Positivity and the attractor dimension in a fourth order reaction-diffusion equation

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Abstract: In this paper we investigate the semilinear partial differential equation

$$u_t = -\alpha u_{xxxx} - u_{xx} + u(1 - u^2)$$

with a view particularly to obtaining some insight into how one might establish positivity preservation results for equations containing fourth order spatial derivatives. The maximum principle cannot be applied to such equations. However, progress can be made by employing some very recent 'best possible' interpolation inequalities due to A.A. Ilyin, in which the interpolation constants are both explicitly known and sharp. These are used to estimate the L^∞ distance between u and 1 during the evolution. A positivity preservation result can be obtained under certain restrictions on the initial datum. We also establish an explicit two-sided estimate for the fractal dimension of the attractor which is sharp in terms of the physical parameters.

Short title: Fourth order reaction-diffusion equation.

Keywords: Interpolation Inequalities, Positivity, Attractors, Fractal Dimension.

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1 Introduction

Partial differential equations are frequently used as mathematical models for the evolution of some physical or biological quantity which is positive, such as a population. A realistic mathematical model for such a process should have a positivity preserving property, i.e., if the solution starts positive, it should stay positive. For example, in many of the mathematical models used in population dynamics the highest order spatial derivative term is a Laplacian term. Establishing a positivity preservation result for such a model equation or system is usually a matter of applying the maximum principle (Protter & Weinberger 1984) or some related result such as invariant set theory for reaction-diffusion systems (Grindrod 1991).

However, the mathematical modelling of some processes involves the use of higher order spatial derivatives and in such cases the maximum principle cannot be used. In such cases positivity preservation is extremely difficult to establish and, in fact, may not hold for arbitrary positive initial data. However, one example for which positivity preservation is believed to hold is the equation

$$w_t = -w_{xxxx} + \left(\frac{w_x^2}{w}\right)_{xx} \tag{1.1}$$

under periodic boundary conditions, which arises in the study of interface fluctuations and has been considered by Bleher et al (1994). Actually, these authors were not able to rigorously establish positivity preservation for arbitrary positive initial data, but they did establish a sharp connection between positivity preservation and global existence. They argued that positivity preservation probably does hold since (1.1) is the scaling limit of a positivity preserving discrete recursion. Also, they were able to show that solutions do remain positive under various restrictions on the initial data.

Other important contributions to the study of positivity in high order equations are described in Bernis & Friedman (1990), Dal Passo et al (1988) and Bartuccelli et al (1998a,b) and the references therein.

In general, fourth order equations do not exhibit positivity preservation without some restriction on the initial data beyond the requirement that it be positive. And, even then, the establishment of really useful positivity theorems for such equations is a deep and challenging problem. The same applies to proving results on global convergence. In reaction-diffusion equations with only the Laplacian term, comparison theorems are frequently used for this purpose, but these rely on the maximum principle.

In this paper we shall discuss global convergence, positivity and the fractal dimension of the attractor for the partial differential equation

$$u_t = -\alpha u_{xxxx} - u_{xx} + u(1 - u^2) \tag{1.2}$$

for $x \in [0, L]$, $t > 0$, subject to the initial condition $u(x, 0) = u_0(x)$ and periodic boundary conditions at the ends $x = 0, L$. The parameter α is positive. Equation (1.2) is a generalisation of the well-known KPP equation (Peletier & Troy 1996, Ashwin et al 2001) which is used in population dynamics. The ideas to be described in this paper should also be applicable to other equations with high order spatial derivatives and with polynomial nonlinearities. For example, in Bartuccelli et al (1998a,b) some progress is made but in these papers positivity is only established for sufficiently large times t, leaving open the possibility that the solution may go negative during the transients. In the present paper our positivity result is for all times t.

Our strategy for studying positivity involves centering the equation on the uniform steady state solution $u \equiv 1$ and then showing that, under some restrictions on the initial data and the parameters α and L, solutions of the transformed equation are bounded, in absolute value, by 1. To show this we use ideas of energy methods together with some very recent interpolation inequalities due to Ilyin (1998), in which the interpolation constants are explicitly known and are the best possible. This strategy can also be employed to show (under certain other conditions) that solutions of (1.2) converge to $u = 1$, uniformly for $x \in [0, L]$.

We introduce $v(x, t)$, defined by

$$u(x, t) = 1 + v(x, t) \tag{1.3}$$

where u satisfies (1.2). If we can show that $v(x, t)$ satisfies

$$\|v(\cdot, t)\|_\infty \leq 1 \qquad \text{for all } t \tag{1.4}$$

where

$$\|v(\cdot\,, t)\|_\infty = \sup_{x \in [0,L]} |v(x, t)|$$

then $u(\cdot, t)$ is a nonnegative function for all t. If we can show that $\|v(\cdot, t)\|_\infty \to 0$ as $t \to \infty$ then we have uniform convergence.

2 Convergence

In this section we establish a convergence result for (1.2). A linearised analysis shows that the $u = 1$ solution of (1.2) is unstable when $\alpha < 1/8$ and the domain length L is sufficiently large. For (nonlinear) stability of $u = 1$, the condition on α that emerges from our analysis is $\alpha > 1/4$, as we shall show. We introduce the time-dependent quantities

$$J_n := \left\| \frac{\partial^n v}{\partial x^n} \right\|_2^2 = \int_0^L \left(\frac{\partial^n v}{\partial x^n} \right)^2 \mathrm{d}x \tag{2.1}$$

where v denotes the function defined by (1.3).

Substituting (1.3) into (1.2) we obtain for v the equation

$$v_t = -\,\alpha\, v_{xxxx} - v_{xx} - 2v - 3v^2 - v^3. \tag{2.2}$$

Our aim is now to find conditions which will ensure that $\|v\|_\infty \to 0$ as $t \to \infty$ which establishes uniform convergence of solutions of (1.2) to the $u = 1$ solution thereof.

We first obtain appropriate estimates for the L^2-norm of the solution of (2.2) and also of its derivative $\partial v/\partial x$. Using these estimates together with a new interpolation inequality, we then estimate $\|v\|_\infty$ in a way that only involves the parameters of our equation and an interpolation constant whose value is explicitly known.

We start our analysis by investigating the evolution of the L^2-norm of the solution v of (2.2), namely J_0. Differentiating J_0 with respect to time and inserting the right hand side of (2.2) gives

$$\frac{1}{2}\dot{J}_0 = -\alpha J_2 + J_1 - 2J_0 - 3\int_0^L v^3\,\mathrm{d}x - \int_0^L v^4\,\mathrm{d}x \tag{2.3}$$

where the dot denotes differentiation with respect to time. The $-\alpha J_2$ and the J_1 terms have been obtained by integration by parts (notice that the integrated terms vanish since the boundary conditions are periodic).

By using

$$J_1 \le J_2^{1/2} J_0^{1/2} \le \frac{\alpha}{2} J_2 + \frac{1}{2\alpha} J_0$$

and

$$-3\int_0^L v^3\,\mathrm{d}x \le 3\|v\|_\infty \int_0^L v^2\,dx = 3\|v\|_\infty J_0,$$

equation (2.3) becomes

$$\frac{1}{2}\dot{J}_0 \le -\frac{\alpha}{2} J_2 - \left(2 - \frac{1}{2\alpha}\right) J_0 + 3\|v\|_\infty J_0 - \int_0^L v^4\,\mathrm{d}x. \tag{2.4}$$

Next, the term $\|v\|_\infty$ has to be estimated. To do this we shall use the following sharp interpolation inequality which is discussed in the Appendix (see (5.4), with $l = 2$):

$$\|v\|_\infty \le c\, J_2(v)^{\frac{1}{8}} J_0(v)^{\frac{3}{8}} + L^{-\frac{1}{2}} J_0(v)^{\frac{1}{2}}, \qquad c = \left(\frac{4}{27}\right)^{\frac{1}{8}}. \tag{2.5}$$

Also, by the Cauchy-Schwarz inequality,

$$-\int_0^L v^4\,\mathrm{d}x \le -J_0^2 L^{-1}.$$

Using these estimates in (2.4) yields

$$\frac{1}{2}\dot{J}_0 \le -\frac{\alpha}{2}J_2 - \left(2-\frac{1}{2\alpha}\right)J_0 + 3cJ_2^{\frac{1}{8}}J_0^{\frac{11}{8}} + 3L^{-\frac{1}{2}}J_0^{\frac{3}{2}} - J_0^2L^{-1}. \tag{2.6}$$

Next, we use Young's inequality to estimate $3cJ_2^{\frac{1}{8}}J_0^{\frac{11}{8}}$ as follows:

$$3cJ_2^{\frac{1}{8}}J_0^{\frac{11}{8}} = \alpha^{\frac{1}{8}}J_2^{\frac{1}{8}}\frac{3c}{\alpha^{\frac{1}{8}}}J_0^{\frac{11}{8}} \le \frac{1}{8}\alpha J_2 + \frac{7}{8}\frac{(3c)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}}J_0^{\frac{11}{7}}$$

so

$$\begin{aligned}
\frac{1}{2}\dot{J}_0 &\le -\frac{3\alpha}{8}J_2 - \left(2-\frac{1}{2\alpha}\right)J_0 + \frac{7}{8}\frac{(3c)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}}J_0^{\frac{11}{7}} + 3L^{-\frac{1}{2}}J_0^{\frac{3}{2}} - J_0^2L^{-1} \\
&\le -\left(2-\frac{1}{2\alpha}\right)J_0 + \frac{7}{8}\frac{(3c)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}}J_0^{\frac{11}{7}} + 3L^{-\frac{1}{2}}J_0^{\frac{3}{2}} - J_0^2L^{-1} \qquad (2.7) \\
&:= f(J_0)
\end{aligned}$$

We need to study the solutions of the above differential inequality. By elementary theory they are bounded above by the solutions of the one-dimensional ODE $\dot{J}_0 = 2f(J_0)$. Note that the solutions of (2.2) certainly cannot all satisfy $v \to 0$ as $t \to \infty$ since this equation has uniform steady states $v = -1$ and $v = -2$ in addition to $v = 0$. These other states correspond to the uniform steady states $u = 0$ and $u = -1$ of (1.2). At these states we have $J_0 = L$ and $J_0 = 4L$ respectively and it is easily checked that f is positive at these values. In fact, provided $\alpha > 1/4$, the function f is negative for J_0 small, and for J_0 large, and positive in some intermediate range. A sketch of its graph is given in Fig. 1, in which we have also depicted the behaviour of the solutions of the ODE $\dot{J}_0 = 2f(J_0)$ for various initial data. We may therefore state that if $J_0(t=0) < J^*$, where J^* is the smallest positive root of $f(J) = 0$, then $J_0 \to 0$ as $t \to \infty$. Sufficient conditions to ensure this are that $J_0(t=0) < 1$ and

$$2 - \frac{1}{2\alpha} > \frac{7}{8}\frac{(3c)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}}(J_0(t=0))^{\frac{4}{7}} + 3L^{-\frac{1}{2}}(J_0(t=0))^{\frac{1}{2}}. \tag{2.8}$$

Figure 1 here

Our aim is to establish an estimate for $\|v\|_\infty$ with a view to proving that it tends to zero under some conditions. Estimating $\|v\|_\infty$ will be achieved by employing a certain inequality which will necessitate our having an estimate on J_1, where

$$J_1 = \int_0^L \left(\frac{\partial v}{\partial x}\right)^2 dx.$$

It is easily seen that the evolution equation for J_1 is

$$\frac{1}{2}\,\dot{J}_1 \;=\; -\alpha J_3 + J_2 - 2J_1 - 6\int_0^L v v_x^2\,\mathrm{d}x - 3\int_0^L v^2 v_x^2\,\mathrm{d}x \tag{2.9}$$

and, after some manipulation, this becomes

$$\frac{1}{2}\,\dot{J}_1 \;\leq\; -\frac{\alpha}{2}J_3 - \left(2 - \frac{1}{2\alpha}\right) J_1 - 6\int_0^L v v_x^2\,\mathrm{d}x - 3\int_0^L v^2 v_x^2\,\mathrm{d}x. \tag{2.10}$$

Using the Cauchy-Schwarz inequality and then Young's inequality we have

$$-6\int_0^L v v_x^2\,\mathrm{d}x \leq \left(6\int_0^L v^2 v_x^2\,\mathrm{d}x\right)^{\frac{1}{2}} (6J_1)^{\frac{1}{2}} \leq 3\int_0^L v^2 v_x^2\,\mathrm{d}x + 3J_1,$$

thus

$$\frac{1}{2}\,\dot{J}_1 \;\leq\; -\frac{\alpha}{2}J_3 + \left(1 + \frac{1}{2\alpha}\right) J_1. \tag{2.11}$$

By using the inequality $-J_3 \leq -\dfrac{J_1^3}{J_0^2}$ (see Bartuccelli et al (1993)) we have

$$\frac{1}{2}\,\dot{J}_1 \;\leq\; -\frac{\alpha}{2}\frac{J_1^3}{J_0^2} + \left(1 + \frac{1}{2\alpha}\right) J_1. \tag{2.12}$$

Taking into account that, under the previously stated conditions, $J_0 \to 0$ as $t \to \infty$, we find from (2.12) that

$$J_1(t) \leq \text{constant, for all } t \geq 0. \tag{2.13}$$

From the Appendix (see (5.4) with $l = 1$) we have that

$$\|v\|_\infty \leq J_1^{\frac{1}{4}} J_0^{\frac{1}{4}} + L^{-\frac{1}{2}} J_0^{\frac{1}{2}} \to 0 \quad \text{as } t \to \infty \tag{2.14}$$

and therefore, from (1.3), that $\lim_{t\to\infty} u(x,t) = 1$, uniformly in x.

We may summarise our findings in the following theorem, which gives a condition on the initial conditions which is sufficient for convergence. Of course, we have formulated this theorem in terms of the original state variable u. The existence of J^* is guaranteed by earlier remarks.

Theorem 1 *If $\alpha > 1/4$ and the initial data satisfies*

$$\int_0^L (u(x,0) - 1)^2\, dx < J^*,$$

where J^* is the smallest positive root of

$$f(J) := -\left(2 - \frac{1}{2\alpha}\right) J + \frac{7}{8}\frac{(3c)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}} J^{\frac{11}{7}} + 3L^{-\frac{1}{2}} J^{\frac{3}{2}} - J^2 L^{-1} = 0,$$

and $c = (4/27)^{1/8}$, then the solution $u(x,t)$ of (1.2) satisfies

$$\lim_{t \to \infty} u(x,t) = 1$$

uniformly for $x \in [0, L]$.

Corollary 1 *If the initial data satisfies*

$$\int_0^L (u(x,0) - 1)^2 \, dx < 1$$

and

$$2 - \frac{1}{2\alpha} > \frac{7}{8}\frac{(3c)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}} \left(\int_0^L (u(x,0) - 1)^2 \, dx\right)^{\frac{4}{7}} + 3L^{-\frac{1}{2}} \left(\int_0^L (u(x,0) - 1)^2 \, dx\right)^{\frac{1}{2}}$$

then the same conclusion holds.

3 Positivity of solutions

We have shown (theorem 1) that under certain restrictions on the L^2 norm of the initial datum, the solution of (1.2) converges uniformly to 1. The fact that the convergence is uniform enables us to further deduce that the solution must be positive for all t sufficiently large, but not necessarily for all t. Establishing a positivity preservation result along the lines that solutions which "start positive, stay positive" is a much more difficult problem for any fourth order problem since the maximum principle and related theorems do not apply and, furthermore, positivity preservation can only realistically be anticipated under some restriction on the initial datum.

In the previous section our strategy for proving convergence of the solution to 1 was to show that

$$\|u(\cdot,t) - 1\|_\infty \to 0 \qquad \text{as} \qquad t \to \infty,$$

and this was achieved by estimating $\|u(\cdot,t) - 1\|_\infty$ using the best known interpolation inequalities available to date. By refining this strategy further our aim in this section is to show that under certain other restrictions on the initial data one can establish that, for *all* values of t (not just as $t \to \infty$)

$$\|u(\cdot,t) - 1\|_\infty \le 1.$$

If we can establish the above inequality, then we have that $u(x,t) \geq 0$ for all x and t and, therefore, that we have positivity preservation. This strategy can be applied in principle to any other partial differential equation.

Some of the hypotheses of theorem 1 will still be needed, though we can relax some of the strict inequalities. We assume that $\alpha > 1/4$ and that

$$J_0(t=0) = \int_0^L (v(x,0))^2\,dx \leq J^*,$$

where J^* is as previously defined, and as labelled in Fig. 1. Under this assumption, we have that

$$J_0(t) \leq J^* \qquad \text{for all } t \geq 0. \tag{3.1}$$

We shall use this estimate in (2.12) to deduce that

$$\frac{1}{2}\dot{J}_1 \leq -\frac{\alpha}{2}\frac{J_1^3}{J^{*2}} + \left(1 + \frac{1}{2\alpha}\right) J_1, \tag{3.2}$$

which is an *autonomous* differential inequality, solutions of which will be bounded above by solutions of the corresponding differential equation. In particular it is easily seen that if

$$J_1(t=0) \leq J^* \sqrt{\frac{2}{\alpha} + \frac{1}{\alpha^2}}$$

then

$$J_1(t) \leq J^* \sqrt{\frac{2}{\alpha} + \frac{1}{\alpha^2}} \qquad \text{for all } t \geq 0. \tag{3.3}$$

Recall that $u - 1 = v$ and that we have the interpolation inequality

$$\|v(\cdot,t)\|_\infty \leq J_1^{\frac{1}{4}} J_0^{\frac{1}{4}} + L^{-\frac{1}{2}} J_0^{\frac{1}{2}} \tag{3.4}$$

so it is sufficient to show that the right hand side of (3.4) is bounded above by 1 for all $t \geq 0$. In view of the bounds we have for J_0 and J_1, given by (3.1) and (3.3) respectively, it is therefore sufficient to show that

$$(J^*)^{1/4}\left(\frac{2}{\alpha} + \frac{1}{\alpha^2}\right)^{1/8} (J^*)^{1/4} + L^{-1/2}(J^*)^{1/2} \leq 1$$

or, equivalently, that

$$J^* \leq \frac{1}{\left((2/\alpha + 1/\alpha^2)^{1/8} + L^{-1/2}\right)^2}. \tag{3.5}$$

Recall that J^* is defined as the smallest positive root of $f(J) = 0$, where $f(J)$ is the function defined in the statement of theorem 1. It is not immediately obvious that inequality (3.5) can be satisfied for any parameter combination. But, because of the properties of $f(J)$ (see Fig. 1), the inequality will certainly be satisfied if

$$f\left(\frac{1}{\left((2/\alpha+1/\alpha^2)^{1/8}+L^{-1/2}\right)^2}\right)\geq 0$$

and this is equivalent to

$$G(\alpha,L)\geq 0$$

where

$$G(\alpha,L)=-\frac{\left(2-\dfrac{1}{2\alpha}\right)}{\left(\left(2/\alpha+1/\alpha^2\right)^{1/8}+L^{-1/2}\right)^2}+\frac{7(3c)^{8/7}}{8\alpha^{1/7}\left(\left(2/\alpha+1/\alpha^2\right)^{1/8}+L^{-1/2}\right)^{22/7}}$$
$$+\frac{3}{\sqrt{L}\left(\left(2/\alpha+1/\alpha^2\right)^{1/8}+L^{-1/2}\right)^3}-\frac{1}{L\left(\left(2/\alpha+1/\alpha^2\right)^{1/8}+L^{-1/2}\right)^4} \tag{3.6}$$

and $c=(4/27)^{1/8}$. Extensive numerical investigations demonstrate that for any fixed value of L, $G(\alpha,L)$ is positive until α is extremely large and is then negative and tends to zero from below as $\alpha\to\infty$. Some asymptotic analysis confirms this; we can show that as $\alpha\to\infty$,

$$G(\alpha,L)\sim\frac{1}{\alpha^{1/8}}\left(-2^{1/8}L^{3/2}+\frac{7}{648}9^{6/7}(4^{1/8}27^{7/8})^{8/7}L^{11/7}\frac{1}{\alpha^{1/56}}\right).$$

Of course, we always have to assume that $\alpha>1/4$. Therefore, what we have to assure ourselves of is that for any given reasonable L, there will always be a large range of values of α for which $G(\alpha,L)\geq 0$. This is certainly the case. For example if $L=1$, either a numerical study of the equation $G(\alpha,L)=0$, or its asymptotic approximation as $\alpha\to\infty$, predicts that the value of α at which $G(\alpha,L)>0$ ceases to hold is of the order of 10^{18} !

We can now summarise our findings in the following theorem, formulated in terms of the original state variable u.

Theorem 2 *Let $\alpha>1/4$ and let the initial data satisfy*

$$\int_0^L (u(x,0)-1)^2\,dx\leq J^*,$$

where J^* is the smallest positive root of $f(J)=0$ where $f(J)$ is as in the statement of theorem 1. Assume that the initial data also satisfies

$$\int_0^L\left(\frac{\partial u}{\partial x}(x,0)\right)^2 dx\leq J^*\sqrt{\frac{2}{\alpha}+\frac{1}{\alpha^2}}$$

and assume that α and L are such that

$$J^* \leq \frac{1}{\left((2/\alpha + 1/\alpha^2)^{1/8} + L^{-1/2}\right)^2}.$$

Then the solution $u(x,t)$ of (1.2) satisfies $u(x,t) \geq 0$ for all $t \geq 0$ and all $x \in [0, L]$.

Remark. One could inquire whether another positivity preservation theorem could be established by following through the above methodology but using J^{**}, the larger of the two positive roots of $f(J) = 0$, rather than J^*. This leads us to an inequality like (3.5) but with J^{**} in place of J^*. But unfortunately the resulting inequality is not satisfied by any parameter combination.

4 Attractors and their dimension

It will be convenient for us to rewrite equation (1.2) as

$$u_t = -\alpha u_{xxxx} - \beta u_{xx} + \lambda u - u^3, \quad u(0) = u_0. \tag{4.1}$$

Following Babin & Vishik (1989) (see section I.5) we can easily show that for each $u_0 \in H$, where $H = L_2(0, L)$, and $T > 0$ the problem (4.1) has a unique solution $u \in C([0,T]; H)$ and the corresponding semigroup $S_t u_0 = u(t)$ has a global attractor $\mathcal{A} \subset \subset H$. In this section we derive a two-sided estimate of the fractal dimension of it. (See the necessary definitions in Babin & Vishik (1989), Temam (1997) and Chepyzhov & Ilyin (2001)). See also Constantin & Foias (1985) and Constantin et al (1988) where attractor dimensions are computed in other contexts.

It can be shown (see Babin & Vishik (1989)) that the semigroup $S_t u$ is uniformly (quasi)differentiable on $\mathcal{A}$ in H and the differential is the solution operator of the first variation equation

$$U_t = -\alpha U_{xxxx} - \beta U_{xx} + \lambda U - 3u(t)^2 U \equiv \mathcal{L}(u(t)) \cdot U \tag{4.2}$$

corresponding to (4.1). Here $u(t) = S_t u_0$.

We recall from Temam (1997) that the sums of the first m global Lyapunov exponents (which we denote by $q(m)$) for the attractor $\mathcal{A}$ satisfy the estimate

$$q(m) \leq \limsup_{t\to\infty} \sup_{u_0 \in X} \sup_{\substack{\xi_i \in H \\ i=1,\dots m}} \left(\frac{1}{t} \int_0^t \operatorname{Tr} \mathcal{L}(S_\tau u_0) \circ Q_m(\tau) d\tau \right), \tag{4.3}$$

where $Q_m(t)$ is the orthogonal projection in H onto $\operatorname{Span}(U_1(\tau), \dots, U_m(\tau))$. Here the U_j are the solutions of the problem (4.2) with $U_j(0) = \xi_j$.

Therefore we have

$$\begin{aligned}\operatorname{Tr}\mathcal{L}(S_tu_0)\circ Q_m(t) = -\alpha\sum_{i=1}^m(\Delta^2\varphi_i,\varphi_i) - \beta\sum_{i=1}^m(\Delta\varphi_i,\varphi_i) + \\ \lambda\sum_{i=1}^m(\varphi_i,\varphi_i) - 3u(t)^2\sum_{i=1}^m(\varphi_i,\varphi_i) \le \\ -\alpha\sum_{i=1}^m\|\Delta\varphi_i\|^2 + \beta\sum_{i=1}^m\|\nabla\varphi_i\|^2 + \lambda m,\end{aligned} \tag{4.4}$$

where the $\varphi_i = \varphi_i(t)$ are an orthonormal basis in $\operatorname{Span}(U_1(t),\dots,U_m(t))$.

Since $\|\nabla\varphi_i\|^2 \le \|\varphi_i\|\,\|\Delta\,\varphi_i\| \le \varepsilon\|\Delta\,\varphi_i\|^2 + (4\varepsilon)^{-1}\|\varphi_i\|^2$ we set $\varepsilon = \alpha/(2\beta)$ and find that

$$\operatorname{Tr}\mathcal{L}(S_tu_0)\circ Q_m(t) \le \frac{\alpha}{2}\sum_{i=1}^m\|\Delta\varphi_i\|^2 + \left(\frac{\beta^2}{2\alpha}+\lambda\right)m.$$

Next, in view of the variational principle (see Temam (1997), Lemma VI.2.1),

$$\sum_{i=1}^m\|\Delta\,\varphi_i\|^2 \ge \sum_{i=1}^m\lambda_i, \tag{4.5}$$

where the λ_i are the eigenvalues of the operator Δ^2 in H written in non-decreasing order. Obviously,

$$\{\lambda_i\}_{i=1}^\infty = 0 \cup \left\{(2\pi/L)^4\,i^4, (2\pi/L)^4\,i^4\right\}_{i=1}^\infty . \tag{4.6}$$

Therefore if we write $m-1$ as $m-1=2k+p$, where $p=0,1$, we find that

$$\sum_{i=1}^m\lambda_i = \left(\frac{2\pi}{L}\right)^4\left(p\sum_{i=1}^{k+1}i^4 + (2-p)\sum_{i=1}^k i^4\right) \ge$$

$$\left(\frac{2\pi}{L}\right)^4\frac{1}{5}\left(p(k+1)^5+(2-p)k^5\right) = \left(\frac{2\pi}{L}\right)^4\frac{2}{5}\left(\frac{p}{2}(k+1)^5+\frac{2-p}{2}k^5\right) \ge$$

$$\left(\frac{2\pi}{L}\right)^4\frac{2}{5}\left(k+\frac{p}{2}\right)^5 = \frac{1}{5}\left(\frac{\pi}{L}\right)^4(m-1)^5,$$

where we used the inequality $\sum_{i=1}^k i^4 \ge \frac{1}{5}k^5$ and the fact that the function $k\to k^5$ is convex.

Hence the numbers $q(m)$ (see (4.3)) satisfy the following estimate:

$$q(m) \le f(m) = -\frac{\alpha}{10}\left(\frac{\pi}{L}\right)^4(m-1)^5 + \left(\frac{\beta^2}{2\alpha}+\lambda\right)m\,. \tag{4.7}$$

The function $f(m)$ is concave. Therefore, in view of the estimate of the fractal dimension established in Chepyzhov & Ilyin (2001) we have

$$\dim_F\mathcal{A} \le m_0, \tag{4.8}$$

where m_0 is the unique root of the equation $f(m) = 0$. (The condition that $f(m)$ be concave is not necessary; see Ilyin (2000)). However, the root of the equation

$$-(m-1)^5 + Cm = 0, \quad C \gg 1$$

has the following asymptotics

$$m_0 = C^{\frac{1}{4}} + \frac{5}{4} + \cdots,$$

and for all $C > 0$ we have $m_0 < C^{\frac{1}{4}} + \frac{5}{4}$. Therefore, by (4.8),

$$\dim_F \mathcal{A} \leq 10^{\frac{1}{4}} \frac{L}{\pi} \left(\frac{\beta^2}{2\alpha^2} + \frac{\lambda}{\alpha} \right)^{\frac{1}{4}} + \frac{5}{4}. \tag{4.9}$$

We now give a lower bound for the dimension of the attractor. We consider the stationary solution $u = 0$ and the corresponding linearized operator

$$A\,u = -\alpha u_{xxxx} - \beta u_{xx} + \lambda u. \tag{4.10}$$

We shall find the number of unstable (positive) eigenvalues of A. Then this number is equal to the dimension of the unstable manifold of equation (4.1) passing through $u = 0$. Since this unstable manifold is contained in the attractor (Babin & Vishik 1989, section VI.4), we obtain the required lower bound. Since the spectrum of the operator A is known explicitly (see (4.6)), we see that the dimension of the unstable manifold is $2K + 1$, where K is the number of positive integers k satisfying the inequality

$$-\alpha \left(\frac{2\pi}{L} \right)^4 k^4 + \beta \left(\frac{2\pi}{L} \right)^2 k^2 + \lambda > 0. \tag{4.11}$$

Therefore

$$2K + 1 = 2 \left[\frac{L}{2\pi} \left(\frac{\beta}{2\alpha} + \sqrt{\frac{\beta^2}{4\alpha^2} + \frac{\lambda}{\alpha}} \right)^{\frac{1}{2}} \right] + 1, \tag{4.12}$$

where $[x]$ is the number of positive integers strictly less than x.

Combining (4.9) and (4.12) we finally obtain the following result.

Theorem 2. *Equation* (1.2) *with periodic boundary conditions has a global attractor whose fractal dimension satisfies the following two-sided estimate:*

$$2 \left[\frac{L}{2\pi} \left(\frac{\beta}{2\alpha} + \sqrt{\frac{\beta^2}{4\alpha^2} + \frac{\lambda}{\alpha}} \right)^{\frac{1}{2}} \right] + 1 \leq \dim_F \mathcal{A} \leq 10^{\frac{1}{4}} \frac{L}{\pi} \left(\frac{\beta^2}{2\alpha^2} + \frac{\lambda}{\alpha} \right)^{\frac{1}{4}} + \frac{5}{4}. \tag{4.13}$$

Remark 1. We observe that estimate (4.13) is sharp in terms of the physical parameters in the sense that for all positive α, β, and λ the following inequality holds:

$$2^{\frac{1}{4}} \geq \frac{\left(\frac{\beta}{2\alpha} + \sqrt{\frac{\beta^2}{4\alpha^2} + \frac{\lambda}{\alpha}} \right)^{\frac{1}{2}}}{\left(\frac{\beta^2}{2\alpha^2} + \frac{\lambda}{\alpha} \right)^{\frac{1}{4}}} \geq 1.$$

5 Appendix

In Ilyin (1998) the following inequality was proved:

$$\|u\|_\infty \le c(l) J_0(u)^{\frac{2l-1}{4l}} J_l(u)^{\frac{1}{4l}}, \quad l > \frac{1}{2}, \tag{5.1}$$

where

$$c(l) = \left(\frac{1}{(2l-1)^{\frac{2l-1}{2l}} \sin \frac{\pi}{2l}} \right)^{\frac{1}{2}}, \tag{5.2}$$

and u is a periodic function on $[0, L]$ with zero average:

$$\int_0^L u(x)\, dx = 0. \tag{5.3}$$

It was also shown in Ilyin (1998) that the constant $c(l)$ in (5.1) is sharp and there are no non-trivial extremal functions for which inequality (5.1) becomes an equality.

Suppose now that $\int_0^L u(x)\, dx \neq 0$ and $u(x) = \bar{u} + u'(x)$, where $\bar{u} = \text{const} \neq 0$ and $\int_0^L u'(x)\, dx = 0$. Then, using (5.1) and the inequality

$$|\bar{u}| = L^{-1} \left| \int_0^L u(x)\, dx \right| \le L^{-\frac{1}{2}} \|u\|,$$

we obtain

$$\|u\|_\infty \le |\bar{u}| + \|u'\|_\infty \le L^{-\frac{1}{2}} J_0(u)^{\frac{1}{2}} + c(l) J_0(u)^{\frac{2l-1}{4l}} J_l(u)^{\frac{1}{4l}}. \tag{5.4}$$

By substituting $u = 1$ in (5.4) we see that the constant $L^{-\frac{1}{2}}$ is sharp.

To see that the constant $c(l)$ in (5.4) is sharp we set $L = 2\pi$ and consider the sequence

$$u_\mu(x) = \mu^{2l-1} \sum_{n=1}^{\infty} \frac{\cos nx}{\mu^{2l} + n^{2l}}$$

and calculate all the three norms of u_μ as $\mu \to \infty$. We consider the case $l = 1$, the general case being completely analogous. For

$$u_\mu(x) = \mu \sum_{n=1}^{\infty} \frac{\cos nx}{\mu^2 + n^2}$$

we have

$$\|u_\mu\|_\infty = u_\mu(0) = \mu \sum_{n=1}^{\infty} \frac{1}{\mu^2 + n^2} = \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{1}{1 + (n/\mu)^2} \to \int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2}. \tag{5.5}$$

We calculate $J_0(u_\mu)$ and $J_1(u_\mu)$ in the similar way. We have

$$\mu J_0(u_\mu) = \mu^3\pi \sum_{n=1}^{\infty} \frac{1}{(\mu^2+n^2)^2} = \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{1}{(1+(n/\mu)^2)^2} \to \pi \int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{\pi^2}{4} \tag{5.6}$$

and

$$\mu^{-1} J_1(u_\mu) = \mu\pi \sum_{n=1}^{\infty} \frac{n^2}{(\mu^2+n^2)^2} = \frac{1}{\mu} \sum_{n=1}^{\infty} \frac{(n/\mu)^2}{(1+(n/\mu)^2)^2} \to \pi \int_0^\infty \frac{x^2dx}{(1+x^2)^2} = \frac{\pi^2}{4} \tag{5.7}$$

as $\mu \to \infty$. If we substitute asymptotic equalities (5.5), (5.6), and (5.7) into inequality (5.4), then we see that the constant $c(1) = 1$ in (5.4) is sharp.

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Figure Caption

Fig. 1. Qualitative sketch of the graph of $f(J_0)$ against J_0, showing also the behaviour of the solutions of the ODE $\dot{J}_0 = 2f(J_0)$ for various initial data.