - k_0^2}\,. \tag{6.2}$$

Start with the formula for the first coefficient in (6.1)

$$F_0^{(m)} = \frac{1}{h_0}\int_0^{h_0} f_m(y)\psi_0(y)\,\mathrm{d}y = \frac{1}{N_0 h_0}\int_0^{h_0} f_m(y)\cosh(k_0 y)\,\mathrm{d}y\,. \tag{6.3}$$

The key integral is

$$\frac{(k_0^2-\alpha_m^2)}{k_0^2}\int_0^{h_0}\frac{\cosh(\alpha_m y)}{\cosh(\alpha_m h_0)}\cosh(k_0 y)\,\mathrm{d}y = \frac{1}{k_0}\sinh(k_0 h_0) - \frac{\alpha_m}{k_0^2}\tanh(\alpha_m h_0)\cosh(k_0 h_0)\,.$$

which can be verified using integration by parts twice. Now use (3.5) and the identity (A-3)

$$\frac{(k_0^2-\alpha_m^2)}{k_0^2}\int_0^{h_0}\frac{\cosh(\alpha_m y)}{\cosh(\alpha_m h_0)}\cosh(k_0 y)\,\mathrm{d}y = \frac{1}{k_0}\left(\frac{K-K_m}{K}\right)\sinh(k_0 h_0)\,.$$

Hence

$$\frac{1}{h_0}\int_0^{h_0}\left(\frac{K}{K_m-K}\right)\frac{\cosh(\alpha_m y)}{\cosh(\alpha_m h_0)}\psi_0(y)\,\mathrm{d}y = -\frac{k_0^2}{(k_0^2-\alpha_m^2)}\frac{\sinh(k_0 h_0)}{N_0 k_0 h_0} = c_0\frac{k_0^2}{\alpha_m^2 - k_0^2}\,,$$

using (B-2). The integrand multiplying ψ_0 is $f_m(y) - 1$ and so

$$c_0 \frac{k_0^2}{\alpha_m^2 - k_0^2} = \frac{1}{h_0}\int_0^{h_0} (f_m(y) - 1)\psi_0(y)\,\mathrm{d}y = \frac{1}{h_0}\int_0^{h_0} f_m(y)\psi_0(y)\,\mathrm{d}y - c_0\,.$$

or

$$F_0^{(m)} = \frac{1}{h_0}\int_0^{h_0} f_m(y)\psi_0(y)\,\mathrm{d}y = c_0 \frac{\alpha_m^2}{\alpha_m^2 - k_0^2}\,,$$

confirming (6.2).

6.2 Computing $F_n^{(m)}$

In this subsection it is proved that

$$F_n^{(m)} = c_n \frac{\alpha_m^2}{k_n^2 + \alpha_m^2}\,. \tag{6.4}$$

Start with the formula for the $n-$th coefficient

$$F_n^{(m)} = \frac{1}{h_0}\int_0^{h_0} f_m(y)\psi_n(y)\,\mathrm{d}y = \frac{1}{N_n h_0}\int_0^{h_0} f_m(y)\cos(k_n y)\,\mathrm{d}y\,. \tag{6.5}$$

The key integral is

$$\frac{(k_n^2+\alpha_m^2)}{\alpha_m^2}\int_0^{h_0} \frac{\cosh(\alpha_m y)}{\cosh(\alpha_m h_0)}\cos(k_n y)\,\mathrm{d}y = \frac{k_n}{\alpha_m^2}\sin(k_n h_0) + \frac{1}{\alpha_m}\tanh(\alpha_m h_0)\cos(k_n h_0)\,,$$

which can be verified directly using integration by parts twice. Now use (3.5) and the infinite set of identities (A-5)

$$\frac{(k_n^2+\alpha_m^2)}{\alpha_m^2}\int_0^{h_0} \frac{\cosh(\alpha_m y)}{\cosh(\alpha_m h_0)}\cos(k_n y)\,\mathrm{d}y = \frac{k_n}{\alpha_m^2}\left(\frac{K-K_m}{K}\right)\sin(k_n h_0)\,.$$

Hence

$$\frac{1}{h_0}\int_0^{h_0} \left(\frac{K}{K_m - K}\right)\frac{\cosh(\alpha_m y)}{\cosh(\alpha_m h_0)}\psi_n(y)\,\mathrm{d}y = -\frac{k_n^2}{k_n^2+\alpha_m^2}\frac{\sin(k_n h_0)}{k_n h_0 N_n} = -c_n\frac{k_n^2}{k_n^2+\alpha_m^2}\,,$$

using (B-2). The integrand multiplying ψ_n is then $f_m(y) - 1$ and so

$$-c_n \frac{k_n^2}{k_n^2+\alpha_m^2} = \frac{1}{h_0}\int_0^{h_0} (f_m(y) - 1)\psi_n(y)\,\mathrm{d}y = \frac{1}{h_0}\int_0^{h_0} f_m(y)\psi_n(y)\,\mathrm{d}y - c_n\,,$$

or

$$F_n^{(m)} = \frac{1}{h_0}\int_0^{h_0} f_m(y)\psi_n(y)\,\mathrm{d}y = c_n \frac{\alpha_m^2}{k_n^2+\alpha_m^2}\,,$$

confirming (6.5).

6.3 Summary of the expansion for $f_n(y)$

To summarize, the expansion in terms of vertical eigenfunctions of $f_m(y)$ is

$$f_m(y) = \frac{\alpha_m^2}{\alpha_m^2 - k_0^2} c_0\psi_0(y) + \sum_{n=1}^{\infty} \frac{\alpha_m^2}{\alpha_m^2 + k_n^2}\, c_n\psi_n(y)\,. \tag{6.6}$$

7 Transforming: cosine to vertical eigenfunction

Start with the cosine expansion (3.4) and (3.6). Substititute the vertical eigenfunction expansion for $f_n(z)$, reverse the order of summation, and then substitute the cosine expansion of the sequence of functions $A_n(x)$. The result is the vertical eigenfunction represenatation. Carrying out these steps:

$$\begin{aligned}
\widehat{\phi}(x,y) &= \big(x - \tfrac{1}{2}L\big) + \sum_{n=0}^{\infty} a_n \frac{\cosh(\alpha_n y)}{\cosh(\alpha_n h_0)} \cos(\alpha_n x) \\
&= \sum_{n=0}^{\infty} p_n \cos(\alpha_n x) + \sum_{n=0}^{\infty} p_n \left(\frac{K}{K_n - K}\right) \frac{\cosh(\alpha_n y)}{\cosh(\alpha_n h_0)} \cos(\alpha_n x) \\
&= \sum_{n=0}^{\infty} p_n f_n(y) \cos(\alpha_n x) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_n F_m^{(n)} \psi_m(y) \cos(\alpha_n x) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_m F_n^{(m)} \psi_n(y) \cos(\alpha_m x) \\
&= \sum_{m=0}^{\infty} p_m F_0^{(m)} \psi_0(y) \cos(\alpha_m x) + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} p_m F_n^{(m)} \psi_n(y) \cos(\alpha_m x) \\
&= c_0 \psi_0(y) \sum_{m=0}^{\infty} p_m \frac{\alpha_m^2}{\alpha_m^2 - k_0^2} \cos(\alpha_m x) + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} p_m c_n \frac{\alpha_m^2}{k_n^2 + \alpha_m^2} \psi_n(y) \cos(\alpha_m x) \\
&= \psi_0(y) \sum_{m=0}^{\infty} u_m \cos(\alpha_m x) + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} U_m^{(n)} \psi_n(y) \cos(\alpha_m x) \\
&= \left[\sum_{m=0}^{\infty} u_m \cos(\alpha_m x)\right] \psi_0(y) + \sum_{n=1}^{\infty} \left[\sum_{m=0}^{\infty} U_m^{(n)} \cos(\alpha_m x)\right] \psi_n(y) \\
&= A_0(x)\psi_0(y) + \sum_{n=1}^{\infty} A_n(x)\psi_n(y)\,,
\end{aligned}$$

which is the representation of $\widehat{\phi}$ in terms of vertical eigenfunction expansion. This completes the transformation from the cosine expansion to the vertical eigenfunction expansion.

8 Transforming the characteristic equation

In constructing the characteristic equation in the coupled vessel-fluid problem in [2] the integral of the horizontal momentum is important. Here, the equivalence of representations in §7 is used to give two equivalent representations of the total horizontal momentum.

Start by integrating (6.1) over the interval $[0, h_0]$,

$$\frac{1}{h_0} \int_0^{h_0} \left[1 + \left(\frac{K}{K_m - K}\right) \frac{\cosh(\alpha_m y)}{\cosh(\alpha_m h_0)}\right] \mathrm{d}y = \sum_{n=0}^{\infty} F_n^{(m)} \frac{1}{h_0} \int_0^{h_0} \psi_n(y)\, \mathrm{d}y\,,$$

giving

$$1 + \left(\frac{K K_m}{K_m - K}\right) \frac{1}{\alpha_m^2 h_0} = \sum_{n=0}^{\infty} c_n F_n^{(m)} = \frac{c_0^2 \alpha_m^2}{\alpha_m^2 - k_0^2} + \sum_{n=1}^{\infty} \frac{c_n^2 \alpha_m^2}{\alpha_m^2 + k_n^2}\,.$$

Multiply both sides by $2p_m$ and sum over m

$$\sum_{m=0}^{\infty} 2p_m \left[1 + \left(\frac{KK_m}{K_m - K} \right) \frac{1}{\alpha_m^2 h_0} \right] = -\frac{8}{L} c_0^2 \sum_{m=0}^{\infty} \frac{1}{\alpha_m^2 - k_0^2} - \frac{8}{L} \sum_{n=1}^{\infty} c_n^2 \sum_{m=0}^{\infty} \frac{1}{\alpha_m^2 + k_n^2} .$$

To simplify further we need the following two identities

$$\frac{\tan(z)}{z} = 8 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2\pi^2 - 4z^2} \quad \text{and} \quad \frac{\tanh(z)}{z} = 8 \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2\pi^2 + 4z^2} .$$

These two identities can be proved by substituting $x = 0$ in (5.3) and (5.5) respectively. Using these identities

$$\sum_{m=0}^{\infty} 2p_m \left[1 + \left(\frac{KK_m}{K_m - K} \right) \frac{1}{\alpha_m^2 h_0} \right] = -2\frac{c_0^2}{k_0} \tan\left(\tfrac{1}{2} k_0 L\right) - \sum_{n=1}^{\infty} 2\frac{c_n^2}{k_n} \sum_{m=0}^{\infty} \tanh\left(\tfrac{1}{2} k_n L\right) .$$

Since

$$\sum_{m=0}^{\infty} p_m = -\tfrac{1}{2} L \,, \tag{8.1}$$

which follows by setting $x = 0$ in (3.3), this formula simplifies to

$$1 - \frac{1}{h_0 L} \sum_{m=0}^{\infty} \frac{2p_m}{\alpha_m^2} \left(\frac{KK_m}{K_m - K} \right) = 2\frac{c_0^2}{k_0 L} \tan\left(\tfrac{1}{2} k_0 L\right) + \sum_{n=1}^{\infty} 2\frac{c_n^2}{k_n L} \tanh\left(\tfrac{1}{2} k_n L\right) . \tag{8.2}$$

8.1 The horizontal momentum of the fluid

The total horizonatal momentum of the fluid is

$$\mathsf{M}^{\text{horz}} := \int_0^L \int_0^{h_0} \rho \widehat{\phi_x} \,\mathrm{d}y\mathrm{d}x \,.$$

First compute the integrals on the right-hand side using the cosine representation and $m_f = \rho h_0 L$,

$$\begin{aligned}
\mathsf{M}^{\text{horz}} &= \int_0^L \int_0^{h_0} \rho \widehat{\phi_x} \,\mathrm{d}y\mathrm{d}x \\
&= m_f - \rho \sum_{n=0}^{\infty} a_n \alpha_n \left[\int_0^{h_0} \frac{\cosh(\alpha_n y)}{\cosh(\alpha_n h_0)} \,\mathrm{d}y \right] \int_0^L \sin(\alpha_n x)\mathrm{d}x \\
&= m_f + \rho \sum_{n=0}^{\infty} \frac{a_n}{\alpha_n} \tanh(\alpha_n h_0)(\cos(\alpha_n L) - 1) \\
&= m_f - 2\rho \sum_{n=0}^{\infty} \frac{a_n}{\alpha_n^2} K_n \\
&= m_f \left[1 - \frac{1}{h_0 L} \sum_{n=0}^{\infty} 2\frac{p_n}{\alpha_n^2} \left(\frac{KK_n}{K_n - K} \right) \right]
\end{aligned}$$

Now compute the total horizontal momentum using the vertical eigenfunctions

$$
\begin{aligned}
\mathsf{M}^{\text{horz}} &= \int_0^L \int_0^{h_0} \rho \widehat{\phi}_x \,\mathrm{d}y\mathrm{d}x \\
&= \int_0^L \int_0^{h_0} \rho \big[A_0'(x)\psi_0(y) + \sum_{n=1}^{\infty} A_n'(x)\psi_n(y)\big] \,\mathrm{d}y\mathrm{d}x \\
&= \rho\big[A_0(L) - A_0(0)\big] \int_0^{h_0} \psi_0(y)\,\mathrm{d}y + \sum_{n=1}^{\infty} \rho\big[A_n(L) - A_n(0)\big] \int_0^{h_0} \psi_n(y)\,\mathrm{d}y \\
&= \rho h_0 c_0 \big[A_0(L) - A_0(0)\big] + \sum_{n=1}^{\infty} \rho h_0 c_n \big[A_n(L) - A_n(0)\big] \\
&= 2\rho h_0 \frac{c_0^2}{k_0} \tan\left(\tfrac{1}{2}k_0 L\right) + \sum_{n=1}^{\infty} 2\rho h_0 \frac{c_n^2}{k_n} \tanh\left(\tfrac{1}{2}k_n L\right) \\
&= m_f \left[2\frac{c_0^2}{k_0 L} \tan\left(\tfrac{1}{2}k_0 L\right) + \sum_{n=1}^{\infty} 2\frac{c_n^2}{k_n L} \tanh\left(\tfrac{1}{2}k_n L\right)\right].
\end{aligned}
$$

It follows from (8.2) that these two representations are equal. This equivalence of the two representations of the total horizontal momentum is used in [2].

— Appendix —

A The vertical eigenfunctions

The eigenvalue problem for the vertical eigenfunctions is

$$
\begin{aligned}
-\psi_{yy} &= \lambda\psi\,, \quad 0 < y < h_0\,, \\
\psi_y &= 0 \quad \text{at } y = 0\,, \\
\psi_y &= K\psi \quad \text{at } y = h_0\,,
\end{aligned}
\tag{A-1}
$$

with λ the eigenvalue parameter, and $K = \omega^2/g$ is treated as a given real parameter. Here the properties of the eigenvalues and eigenfunctions are recorded following [6].

The first eigenvalue λ_0 is negative. Therefore define

$$\lambda_0 = -k_0^2\,, \tag{A-2}$$

where, for fixed K and h_0, k_0 is the unique root of

$$k_0 \tanh(k_0 h_0) - K = 0\,. \tag{A-3}$$

In addition there is a countable number of positive eigenvalues

$$\lambda_n = k_n^2\,, \quad n = 1, 2, \dots\,, \tag{A-4}$$

with the sequence k_n determined by

$$k_n \tan(k_n h_0) + K = 0\,, \quad n = 1, 2, \dots\,. \tag{A-5}$$

The associated eigenfunctions are

$$\psi_0(y) = \frac{1}{N_0}\cosh(k_0 y) \quad \text{and} \quad \psi_n(y) = \frac{1}{N_n}\cos(k_n y)\,, \quad n = 1, 2, \dots\,, \tag{A-6}$$

with

$$N_0 = \sqrt{\frac{1}{2}\left(1 + \frac{\sinh 2k_0 h_0}{2k_0 h_0}\right)} \quad \text{and} \quad N_n = \sqrt{\frac{1}{2}\left(1 + \frac{\sin 2k_n h_0}{2k_n h_0}\right)}\,. < \tag{A-7}$$

The coefficients N_0 and N_n are chosen so that the eigenfunctions have unit norm,

$$\frac{1}{h_0}\int_0^{h_0} \psi_n(y)^2 \,\mathrm{d}y = 1\,, \quad n = 0, 1, 2, \dots\,. \tag{A-8}$$

The set $\{\psi_0(y), \psi_1(y), \dots\}$ is complete on the interval $[0, h_0]$. Hence any square-integrable function $g(y)$ on this interval can be expanded in a series

$$g(y) = \sum_{n=0}^{\infty} g_n \psi_n(y)\,, \tag{A-9}$$

with the coefficients determined using orthogonality of the eigenfunctions,

$$g_n = \frac{1}{h_0}\int_0^{h_0} g(y)\,\psi_n(y)\,\mathrm{d}y\,. \tag{A-10}$$

B Anomalies in the eigenfunction expansions

The theory of Appendix A can be used to expand the function "1"

$$1 = \sum_{n=0}^{\infty} c_n \psi_n(y)\,, \tag{B-1}$$

with coefficients

$$c_0 = \frac{1}{N_0}\frac{\sinh k_0 h_0}{k_0 h_0} \quad \text{and} \quad c_n = \frac{1}{N_n}\frac{\sin k_n h_0}{k_n h_0}\,. \tag{B-2}$$

The expansion of "1" highlights a difficulty with the vertical eigenfunction expansion. Let $g(y)$ be an arbitrary function on the interval $0 < y < h_0$ with expansion in terms of vertical eigenfunctions as in (A-9) and (A-10). Then

$$\begin{aligned} g'(y) - Kg(y) &= \sum_{n=0}^{\infty} g_n \psi_n'(y) - K\sum_{n=0}^{\infty} g_n \psi_n(y) \\ &= \sum_{n=0}^{\infty} g_n(\psi_n'(y) - K\psi_n(y))\,. \end{aligned}$$

Now, using the fact that the vertical eigenfunctions satisfy the identity

$$\psi_n'(h_0) - K\psi_n(h_0) = 0\,,$$

it follows that

$$g'(h_0) - Kg(h_0) = \sum_{n=0}^{\infty} g_n(\psi_n'(h_0) - K\psi_n(h_0)) = 0\,. \tag{B-3}$$

Therefore if $g'(h_0) - Kg(h_0) \neq 0$ then there is a discontinuity at $y = h_0$ in the representation in terms of vertical eigenfunctions. This happens in the representation for "1" since with $g(y) = 1$ the left hand side of (B-3) is $-K$ which is nonzero but the sum on the right-hand side vanishes.

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