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# LENGTH SCALES AND POSITIVITY OF SOLUTIONS OF A CLASS OF REACTION-DIFFUSION EQUATIONS

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Abstract. In this paper, the sharpest interpolation inequalities are used to find a set of length scales for the solutions of the following class of dissipative partial differential equations

$$u_t = -\alpha_k(-1)^k\nabla^{2k}u + \sum_{j=1}^{k-1}\alpha_j(-1)^j\nabla^{2j}u + \nabla^2(u^m) + u(1-u^{2p}),$$

with periodic boundary conditions on a one-dimensional spatial domain. The equation generalises nonlinear diffusion model for the case when higher-order differential operators are present. Furthermore, we establish the asymptotic positivity as well as the positivity of solutions for all times under certain restrictions on the initial data. The above class of equations reduces for some particular values of the parameters to classical models such as the KPP-Fisher equation which appear in various contexts in population dynamics.

1. **Introduction.** The problems addressed in this paper concern the analysis of the length scales and the positivity of solutions for a class of nonlinear dissipative partial differential equations (PDEs).

We consider the class of PDEs

$$u_t = -\alpha_k(-1)^k\nabla^{2k}u + \sum_{j=1}^{k-1}\alpha_j(-1)^j\nabla^{2j}u + \nabla^2(u^m) + u(1-u^{2p}), \tag{1}$$

in one spatial dimension, where $\alpha_k > 0$, $\alpha_j > 0$, $k, m, p$ positive integers, $k > 1$, $m \geq 1$, $p \geq 1$ and $u = u(x,t)$, $t \geq 0$, $x \in \Omega = [0, L]$ with periodic boundary conditions. A particular case of this equation with $k = 2, m = 0, p = 1$ is known in

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physical literature as the Swift-Hohenberg equation [18], which is used to describe the Rayleigh-Bénard convection. A similar equation has been used by Pomeau and Manneville [16] in the study of cellular flows just past the onset of instability. For the same values of parameters and positive second term, this is the Extended Fisher-Kolmogorov equation used as a model of phase transitions, onset of spatio-temporal chaos and other phenomena in bistable physical systems [15].

First we will obtain a set of dissipative length scales directly involving the solutions of the PDEs. The length scales provide better understanding of the spatio-temporal dynamics of the solutions of dissipative PDEs, and their importance consists in giving accurate information about the smallest features of the flow. Among the first length scales to be used is the famous dissipation length scale obtained by Kolmogorov using the scaling argument for the incompressible fluid flow of the Navier-Stokes equations under a few heuristic assumptions [13]. This length scale is defined as the scale below which the dynamics is completely dominated by dissipation.

We then prove that the solutions of the class of PDEs under investigation preserve positivity. The positivity of solutions must be a fundamental feature for every PDE whose solutions represent some physical or biological quantity that cannot be negative, as, for example, in population dynamics. Here it is important to stress that in many of the models used in population dynamics the highest order spatial derivative term is the Laplacian. In this case, the positivity preservation result can be established by using the maximum principle [17]. However, the maximum principle does not apply in the study of dissipative partial differential equations which contain differential operators of higher order than that of the Laplacian. A number of important contributions to the study of positivity of solutions in higher-order equations can be found in the literature (see, for example, [5, 6, 9, 7], and references therein). In this paper we investigate a particular class of dissipative partial differential equations which possess a uniform steady state solution. For this class, we shall prove that under certain restrictions on initial data solutions are asymptotically stable, as well as stable for all times. These results naturally extend our previous study on positivity of solutions for some particular cases of equation (1) (see [3, 4]).

The class of PDEs under consideration represents a generalised diffusion model of population dynamics [1, 4]. In such a model, the smallest length scale is a measure of the smallest scale in which spatial fluctuations of a population occur. Our length scales are defined in a way similar to Bartuccelli et al. [2] and Doering and Gibbon [10]. The use of recently obtained sharp inequalities [12] allows us to improve the estimates for the bottom rung of the ladder and the length scales.

In the context of population dynamics, an illustrative example with $k = 2, m = 3$ has been considered by Cohen and Murray [8]. They showed that the balance between the stabilizing $-\alpha_k u_{xxxx}$ term and a destabilizing $-\alpha_j u_{xx}$ term leads to the existence of stationary spatially periodic solutions for $\alpha_j$ sufficiently positive.

We note that the first term in (1) always has a stabilizing effect, while the second term can be stabilizing or destabilizing. In order to have spatially structured solutions, for which the introduction of the length scales makes sense, we choose $\alpha_j$ to be positive. The balance between $k$ and $m$ (which is the order of nonlinearity in the equation) produces the necessary "slaving" of the dynamics by the dissipation.

In what follows we use this model as a representative example of a dissipative nonlinear PDE incorporating the features of a certain class of systems.

The outline of the paper is as follows: Section 2 contains the derivation of the ladder theorem for our class of PDEs. Section 3 deals with the estimates for the bottom rung of the ladder. In Section 4 temporal averages are used to derive the length scales. Section 5 is devoted to the proof of the asymptotic positivity of solutions, and Section 6 accomplishes the analysis by proving the positivity of solutions for all times under certain restrictions on the initial data and the parameters in the equation. Concluding remarks are presented in Section 7.

2. **Ladder estimates.** In this section we derive the set of ladder inequalities for (1) and use them to find the absorbing balls needed for the estimates of the length scales. Below we shall use the following notation: for $1 \le s < \infty$ we define $L^s$-norm as

$$\|u\|_s = \left( \int_\Omega |u|^s dx \right)^{1/s}.$$

This is the space of equivalence classes of functions that are $s$-integrable over the bounded domain $\Omega$. Also for $s = \infty$ we define

$$\|u(\cdot,t)\|_\infty = \sup_{x\in\Omega} |u(x,t)| < \infty.$$

We shall especially make frequent use of the following set of quantities in one spatial dimension:

$$L^2 - \text{norm of the N-th derivative: } J_N = \left\| \frac{\partial^N u}{\partial x^N} \right\|_2^2 = \int \left( \frac{\partial^N u}{\partial x^N} \right)^2 dx, \tag{2}$$

$$\text{length scales : } L^{-1} \le \frac{\|u\|_\infty^2}{J_0} \le \left( \frac{J_N}{J_0} \right)^{1/2N} \equiv \ell_N^{-1}. \tag{3}$$

These time-dependent quantities contain the most important information for any dissipative partial differential equation [4, 10]. In particular, having control (boundedness) of all $J_N$ gives a smooth attractor for the solutions of our equation. Therefore, the general strategy employed here consists in using interpolation inequalities to find the upper bounds for the long-time asymptotics and time averages of $J_N$. We also define the time-asymptotic upper bound of the function $f(x,t)$ as

$$\overline{f} = \sup_{x\in\Omega} \lim_{t\to\infty} \sup f(x,t), \tag{4}$$

and the time average as

$$\langle f \rangle = \overline{\lim_{t\to\infty}} \sup_{x\in\Omega} \frac{1}{t} \int_0^t f(x,\tau) d\tau. \tag{5}$$

We use the following properties of time average:

$$\begin{aligned} &\langle f \rangle \le \overline{f}, \\ &\langle fg \rangle \le \langle f^p \rangle^{\frac{1}{p}} \langle g^q \rangle^{\frac{1}{q}} \text{ if } \tfrac{1}{p} + \tfrac{1}{q} = 1. \end{aligned} \tag{6}$$

We shall start with the ladder inequality defining the temporal dynamics of $J_N$. Differentiating the $J_N$ with respect to time and inserting the time derivative from equation (1) one obtains:

$$\frac{1}{2}\dot{J}_N = -\alpha_k(-1)^k \int \nabla^N u \nabla^{N+2k} u dx + \sum_{j=1}^{k-1} \alpha_j (-1)^j \int \nabla^N u \nabla^{N+2j} u dx$$
$$+ \int \nabla^N u \nabla^{N+2}(u^m) dx + \int (\nabla^N u)^2 dx - \int \nabla^N u \nabla^N (u^{2p+1}) dx. \tag{7}$$

Integrating by parts $k$-times the first term in (7) gives:

$$-\alpha_k(-1)^k \int \nabla^N u \nabla^{N+2k} u dx = -\alpha_k \int (\nabla^{N+k} u)^2 dx = -\alpha_k J_{N+k}.$$

In a similar manner the second term in (7) can be transformed into

$$\sum_{j=1}^{k-1} \alpha_j (-1)^j \int \nabla^N u \nabla^{N+2j} u dx = \sum_{j=1}^{k-1} \alpha_j J_{N+j}.$$

In order to estimate the third term in (7) we first integrate it by parts obtaining

$$\int (\nabla^N u)(\nabla^{N+2} u^m) dx = -\int \nabla^{N+1} u \nabla^{N+1}(u^m) dx. \tag{8}$$

Performing a Leibnitz-expansion on $\nabla^{N+1}(u^m)$ we obtain:

$$\nabla^{N+1}(u^m) = \sum_{\substack{\theta \in N^m \\ |\theta| = N+1}} \frac{(N+1)!}{\prod_{l=1}^m (\theta_l)!} \prod_{l=1}^m (\nabla^{\theta_l} u).$$

Therefore,

$$|\nabla^{N+1}(u^m)| \le \sum_{\substack{\theta \in N^m \\ |\theta| = N+1}} \frac{(N+1)!}{\prod_{l=1}^m (\theta_l)!} \prod_{l=1}^m |\nabla^{\theta_l} u|.$$

Substituting this into (8), applying the Cauchy-Schwarz inequality, an $m$-fold Hölder inequality and then the Gagliardo-Nirenberg inequality [11, 14, 19] on each term of the sum above, one obtains:

$$\int (\nabla^N u)(\nabla^{N+2} u^m) dx \le d \sum_{\substack{\theta \in N^m \\ |\theta| = N+1}} J_{N+1}^{\frac{1}{2}} \left( \prod_{l=1}^m \|\nabla^{\theta_l} u\|_{2m} \right)$$
$$\le d J_{N+1}^{\frac{1}{2}} \|\nabla^{N+1} u\|_2^{\sum_{l=1}^m b_l} \cdot \|u\|_\infty^{\sum_{l=1}^m (1-b_l)} = d J_{N+1}^{\frac{1}{2}} J_{N+1}^{\gamma} \|u\|_\infty^{\delta}, \tag{9}$$

where $d$ is a constant arising from the Leibnitz expansion and the Gagliardo-Nirenberg inequality, and $\gamma$ and $\delta$ can be found as

$$\gamma = \frac{1}{2} \text{ and } \delta = m - 1.$$

Therefore,

$$\int \nabla^N u \nabla^{N+2}(u^m) dx \le d J_{N+1} \|u\|_\infty^{m-1}. \tag{10}$$

Using the interpolation inequality (see [10])

$$J_N \le J_{N-s}^{\frac{r}{r+s}} J_{N+r}^{\frac{s}{r+s}} \tag{11}$$

with $s = 1$ and $r = k - 1$ and then applying the Hölder inequality we obtain

$$d J_{N+1} \|u\|_\infty^{m-1} \le d J_N^{\frac{k-1}{k}} J_{N+k}^{\frac{1}{k}} \|u\|_\infty^{m-1} \le \frac{\alpha_k}{2} J_{N+k} + d_1 J_N \|u\|_\infty^{\frac{k(m-1)}{k-1}}, \tag{12}$$

where

$$d_1 = \left(\frac{2}{\alpha_k}\right)^{\frac{1}{k-1}} \left(\frac{d}{k}\right)^{\frac{k}{k-1}} \left(\frac{k}{k-1}\right).$$

In [12] the following sharp interpolation inequality was proved

$$\|u\|_\infty \le c(l) J_0^{\frac{2l-1}{4l}} J_l^{\frac{1}{4l}},\ l > \frac{1}{2}, \tag{13}$$

$$c(l) = \left(\frac{1}{(2l-1)^{(2l-1)/2l} \sin\frac{\pi}{2l}}\right)^{\frac{1}{2}}.$$

Applying this inequality with $l = N$ to the last term in (12) we have

$$dJ_{N+1}\|u\|_\infty^{m-1} \le \frac{\alpha_k}{2} J_{N+k} + d_1 c(N)^{m-1} J_N^{1+\frac{m-1}{4N}} J_0^{\frac{(2N-1)(m-1)}{4N}}.$$

Similar argument applied to the last term in (7) gives:

$$-\int \nabla^N u \nabla^N (u^{2p+1}) dx \le c J_N^{1+\frac{p}{2N}} J_0^{\frac{2N-1}{2N}p}. \tag{14}$$

Summarising these calculations we have:

$$\frac{1}{2}\dot{J}_N \le -\frac{\alpha_k}{2} J_{N+k} + \sum_{j=1}^{k-1} \alpha_j J_{N+j} + J_N + c J_N^{1+\frac{p}{2N}} J_0^{\frac{2N-1}{2N}p}$$

$$+ d_1 c(N)^{m-1} J_N^{1+\frac{m-1}{4N}} J_0^{\frac{(2N-1)(m-1)}{4N}}. \tag{15}$$

With the help of the inequality (11) and the Hölder inequality one obtains

$$\frac{1}{2}\dot{J}_N \le -\frac{\alpha_k}{4} J_{N+k} + \left(1 + \sum_{j=1}^{k-1} \phi_j\right) J_N + c J_N^{1+\frac{p}{2N}} J_0^{\frac{2N-1}{2N}p}$$

$$+ d_1 c(N)^{m-1} J_N^{1+\frac{m-1}{4N}} J_0^{\frac{(2N-1)(m-1)}{4N}}, \tag{16}$$

where the constants $\phi_j$ are defined as

$$\phi_j = \frac{k-j}{j} \alpha_j^{\frac{k}{k-j}} \left(\frac{4j(k-1)}{k\alpha_k}\right)^{\frac{j}{k-j}}.$$

By using inequality (11) with $r = k$ and $s = N$ the ladder transforms into:

$$\frac{1}{2}\dot{J}_N \le -\frac{\alpha_k}{4} \frac{J_N^{1+k/N}}{J_0^{k/N}} + \left(1 + \sum_{j=1}^{k-1} \phi_j\right) J_N + c J_N^{1+\frac{p}{2N}} J_0^{\frac{2N-1}{2N}p}$$

$$+ d_1 c(N)^{m-1} J_N^{1+\frac{m-1}{4N}} J_0^{\frac{(2N-1)(m-1)}{4N}}. \tag{17}$$

Since we are looking for the length scales, we need to have control over the terms giving the largest contribution to the value of the length scales. This means that without loss of generality we can neglect the second term in the above expression. Comparing the last two terms in the ladder (17), we distinguish between the following possibilities:

*Case* **A.** This is defined by the inequality $m > 2p + 1$. In this case the last term in (17) gives the largest contribution, and therefore all the smaller order terms can be neglected. Therefore, the ladder becomes

$$\frac{1}{2}\dot{J}_N \leq -\frac{\alpha_k}{4}\frac{J_N^{1+k/N}}{J_0^{k/N}} + d_1 c(N)^{m-1} J_N^{1+\frac{m-1}{4N}} J_0^{\frac{(2N-1)(m-1)}{4N}}. \tag{18}$$

One can easily see that an absorbing ball can be found only for $m < 4k+1$ as

$$\overline{J}_N \leq \left(\frac{4d_1 c(N)^{m-1}}{\alpha_k}\right)^{\frac{4N}{4k-m+1}} \overline{J}_0^{\frac{4k+(2N-1)(m-1)}{4k-m+1}}, \ 2p+1 < m < 4k+1. \tag{19}$$

*Case* **B.** This is the situation when $m < 2p+1$, and therefore the fourth term in (17) dominates. The ladder then takes the form:

$$\frac{1}{2}\dot{J}_N \leq -\frac{\alpha_k}{4}\frac{J_N^{1+k/N}}{J_0^{k/N}} + cJ_N^{1+\frac{p}{2N}} J_0^{\frac{2N-1}{2N}p}. \tag{20}$$

Now, an absorbing ball can be found as

$$\overline{J}_N \leq \left(\frac{4c}{\alpha_k}\right)^{\frac{2N}{2k-p}} \overline{J}_0^{\frac{2(N-1)p-2k}{2k-p}}, \ m < 2p+1, \ p < 2k. \tag{21}$$

*Case* **C.** When $m = 2p+1$ the last two terms in (17) give contributions of the same order, and therefore they can be combined. The absorbing ball in this case has the form

$$\overline{J}_N \leq \left(\frac{4\left[d_1 c(N)^{m-1} + c\right]}{\alpha_k}\right)^{\frac{2N}{2k-p}} \overline{J}_0^{\frac{2(N-1)p-2k}{2k-p}}, \ m = 2p+1, \ p < 2k. \tag{22}$$

In order to have the regularity of solutions, we need to have a control over $\|u\|_\infty^2$, which can be estimated by using the inequality (13) with $l = N$. Thus, one can see that this control is obtained by some form of dynamical control of $\overline{J}_N$. The bounds for this quantity are given by the expressions of the absorbing ball in the cases considered above. The estimate of $\overline{J}_0$ is performed in the next section.

3. **Estimates for the bottom rung.** We have just obtained the absorbing balls through the time-asymptotic bound for the bottom rung $J_0$. In order to find this quantity we differentiate it with respect to time and insert for $u_t$ the right-hand side of (1)

$$\frac{1}{2}\dot{J}_0 = -\alpha_k(-1)^k\int u\nabla^{2k}u dx + \sum_{j=1}^{k-1}\alpha_j(-1)^j\int u\nabla^{2j}u dx$$
$$+\int u\nabla^2(u^m)dx + \int u^2 dx - \int u^{2p+2}dx. \tag{23}$$

Now, integrating by parts the first and the second terms in (23) $k$ and $j$-times respectively, estimating the third term as

$$\int u\nabla^2(u^m)dx = \{\text{ integrating by parts }\} = -m\int u_x^2 u^{m-1}dx \leq mJ_1\|u\|_\infty^{m-1},$$

and the last term as

$$-\int u^{2p+2}dx \leq -L^{-p}J_0^{p+1}, \tag{24}$$

we finally obtain

$$\frac{1}{2}\dot{J}_0 \leq -\alpha_k J_k + \sum_{j=1}^{k-1} \alpha_j J_j + J_0 + mJ_1\|u\|_\infty^{m-1} - L^{-p}J_0^{p+1}. \tag{25}$$

Substituting inequality (11) with $s = N = 1$ and $r = k-1$ in (25) yields

$$\frac{1}{2}\dot{J}_0 \leq -\alpha_k J_k + \sum_{j=1}^{k-1} \alpha_j J_j + J_0 - L^{-p}J_0^{p+1} + mJ_k^{1/k}J_0^{(k-1)/k}\|u\|_\infty^{m-1}. \tag{26}$$

Applying to the last term the Hölder and then Young's inequalities we can see that

$$\frac{1}{2}\dot{J}_0 \leq -\frac{\alpha_k}{2} J_k + \sum_{j=1}^{k-1} \alpha_j J_j + J_0 - L^{-p}J_0^{p+1} + aJ_0\|u\|_\infty^{m-1}, \tag{27}$$

where

$$a = \frac{k-1}{k} m^{\frac{k}{k-1}} \left(\frac{2}{k\alpha_k}\right)^{\frac{1}{k-1}}.$$

Estimating $\|u\|_\infty$ from (13) with $l = k$ one obtains

$$\frac{1}{2}\dot{J}_0 \leq -\frac{\alpha_k}{2} J_k + \sum_{j=1}^{k-1} \alpha_j J_j + J_0 - L^{-p}J_0^{p+1} + ac(k)^{m-1}J_0^{\frac{4k+(2k-1)(m-1)}{4k}} J_k^{\frac{m-1}{4k}}. \tag{28}$$

For $m < 4k+1$ we can apply Young's inequality to the last term in (28):

$$\frac{1}{2}\dot{J}_0 \leq -\frac{\alpha_k}{4} J_k + \sum_{j=1}^{k-1} \alpha_j J_j + J_0 - L^{-p}J_0^{p+1} + \sigma J_0^{\frac{4k+(2k-1)(m-1)}{4k-m+1}}, \tag{29}$$

where

$$\sigma = \frac{4k-m+1}{4k}\left(ac(k)^{m-1}\right)^{\frac{4k}{4k-m+1}} \left(\frac{m-1}{k\alpha_k}\right)^{\frac{m-1}{4k-m+1}}.$$

Substituting (11) with $s = N = j$ and $r = k-j$ in the second term in (29) and applying Young's inequality to every term in the sum, one obtains

$$\frac{1}{2}\dot{J}_0 \leq -\frac{\alpha_k}{8} J_k + \sum_{j=1}^{k-1} \kappa_j J_0 + J_0 - L^{-p}J_0^{p+1} + \sigma J_0^{\frac{4k+(2k-1)(m-1)}{4k-m+1}}, \tag{30}$$

with

$$\kappa_j = \frac{k-j}{k}\alpha_j^{\frac{k}{k-j}} \left(\frac{8j(k-1)}{k\alpha_k}\right)^{\frac{j}{k-j}}.$$

Neglecting the first term in (30), which is negative definite, we obtain

$$\frac{1}{2}\dot{J}_0 \leq \sum_{j=1}^{k-1} \kappa_j J_0 + J_0 - L^{-p}J_0^{p+1} + \sigma J_0^{\frac{4k+(2k-1)(m-1)}{4k-m+1}} := f(J_0). \tag{31}$$

In order to control the time average of $J_0$ we have to choose the power of $J_0$ in the last term less than $(p+1)$, and therefore we require $m < 1 + \dfrac{4kp}{2k+p}$. Since $\dfrac{4kp}{2k+p} < 2p$, then $m < 2p+1$, and we have the *Case* **B** from the previous section. Therefore, in subsequent calculations we will appeal to the formula (21) for the absorbing ball (under the condition $p < 2k$).

By the standard theory the solutions of the above inequality are bounded above by the solutions of the one-dimensional ordinary differential equation $\dot{J}_0 = 2f(J_0)$. Provided $m < 1 + \dfrac{4kp}{2k+p}$ the function $f$ is positive for $J_0$ small and negative for $J_0$ large. Hence, it has a positive root. Thus, we can write that

$$\overline{J}_0 \leq J^*, \tag{32}$$

where $J^*$ is the smallest positive root of

$$f(J_0) = \sum_{j=1}^{k-1} \kappa_j J_0 + J_0 - L^{-p} J_0^{p+1} + \sigma J_0^{\frac{4k+(2k-1)(m-1)}{4k-m+1}} = 0. \tag{33}$$

Substitution of the bound for $J_0$ in the absorbing ball (21) gives

$$\overline{J}_N \leq \left(\frac{4c}{\alpha_k}\right)^{\frac{2N}{2k-p}} (J^*)^{\frac{2(N-1)p-2k}{2k-p}}, \; m < 1 + \frac{4kp}{2k+p}, \; p < 2k. \tag{34}$$

4. **Length scales.** In this section we investigate the length scales appearing in our flow. As it was mentioned in the introduction, information on the length scales is crucial in obtaining the picture of the dynamics involved in the solutions of any dissipative flow. A set of length scales is usually obtained through a set of "relevant modes" defined through the Fourier expansion of the solutions of the equation [4, 10].

We now have to study the dynamics content of the length scales. First we note that

$$\ell_N^{-1} = \left(\frac{J_N}{J_0}\right)^{1/2N} \tag{35}$$

is a function of time; thus it is natural to say that it defines a dynamic length scale. Since its dynamic evolution cannot be easily obtained in general, it is useful to define a time-independent length scale. This is normally achieved by taking time averages or time asymptotic bounds.

We start with the time average estimates of our length scales. To do it we use the ladder (20):

$$\frac{1}{2}\dot{J}_N \leq -\frac{\alpha_k}{4}\frac{J_N^{1+k/N}}{J_0^{k/N}} + cJ_N^{1+\frac{p}{2N}} J_0^{\frac{2N-1}{2N}p}. \tag{36}$$

Dividing this inequality through by $J_N$ and taking time average, as $\langle \dot{J}_N/J_N \rangle$ vanishes, one obtains

$$\left\langle \left(\frac{J_N}{J_0}\right)^{1/2N} \right\rangle \leq \left(\frac{4c}{\alpha_k}\right)^{1/2k} \left\langle J_N^{\frac{p}{4kN}} J_0^{\frac{(2N-1)p}{4kN}} \right\rangle. \tag{37}$$

Now applying the Hölder inequality for $p < 2k$ to the time-averaged term on the right-hand side of (37) we finally obtain the estimate for the time-averaged length scales

$$\langle \ell^{-1} \rangle \leq \left(\frac{4c}{\alpha_k}\right)^{\frac{1}{2k-p}} (J^*)^{\frac{1}{2k-p}}, \; m < 1 + \frac{4kp}{2k+p}, \; p < 2k. \tag{38}$$

5. **Asymptotic positivity of solutions.** In order to study the positivity we center the equation (1) on the uniform steady state solution $u = 1$ and then show that under certain assumptions concerning the initial data, solutions of the transformed equation are bounded (by absolute value) by 1. Therefore, we introduce $v(x,t)$ defined by

$$u(x,t) = 1 + v(x,t) \tag{39}$$

where $u$ satisfies (1). If the function $v(x,t)$ satisfies

$$\|v(\cdot,t)\|_\infty \le 1$$

then clearly $u(\cdot,t)$ is non-negative function for all $t$. Moreover, if we can show that $\|v(\cdot,t)\|_\infty \to 0$ as $t \to \infty$ then we have a uniform convergence (meaning the asymptotic positivity).

We introduce the following time-dependent quantities:

$$H_N := \left\| \frac{\partial^N v}{\partial x^N} \right\|_2^2 = \int \left( \frac{\partial^N v}{\partial x^N} \right)^2 dx, \tag{40}$$

where the integration is over the spatial domain $\Omega := [0, L]$. Substituting (39) into (1) we obtain for $v$ the equation

$$\begin{aligned} v_t = -\alpha_k(-1)^k \nabla^{2k} v + \sum_{j=1}^{k-1} \alpha_j (-1)^j \nabla^{2j} v + \sum_{s=0}^{m} \binom{m}{s} \nabla^2(v^s) \\ - \sum_{l=0}^{2p-1} \binom{2p}{l} [v^{2p-l} + v^{2p-l+1}]. \end{aligned} \tag{41}$$

We start our analysis by investigating the evolution of the $L^2$-norm of the solution $v$ of (41), namely $H_0$. Differentiating $H_0$ with respect to time, and inserting the right hand side of (41), we obtain

$$\begin{aligned} \frac{1}{2}\dot{H}_0 = -\alpha_k H_k + \sum_{j=1}^{k-1} \alpha_j H_j + \sum_{s=0}^{m} \binom{m}{s} \int v \nabla^2(v^s) dx \\ - \sum_{l=0}^{2p-1} \binom{2p}{l} \left[ \int v^{2p-l+1} dx + \int v^{2p-l+2} dx \right]. \end{aligned} \tag{42}$$

Let us consider the second term in the last expression. Substituting (11) with $s = N = j$ and $r = k - j$ in the second term in (42) and applying Young's inequality to every term in the sum, one obtains

$$\begin{aligned} \frac{1}{2}\dot{H}_0 \le -\frac{\alpha_k}{2} H_k + \sum_{j=1}^{k-1} \kappa_j H_0 + \sum_{s=0}^{m} \binom{m}{s} \int v \nabla^2(v^s) dx \\ - \sum_{l=0}^{2p-1} \binom{2p}{l} \left[ \int v^{2p-l+1} dx + \int v^{2p-l+2} dx \right], \end{aligned} \tag{43}$$

where

$$\beta_j = \frac{k-j}{k} \left( \frac{2j(k-1)}{k\alpha_k} \right)^{\frac{j}{k-j}} \alpha_j^{\frac{k}{k-j}} \tag{44}$$

Next, we consider the third term in (42):

$$\sum_{s=0}^{m}\binom{m}{s}\int v\nabla^2(v^s)dx \leq \sum_{s=1}^{m}\binom{m}{s} sH_1\|v\|_\infty^{s-1}. \tag{45}$$

In [3] the following version of the inequality (13) was proved (the constants $c(k)$ are the same as in (13))

$$\|v\|_\infty \leq c(k)H_k^{\frac{1}{4k}}H_0^{\frac{2k-1}{4k}} + L^{-\frac{1}{2}}H_0^{\frac{1}{2}}. \tag{46}$$

Substituting this estimate in (45) and using the binomial expansion we can rewrite it as

$$-\sum_{s=1}^{m}\binom{m}{s}sH_1\sum_{n=0}^{s-1}\binom{s-1}{n}c(k)^n H_0^{\frac{(2k-1)n+2k(s-1-n)}{4k}}H_k^{\frac{n}{4k}}L^{-\frac{s-1-n}{2}}. \tag{47}$$

By introducing the variable $W(s,n)$ as

$$W(s,n) = \binom{m}{s}\binom{s-1}{n}c(k)^n s,$$

and expressing $H_1$ via $H_0$ and $H_k$ from (11), we obtain:

$$\sum_{s=0}^{m}\binom{m}{s}\int v\nabla^2(v^s)dx \leq \sum_{s=1}^{m}\sum_{n=0}^{s-1}W(s,n)H_k^{\frac{4+n}{4k}}H_0^{\frac{2ks+2k-4-n}{4k}}L^{-\frac{s-1-n}{2}}$$

$$\leq \frac{\alpha_k}{4}H_k + \sum_{s=1}^{m}\sum_{n=0}^{s-1}\sigma(s,n)H_0^{\frac{2ks+2k-4-n}{4k-4-n}}, \tag{48}$$

where in the last step we have used the Hölder inequality for $m < 4k-3$ and

$$\sigma(s,n) = \frac{4k-4+n}{4k}\left[\frac{m(m+1)(4+n)}{2k\alpha_k}\right]^{\frac{4+n}{4k-4-n}} W(s,n)^{\frac{4k}{4k-4-n}}L^{\frac{2k(1+n-s)}{4k-4-n}}.$$

Now, we estimate the last two terms in the (42) as

$$-\sum_{l=0}^{2p-1}\binom{2p}{l}\int[v^{2p-l+1}+v^{2p-l+2}]dx = -2pH_0 - L^{-p}H_0^{p+1}$$

$$+\sum_{l=0}^{2p-2}\binom{2p}{l}\|v\|_\infty^{2p-l+1} + \sum_{l=1}^{2p-1}\binom{2p}{l}\|v\|_\infty^{2p-l+2}. \tag{49}$$

Applying the inequality (46) and then the binomial expansion to the last two terms in (49) one finally obtains

$$-\sum_{l=0}^{2p-1}\binom{2p}{l}\int[v^{2p-l+1}+v^{2p-l+2}]dx = -2pH_0 - L^{-p}H_0^{p+1}$$

$$+\sum_{l=0}^{2p-2}\sum_{s=0}^{2p-l+1}\xi_1(l,s)H_0^{\frac{4kp-2kl+2k-s}{4k-s}} + \sum_{l=1}^{2p-1}\sum_{s=0}^{2p-l+2}\xi_2(l,s)H_0^{\frac{4kp-2kl+4k-s}{4k-s}},$$

where

$$\begin{aligned}\xi_1(l,s) &= \tfrac{4k-s}{4k}\left(\tbinom{2p}{l}\tbinom{2p-l+1}{s}\right)^{\frac{4k}{4k-s}}\left[c(k)^{4k}\tfrac{4s(2p-1)(p+3)}{k\alpha_k}\right]^{\frac{s}{4k-s}} L^{-\frac{2k(2p-l+1-s)}{4k-s}},\\ \xi_2(l,s) &= \tfrac{4k-s}{4k}\left(\tbinom{2p}{l}\tbinom{2p-l+2}{s}\right)^{\frac{4k}{4k-s}}\left[c(k)^{4k}\tfrac{4sp(2p+7)}{k\alpha_k}\right]^{\frac{s}{4k-s}} L^{-\frac{2k(2p-l+2-s)}{4k-s}}.\end{aligned} \tag{50}$$

By virtue of all the above calculations we can rewrite (43) in the following form

$$\begin{aligned}\frac{1}{2}\dot{H}_0 \le{}& -\frac{\alpha_k}{4}H_k - \left(2p - \sum_{j=1}^{k-1}\beta_j - \sigma(1,0)\right)H_0 + \sum_{s=2}^{m}\sum_{n=0}^{s-1}\sigma(s,n)H_0^{1+\frac{2ks-2k}{4k-4-n}}\\ &+\sum_{l=0}^{2p-2}\sum_{s=0}^{2p-l+1}\xi_1(l,s)H_0^{\frac{4kp-2kl+2k-s}{4k-s}} + \sum_{l=1}^{2p-1}\sum_{s=0}^{2p-l+2}\xi_2(l,s)H_0^{\frac{4kp-2kl+4k-s}{4k-s}} - L^{-p}H_0^{p+1}\\ \le{}& -\left(2p - \sum_{j=1}^{k-1}\kappa_j - \sigma(1,0)\right)H_0 - L^{-p}H_0^{p+1} + \sum_{s=2}^{m}\sum_{n=0}^{s-1}\sigma(s,n)H_0^{1+\frac{2ks-2k}{4k-4-n}}\\ &+\sum_{l=0}^{2p-2}\sum_{s=0}^{2p-l+1}\xi_1(l,s)H_0^{\frac{4kp-2kl+2k-s}{4k-s}} + \sum_{l=1}^{2p-1}\sum_{s=0}^{2p-l+2}\xi_2(l,s)H_0^{\frac{4kp-2kl+4k-s}{4k-s}} := f(H_0).\end{aligned} \tag{51}$$

Solutions of this differential inequality are bounded from above by the solutions of the autonomous ODE $\dot{H}_0 = 2f(H_0)$. Provided $2p > \sum_{j=1}^{k-1}\beta_j + \sigma(1,0)$, function $f(H_0)$ is negative for $H_0$ small, and for $H_0$ large, and positive for some intermediate values. We note here that the absorbing ball for $H_0$ should exist, what gives a restriction on possible values of $k$, $m$ and $p$. As it can be easily shown, this condition has the form

$$m < 1 + \frac{4p(k-1)}{2k+p}. \tag{52}$$

If this condition holds, it guarantees that $f(H_0)$ will be negative for large values of $H_0$. If also $p < 2k$ then the above inequality ensures the possibility of application of Hölder inequality in (48). Therefore, we can state that if (52) holds and $H_0(t=0) < H^*$, where $H^*$ is the smallest positive root of the equation $f(H) = 0$, then $H_0 \to 0$ as $t \to \infty$.

We have to estimate $\|v\|_\infty$, and afterwards show that it tends to zero under some assumptions on the initial condition. In order to estimate $\|v\|_\infty$, we start with considering the dynamics of $H_1$:

$$\begin{aligned}\frac{1}{2}\dot{H}_1 ={}& -\alpha_k H_{k+1} + \sum_{j=1}^{k-1}\alpha_j H_{j+1} + \sum_{s=0}^{m} C_m^s \int v_x \nabla^3 v^s dx\\ &-\sum_{l=0}^{2p-1}\binom{2p-1}{l}\int v_x^2\left[(2p-l)v^{2p-l-1} + (2p-l+1)v^{2p-l}\right]dx.\end{aligned} \tag{53}$$

Calculation of the second term similar to the one in (42) allows to rewrite the above equation as

$$\frac{1}{2}\dot{H}_1 = -\frac{\alpha_k}{2}H_{k+1} + \sum_{j=1}^{k-1}\mu_j H_0 + \sum_{s=0}^{m}\binom{m}{s}\int v_x \nabla^3 v^s dx$$

$$-\sum_{l=0}^{2p-1}\binom{2p-1}{l}\int v_x^2\left[(2p-l)v^{2p-l-1} + (2p-l+1)v^{2p-l}\right]dx, \qquad (54)$$

with

$$\mu_j = \frac{k-j}{k+1}\left(\frac{2(k-1)(j+1)}{\alpha_k(k+1)}\right)^{\frac{j+1}{k-j}}\alpha_j^{\frac{k+1}{k-j}}.$$

The third term can be estimated as above

$$\sum_{s=0}^{m}\binom{m}{s}\int v_x\nabla^3 v^s dx \le \sum_{s=0}^{m}\binom{m}{s} sH_2\|v\|^{s-1} \le \frac{\alpha_k}{4}H_{k+1} + \sum_{s=1}^{m}\sum_{n=0}^{s-1}\tau(s,n)H_0^{\frac{2ks+2s+2k-n-6}{4k-4-n}}, \qquad (55)$$

where

$$\tau(s,n) = \frac{4k-4-n}{4(k+1)}\left[\frac{m(m+1)(n+8)}{2\alpha_k(k+1)}\right]^{\frac{n+8}{4k-4-n}}\left[\binom{m}{s}\binom{s-1}{n}c(k+1)^n sL^{\frac{1+n-s}{2}}\right]^{\frac{4(k+1)}{4k-4-n}}.$$

Substituting this in (53) one obtains

$$\frac{1}{2}\dot{H}_1 \le -\frac{\alpha_k}{4}H_{k+1} + \sum_{j=1}^{k-1}\mu_j H_0 + \sum_{s=1}^{m}\sum_{n=0}^{s-1}\tau(s,n)H_0^{\frac{2ks+2s+2k-n-6}{4k-4-n}}$$

$$-\sum_{l=0}^{2p-1}\binom{2p-1}{l}\int v_x^2\left[(2p-l)v^{2p-l-1} + (2p-l+1)v^{2p-l}\right]dx. \qquad (56)$$

The last two terms in (53) can also be estimated in a similar manner

$$-\sum_{l=0}^{2p-1}\binom{2p-1}{l}\int v_x^2\left[(2p-l)v^{2p-l-1} + (2p-l+1)v^{2p-l}\right]dx$$

$$\le \sum_{l=0}^{2p-1}\binom{2p-1}{l}H_1\left[(2p-l)\|v\|_\infty^{2p-l-1} + (2p-l+1)\|v\|_\infty^{2p-l}\right]$$

$$\le \frac{\alpha_k}{8}H_{k+1} + \sum_{s=0}^{2p-1}\sum_{n=0}^{2p-s-1}\rho_1(s,n)H_0^{\frac{2k-n+4kp-2ks+4p-2s-2}{4(k+1)}}$$

$$+\sum_{s=0}^{2p-1}\sum_{n=0}^{2p-s}\rho_2(s,n)H_0^{\frac{4k-n+4kp-2ks+4p-2s}{4(k+1)}}, \qquad (57)$$

where we first used the interpolation inequality (46), then the Hölder inequality for $p < 2k$, and $\rho_1(s,n)$, $\rho_2(s,n)$ are given by

$$\rho_1(s,n) = \frac{4k-n}{4(k+1)}\left[\frac{4p(n+4)(2p+1)}{\alpha_k}\right]^{\frac{n+4}{4k-n}} q_1 L^{-\frac{2(2p-s-1-n)(k+1)}{4k-n}},$$

$$\rho_2(s,n) = \frac{4k-n}{4(k+1)}\left[\frac{4p(n+4)(2p+3)}{\alpha_k}\right]^{\frac{n+4}{4k-n}} q_2 L^{-\frac{2(2p-s-n)(k+1)}{4k-n}},$$

$$q_1 = \left[ \binom{2p-1}{s} \binom{2p-s-1}{n} (2p-s)c(k+1)^n \right]^{\frac{4(k+1)}{4k-n}},$$

$$q_2 = \left[ \binom{2p-1}{s} \binom{2p-s}{n} (2p-s+1)c(k+1)^n \right]^{\frac{4(k+1)}{4k-n}}.$$

Substituting this in (56) and using the interpolation inequality (11) with $s = N = 1$, $r = k$, we obtain

$$\frac{1}{2}\dot{H}_1 \leq -\frac{\alpha_k}{8} \frac{H_1^{k+1}}{H_0^k} + \sum_{j=1}^{k-1} \mu_j H_0 + \sum_{s=1}^{m} \sum_{n=0}^{s-1} \tau(s,n) H_0^{\frac{2ks+2s+2k-n-6}{4k-4-n}}$$

$$+ \sum_{s=0}^{2p-1} \sum_{n=0}^{2p-s-1} \rho_1(s,n) H_0^{\frac{2k-n+4kp-2ks+4p-2s-2}{4(k+1)}} + \sum_{s=0}^{2p-1} \sum_{n=0}^{2p-s} \rho_2(s,n) H_0^{\frac{4k-n+4kp-2ks+4p-2s}{4(k+1)}}. \tag{58}$$

Taking into account that, under the previously stated conditions, $H_0 \to 0$ as $t \to \infty$, we find from (58) that

$$H_1(t) \leq \text{const}, \forall t \geq 0. \tag{59}$$

Now we can appeal to the inequality (46) with $k = 1$ to obtain

$$\|v\|_\infty \leq H_1^{\frac{1}{4}} H_0^{\frac{1}{4}} + L^{-\frac{1}{2}} H_0^{\frac{1}{2}} \to 0 \text{ as } t \to \infty, \tag{60}$$

and therefore from (39) $\lim_{t\to\infty} u(x,t) = 1$, uniformly in $x$.

Now we can summarise our finding in the following theorem, that gives a condition on the initial conditions which is sufficient for convergence. The existence of $J^*$ is guaranteed by earlier remarks.

THEOREM 5.1. *If* $p < 2k$, $m < 1 + 4p(k-1)/(2k+p)$, $2p - \sum_{j=1}^{k-1} \beta_j - \sigma(1,0) > 0$, *where* $\sigma(1,0) = \frac{k-1}{k} \left[ \frac{2m^2(m+1)}{k\alpha_k} \right]^{k/(k-1)}$ *and* $\beta_j$ *are defined in (44), and the initial data satisfies*

$$\int_0^L (u(x,0) - 1)^2 dx < H^*, \tag{61}$$

*where* $H^*$ *is the smallest positive root of* $f(H) = 0$ *with* $f(H)$ *defined in (51), then the solution* $u(x,t)$ *of (1) satisfies*

$$\lim_{t\to\infty} u(x,t) = 1$$

*uniformly for* $x \in [0, L]$.

6. **Positivity of solutions.** We have shown that under certain restrictions on the $L^2$-norm of the initial data, the solution of (1) converges uniformly to 1. The fact that the convergence is uniform allows to deduce that the solution must be positive for all $t$ sufficiently large, but not necessarily for all $t$. In this section we want to show that under certain restrictions on the initial data, one can establish that, for *all* values of $t$ (not only $t \to \infty$)

$$\|u(\cdot,t) - 1\|_\infty \leq 1.$$

This inequality ensures that $u(x,t) \geq 0$ for all $x$ and $t$, and therefore it provides the preservation of positivity.

We suppose that the hypotheses of the Theorem 5.1 hold. This means, in particular, that $2p - \sum_{j=1}^{k-1} \kappa_j - \sigma(1,0) > 0$ and that

$$H_0(t=0) = \int_0^L (v(x))^2 dx \leq H^*,$$

where $H^*$ is defined above. Under this assumption, we have that

$$H_0(t) \leq H^*, \ \forall t \geq 0. \tag{62}$$

With this estimate (58) reduces to

$$\frac{1}{2}\dot{H}_1 \leq -\frac{\alpha_k}{8}\frac{H_1^{k+1}}{(H^*)^k} + \sum_{j=1}^{k-1} \mu_j H^* + \sum_{s=1}^{m}\sum_{n=0}^{s-1} \tau(s,n)(H^*)^{\frac{2ks+2s+2k-n-6}{4k-4-n}}$$

$$+ \sum_{s=0}^{2p-1}\sum_{n=0}^{2p-s-1} \rho_1(s,n)(H^*)^{\frac{2k-n+4kp-2ks+4p-2s-2}{4(k+1)}} + \sum_{s=0}^{2p-1}\sum_{n=0}^{2p-s} \rho_2(s,n)(H^*)^{\frac{4k-n+4kp-2ks+4p-2s}{4(k+1)}}, \tag{63}$$

which is an autonomous differential inequality, whose solutions are bounded from above by solutions of the corresponding differential equation. In particular, it is easy to see that if

$$H_1(t=0) \leq \left[\frac{8G}{\alpha_k}(H^*)^k\right]^{\frac{1}{k+1}},$$

then

$$H_1(t) \leq \left[\frac{8G}{\alpha_k}(H^*)^k\right]^{\frac{1}{k+1}}, \ \forall t \geq 0, \tag{64}$$

where by $G$ we denoted the right-hand side of (63) except for the first term.

Recall that $u - 1 = v$, and that we have the interpolation inequality

$$\|v(\cdot,t)\|_\infty \leq H_1^{\frac{1}{4}} H_0^{\frac{1}{4}} + L^{-\frac{1}{2}} H_0^{\frac{1}{2}}, \tag{65}$$

and therefore it is sufficient to show that the right-hand side of (65) is bounded above by 1 for all $t \geq 0$. In view of the bounds for $H_0$ and $H_1$ given by (62) and (64) respectively, positivity of the solutions will be established provided the following inequality holds

$$(H^*)^{\frac{1}{4}} \left[\frac{8G}{\alpha_k}(H^*)^k\right]^{\frac{1}{4(k+1)}} + L^{-\frac{1}{2}}(H^*)^{\frac{1}{2}} \leq 1. \tag{66}$$

Previously $H^*$ was defined as the smallest positive root of $f(H) = 0$ with the function $f(H)$ defined in the statement of the Theorem 5.1. Since the inequality (66) does not necessarily hold for any combination of parameters, therefore exactly these parameters will determine whether the initially positive solution will remain positive for all times. Our findings are summarised in the following theorem.

THEOREM 6.1. *Let $p < 2k$, $m < 1 + 2p(k-1)/k$, $2p - \sum_{j=1}^{k-1} \beta_j - \sigma(1,0) > 0$, where $\sigma(1,0) = \frac{k-1}{k}\left[\frac{2m^2(m+1)}{k\alpha_k}\right]^{k/(k-1)}$ and $\beta_j$ are defined in (44), and let the initial* *data satisfies*

$$\int_0^L (u(x,0)-1)^2 dx < H^*,$$

*where $H^*$ is the smallest positive root of $f(H)=0$, where $f(H)$ is as in the statement of the Theorem 5.1. Assume that the initial data also satisfies*

$$\int_0^L \left(\frac{\partial u}{\partial x}(x,0)\right)^2 dx \leq \left[\frac{8G}{\alpha_k}(H^*)^k\right]^{\frac{1}{k+1}},$$

*and that the parameters $\alpha_1, ..., \alpha_k, L$ are such that the following inequality holds*

$$(H^*)^{\frac{1}{4}} \left[\frac{8G}{\alpha_k}(H^*)^k\right]^{\frac{1}{4(k+1)}} + L^{-\frac{1}{2}}(H^*)^{\frac{1}{2}} \leq 1,$$

*where $G$ is the right-hand side of (63) except for the first term. Then the solution $u(x,t)$ of (1) satisfies $u(x,t) \geq 0$ for all $t \geq 0$ and all $x \in [0,L]$.*

7. **Concluding remarks.** In this paper the problem of estimating the length scales and proving the positivity of solutions of a higher-order dissipative partial differential equation has been addressed. We have obtained a set of ladder inequalities for our equation, and used them to make the estimates for the length scales. The relation between $k$, $m$ and $p$ is found, which provides the existence of the attracting set, and thus the subsequent smoothing of the dynamics by dissipation. Recently found sharp interpolation inequalities with explicit constants, allowed us to improve some of the estimates, and thereby the length scales.

It has also been shown that under certain restrictions on the parameters in the equation and the initial data, solutions which are initially positive, will remain positive for all times. Asymptotic positivity of solutions is also established for some initial data. The obtained results can be applied to a wide range of dissipative PDEs arising in many fields of applied mathematics, such as mathematical biology and particle dynamics. Specifically we would like to stress that fixing particular values for the parameters $k$, $m$ and $p$ one can obtain *explicit* estimates for various classical reaction-diffusion type equations.

## REFERENCES


[1] D. G. Aronson and L. A. Peletier, Large time behaviour of solutions of the porous medium equation in bounded domains, J. Diff. Eqns., 39 (1981), 378–412.

[2] M. V. Bartuccelli, J. D. Gibbon and M. Oliver, Length scales in solutions of the complex Ginzburg-Landau equation, Phys. D, 89 (1996), 267–286.

[3] M. V. Bartuccelli, S. A. Gourley and A. A. Ilyin, Positivity and the attractor dimension in a fourth order reaction-diffusion equation, Proc. R. Soc. London A, 458 (2002), 1431–1446.

[4] M. V. Bartuccelli, S. A. Gourley and C. J. Woolcock, Length scales in solutions of a generalized diffusion model, Phys. Scripta, 57 (1998), 9–19.

[5] F. Bernis and A. Friedman, Higher order nonlinear degenerate parabolic equations, J. Diff. Eqns., 83 (1990), 179–206.

[6] P. M. Bleher, J. L. Lebowitz and E. R. Speer, Existence and positivity of solutions of a fourth-order nonlinear PDE describing internal fluctuations, Comm. Pure Appl. Math., 47 (1994), 923–942.

[7] J. A. Carrillo, A. Jüngel, S. Tang, Positive Entropic Schmes for a Nonlinear Fourth-Order Parabolic Equation, Discr. Cont. Dyn. Sys., 3 (2003), 1–20.

[8] D. S. Cohen and J. D. Murray, A generalized diffusion model for growth and dispersal in a population, J. Math. Biol., 12 (1981), 321–347.

[9] R. Dal Passo, H. Garcke and G. Grun, On a fourth-order degenerate parabolic equation: global entropy estimates, existence and qualitative behaviour of solutions, SIAM J. Math. Anal., 29 (1988), 321–342.
[10] C. R. Doering and J. D. Gibbon, "Applied analysis of the Navier-Stokes equations," Cambridge University Press, Cambridge, 1995.
[11] E. Gagliardo, Ulteriori proprieta' di alcune classi di funzioni in piu variabili, Ricerche Mat., 8 (1959), 24–51.
[12] A. A. Ilyin, Best constants in multiplicative inequalities for sup-norms, J. London Math. Soc., 58 (1998), 84–96.
[13] A. N. Kolmogorov, Local structure of turbulence in an incompressible fluid at very high Reynolds number, Dokl. Akad. Nauk USSR, 30 (1941), 299–303.
[14] L. Nirenberg, On elliptic partial differential equations, Ann. Scuola Norm. Sup. Pisa, 13 (1959), 115–162.
[15] L. A. Peletier and W. C. Troy, "Spatial patterns: higher order models in physics and mechanics," Birkhäuser, Boston, 2001.
[16] Y. Pomeau and P. Manneville, Wavelength selection in cellular flows, Phys. Lett. A, 75 (1980), 296–298.
[17] M. H. Protter and H. F. Weinberger, "Maximum principles in differential equations," Springer Verlag, New York, 1984.
[18] J. B. Swift and P. C. Hohenberg, Hydrodynamic fluctuations at the convective instability, Phys. Rev. A, 15 (1977), 319–328.
[19] H. Zhao, Large Time Decay Estimates of Solutions of Nonlinear Parabolic Equations, Discr. Cont. Dyn. Sys., 8 (2002), 69–114.

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