On the Asymptotic Positivity of Solutions for the Extended Fisher-Kolmogorov Equation with Nonlinear Diffusion

Bartuccelli*
Department of Mathematics and Statistics
University of Surrey
Guildford GU2 7XH, England

Abstract

The objective of this paper aims to prove positivity of solutions for a semilinear dissipative partial differential equation with nonlinear diffusion. The equation is a generalised model of the well known Fisher-Kolmogorov equation and represents a class of dissipative partial differential equations containing differential operators of higher order than the Laplacian. It arises in a variety of meaningful physical situations including gas flows, diffusion of an electron-ion plasma and the dynamics of biological populations whose mobility is density dependent. In all these situations the solutions of the equation must be positive functions.

Keywords: Dissipative Equations, Inequalities, Ladder Methods, Positivity.

1 Introduction

An important question in the study of the behaviour of solutions of dissipative partial differential equations is whether solutions with positive initial

*E.mail: m.bartuccelli@surrey.ac.uk

data will remain positive at any later time. In dissipative equations such as reaction-diffusion equations where the highest order differential operator is the Laplacian, it is often possible to establish positivity results by using the maximum-principle[15]. However, the maximum principle does not apply in the study of dissipative partial differential equations which contain differential operators of higher order than that of the Laplacian, and, consequently, it is a very difficult problem to establish results on positivity of solutions. In this paper we shall investigate a particular class of dissipative partial differential equations which possess a uniform steady state solution. For this class of equation, by using the sharpest interpolation inequalities to date, we shall prove under a certain condition that solutions are eventually (asymptotically) positive.

Positivity of solutions is important in many fields of applied mathematics, especially in mathematical biology and in particle dynamics. In such fields the solution of the equation may represent some physical quantity that cannot be negative, so a basic requirement of any realistic model is that its solutions should remain positive for all times. There are a number of important contributions to this problem in the literature (see for example Refs. 1,8,9,10 and references therein). The present paper only establishes *asymptotic* positivity of solutions of our equation. Nevertheless achieving such a partial result constitutes real progress in an equation where maximum principles are not available.

The class of equations which we study naturally extend the result on positivity of solutions already established for a particular class of equations[4], (see also Refs. 5-7).

We shall consider the equation

$$u_t = -\alpha u_{xxxx} - u_{xx} + (u^3)_{xx} + u(1-u^2), \tag{1}$$

where α is a positive parameter, u is a scalar valued function of (x,t). The equation (1) defines a particular reaction-diffusion system, where the $(u^3)_{xx}$ term is the nonlinear diffusion and the $u(1-u^2)$ term is the reaction. For the above equation we shall use periodic boundary conditions on the spatial domain $\Omega = [0,1]$.

We define a set of quantities as follows:

$$H_0 := \|u\|_2^2 = \int u^2 \,\mathrm{d}x, \quad H_1 := \|u_x\|_2^2 = \int u_x^2 \,\mathrm{d}x, \quad \overline{\|u\|}_\infty := sup_{x\in[0,1]}|u(x)|, \tag{2}$$

where the integration is over the spatial domain $\Omega = [0,1]$.

Below we shall use Energy and Ladder methods (see Refs. 2,3,12,14,16), and interpolation inequalities[4,12,14], to obtain estimates for the above quantities, which we use for proving regularity and positivity of solutions for our equation. The reason why we are using this quantities is that it is well known[12,14] that in one spatial dimension it is sufficient to have control on the H_1 norm in order to be able to control the $\|u\|_\infty$ of the solutions, thereby giving a-priori estimates on any solution of our model. We begin with the analysis of H_0. We first differentiate it with respect to time and we then insert the right hand side of (1) obtaining,

$$\frac{1}{2}\dot{H}_0 \;=\; -\,\alpha\, H_2 + \; H_1 + \; 6\int (u^2 u_x^2)\,\mathrm{d}x + 3\int (u^3 u_{xx})\,\mathrm{d}x + \; H_0 - \int u^4 \,\mathrm{d}x \tag{3}$$

where the dot denotes differentiation with respect to time. The $-\alpha H_2$ and the H_1 terms have been obtained by integration by parts (notice that the integrated terms vanish on periodic boundary conditions).

Now we make some transformation of some of the terms in (3): we interpolate the second term on the right-hand side of (3) to get

$$H_1 \leq H_2^{1/2} H_0^{1/2} \leq \frac{1}{2}(\alpha H_2 + \frac{H_0}{\alpha}),$$

we integrate the fourth term by parts and we add the result to the third term to get $-3\int (u^2 u_x^2)\,\mathrm{d}x$, we also notice that $-\int u^4\,\mathrm{d}x \leq -H_0^2$. Therefore (3) becomes

$$\frac{1}{2}\dot{H}_0 \leq -\frac{\alpha}{2} H_2 + (1+\frac{1}{2\alpha})H_0 - H_0^2. \tag{4}$$

By ignoring the $-\dfrac{\alpha}{2} H_2$ term and by using an absorbing ball argument we finally obtain that the long time asymptotic behaviour (denoted here and throughout by an overbar) of H_0 is given by

$$\overline{H_0} \leq \frac{2\alpha+1}{2\alpha}. \tag{5}$$

As we are looking for regularity for the solutions of our model we need to have estimates on the H_1 seminorm. Similarly, differentiating it with respect to time and inserting the right hand side of (1) we obtain

$$\frac{1}{2}\dot{H}_1 = -\alpha H_3 + H_2 + 6\int u_x (u u_x^2)_x\,\mathrm{d}x + 3\int u_x (u^2 u_{xx})_x\,\mathrm{d}x + H_1 - \int u_x (u^3)_x\,\mathrm{d}x. \tag{6}$$

Computing the various derivatives one can see that (6) can be cast into

$$\frac{1}{2}\dot{H}_1 = -\alpha H_3 + H_2 + 6\int u_x^4\,\mathrm{d}x + 18\int u u_x^2 u_{xx}\mathrm{d}x \tag{7}$$

$$+3\int u^2 u_x u_{xxx}\mathrm{d}x + H_1 - 3\int u^2 u_x^2\,\mathrm{d}x. \tag{8}$$

An integration by parts on the $18\int u u_x^2 u_{xx}\mathrm{d}x$ term shows that $6\int u_x^4\,\mathrm{d}x + 18\int u u_x^2 u_{xx}\,\mathrm{d}x = 0$. Therefore (7) becomes

$$\frac{1}{2}\dot{H}_1 = -\alpha H_3 + H_2 + 3\int u^2 u_x u_{xxx}\mathrm{d}x + H_1 - 3\int u^2 u_x^2\,\mathrm{d}x. \tag{9}$$

An integration by parts on the $3\int u^2 u_x u_{xxx} \mathrm{d}x$ gives

$$3\int u^2 u_x u_{xxx} \mathrm{d}x \;=2\int u_x^4 \mathrm{d}x - 3\int u^2 u_{xx}^2 \mathrm{d}x,$$

and therefore $3\int u^2 u_x u_{xxx} \mathrm{d}x \leq 2\int u_x^4 \mathrm{d}x$. We now interpolate it by using the Gagliardo-Nirenberg inequality to obtain $2\int u_x^4 \mathrm{d}x \;\leq\; c\, H_3^{1/4} H_1^{7/4}$. The constant c appearing on the right hand side in the above inequality and in similar inequalities used below depends on the spatial dimension and the volume of integration but not on the solutions of the equation; we shall use the same letter c to denote constants which may assume occasionally different values.

Now split the result by using Young's inequality

$$c\, H_3^{1/4} H_1^{7/4} \;\leq\; \frac{\alpha}{4} H_3 + \frac{c}{\alpha^{1/3}} H_1^{7/3}.$$

Thus (9) becomes

$$\frac{1}{2}\,\dot{H}_1 \;\leq\; -\,\alpha\, H_3 +\; H_2 +\; H_1 + \frac{\alpha}{4} H_3 + \frac{c}{\alpha^{1/3}} H_1^{7/3}. \tag{10}$$

By using $H_2 \;\leq\; H_3^{1/2} H_1^{1/2}$ we arrive at

$$\frac{1}{2}\,\dot{H}_1 \;\leq\; -\,\frac{\alpha}{4}\, H_3 + (1+\frac{1}{2\alpha}) H_1 + \frac{c}{\alpha^{1/3}} H_1^{7/3}. \tag{11}$$

Finally by using an absorbing ball argument one can see that to leading orders the long time asymptotic behaviour of H_1 is given by

$$\overline{H_1} \;\leq\; c\,\frac{\overline{H_0^3}}{\alpha^2}. \tag{12}$$

By using the estimate we found in (5) for $\overline{H_0}$ we have

$$\overline{H_1} \;\leq\; \frac{c}{\alpha^2}\left(\frac{2\alpha}{2\alpha+1}\right)^3. \tag{13}$$

Using the Gagliardo-Nirenberg inequality[12,14] for the $\|u\|_\infty := sup_{x\in[0,1]}|u(x)|$ norm of the solution, namely $\|u\|_\infty^2 \le cH_1^{\frac{1}{2}}H_0^{\frac{1}{2}} + H_0$ we can see that the asymptotic behaviour of the solutions of our model obeys the estimates

$$\overline{\|u\|}_\infty^2 \le \frac{c}{\alpha}\left(\frac{2\alpha}{2\alpha+1}\right)^2 + \frac{2\alpha}{2\alpha+1}. \tag{14}$$

2 Positivity of Solutions

In this section we are going to show that solutions of our model, with appropriate initial data, will eventually be positive (positive for all t exceeding some *finite* value). Our strategy first involves centering the equation on the uniform steady state solution $u \equiv 1$ and then showing that for all t sufficiently large solutions of the transformed equation are bounded (in absolute value) by unity. We therefore introduce $v(x,t)$, defined by

$$u(x,t) = 1 + v(x,t) \tag{15}$$

where u satisfies (1). If the function $v(x,t)$ satisfies

$$\|v\|_\infty \le 1$$

then clearly u is a non-negative function. Substituting (15) into (1) we obtain for v the equation

$$v_t = -\,\alpha\, v_{xxxx} + 2v_{xx} + 6v_x^2 + 6vv_x^2 + 3v^2v_{xx} + 6vv_{xx} - 2v - 3v^2 - v^3. \tag{16}$$

We need to find conditions which will ensure that $\overline{\|v\|}_\infty \le 1$. This is sufficient to ensure asymptotic positivity. Our strategy is to first obtain appropriate estimates for equation (16), and then to use these estimate to compute $\overline{\|v\|_\infty}$ such that $\overline{\|v\|_\infty} \le 1$ is satisfied. We start by defining quantities similar to the H_0 and H_1, namely

$$J_0 := \|v\|_2^2 = \int v^2\,\mathrm{d}x, \quad J_1 := \|v_x\|_2^2 = \int v_x^2\,\mathrm{d}x, \quad \|v\|_\infty := sup_{x\in[0,1]}|v(x)|, \tag{17}$$

where, as above, the integration is over the spatial domain $\Omega := [0,1]$. Using the same strategy previously applied to H_0 and H_1, we obtain

$$\frac{1}{2}\,\dot{J}_0 = \int v\left[-\,\alpha\,v_{xxxx} + 2v_{xx} + 6v_x^2 + 6vv_x^2 + 3v^2v_{xx} + 6vv_{xx} - 2v - 3v^2 - v^3\right]\,\mathrm{d}x. \tag{18}$$

Performing similar arguments used above we finally obtain

$$\frac{1}{2}\,\dot{J}_0 \;\le\; -\frac{\alpha}{2}J_2 - 2J_1 - 2J_0 - 3\int v^3\,\mathrm{d}x - \left(1 - \frac{9}{2\alpha}\right)\int v^4\,\mathrm{d}x. \tag{19}$$

We want the last term to be negative definite and therefore we shall assume that $\alpha > \frac{9}{2}$. By using $J_1 \le J_2^{1/2}J_0^{1/2}$ and $-3\int v^3\,\mathrm{d}x \le 3\|v\|_\infty J_0$ we arrive at

$$\frac{1}{2}\,\dot{J}_0 \;\le\; -\frac{\alpha}{2}J_2 - 2J_1 - 2J_0 + 3\|v\|_\infty J_0 - \left(1 - \frac{9}{2\alpha}\right)\int v^4\,\mathrm{d}x. \tag{20}$$

In order to manipulate the term $\|v\|_\infty$ we make use of the following sharp interpolation inequality[13]:

$$\|v\|_\infty \le\; \beta\,J_2(v)^{\frac{1}{8}}J_0(v)^{\frac{3}{8}} + J_0(v)^{\frac{1}{2}}, \qquad \beta = \left(\frac{4}{27}\right)^{\frac{1}{8}}. \tag{21}$$

Also, by the Cauchy-Schwarz inequality we have that $-\int v^4\,\mathrm{d}x \;\le\; -J_0^2$. Substituting into (20) we obtain

$$\frac{1}{2}\,\dot{J}_0 \;\le\; -\frac{\alpha}{2}J_2 - 2J_1 - 2J_0 + 3\beta J_2^{\frac{1}{8}}J_0^{\frac{11}{8}} + 3J_0^{\frac{3}{2}} - \left(1 - \frac{9}{2\alpha}\right)J_0^2. \tag{22}$$

Now by using Young's inequality we split the $3\beta J_2^{\frac{1}{8}}J_0^{\frac{11}{8}}$ as follows;

$$3\beta J_2^{\frac{1}{8}}J_0^{\frac{11}{8}} = \alpha^{\frac{1}{8}}J_2^{\frac{1}{8}}\frac{3\beta}{\alpha^{\frac{1}{8}}}J_0^{\frac{11}{8}} \le \frac{1}{8}\alpha J_2 + \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}}J_0^{\frac{11}{7}}.$$

Therefore (20) becomes

$$\frac{1}{2}\,\dot{J}_0 \;\leq\; \frac{-3\alpha}{8}J_2 + \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}}J_0^{\frac{11}{7}} + 3J_0^{\frac{3}{2}} - J_0^2. \tag{23}$$

By neglecting the negative definite term $\dfrac{-3\alpha}{8}J_2$, we finally have

$$\frac{1}{2}\,\dot{J}_0 \;\leq\; \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}}J_0^{\frac{11}{7}} + 3J_0^{\frac{3}{2}} - \left(1 - \frac{9}{2\alpha}\right)J_0^2 := f(J_0). \tag{24}$$

We need to study the solutions of the above differential inequality. By elementary theory they are bounded above by the solutions of the one-dimensional ODE $\dot{J}_0 = 2f(J_0)$. Note that the solutions of (16) certainly cannot all satisfy $v \to 0$ since this equation has uniform steady states $v = -1$ and $v = -2$ in addition to $v = 0$. These other states correspond to the uniform steady states $u = 0$ and $u = -1$ of (1). At these states it is easily checked that f is positive. In fact, provided $\alpha > 9/2$, the function f is negative for J_0 small, and for J_0 large, and positive in some intermediate range. We may therefore conclude that if $J_0(t = 0) < J^*$, where J^* is the smallest positive root of $f(J) = 0$, then $J_0 \to 0$ as $t \to \infty$. A sufficient condition to ensure this is that

$$2 \;>\; \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}}J_0^{\frac{4}{7}} + 3J_0^{\frac{1}{2}}. \tag{25}$$

In order to estimate the $\|v\|_\infty$ of our perturbed equation (16) we need to have control on the J_1 norm as well. The evolution equation for J_1 is given by

$$\begin{aligned}\frac{1}{2}\,\dot{J}_1 \;\leq\; & -\alpha J_3 - 2J_2 - 2J_1 - 6\int v v_x^2\,\mathrm{d}x - 3\int v^2 v_x^2\,\mathrm{d}x + 18\int v_x^2 v_{xx}\,\mathrm{d}x \\ & +6\int v_x^4\,\mathrm{d}x + 18\int v v_x^2 v_{xx}\,\mathrm{d}x + 3\int v^2 v_x v_{xxx}\,\mathrm{d}x + 6\int v v_x v_{xxx}\,\mathrm{d}x. \end{aligned} \tag{26}$$

Analysing the various terms one can see that on periodic boundary conditions $18\int v_x^2 v_{xx}\,\mathrm{d}x = 0$, $6\int v_x^4\,\mathrm{d}x + 18\int v v_x^2 v_{xx}\,\mathrm{d}x = 0$. Also, by using the Cauchy-Schwarz inequality on the term $-6\int vv_x^2\,\mathrm{d}x$ and then splitting it by using a Young inequality, we obtain $-6\int vv_x^2\,\mathrm{d}x \le 3\int v^2v_x^2\,\mathrm{d}x + 3J_1$; the first term cancels out with the similar term in (26); an integration by parts on the $3\int v^2v_xv_{xxx}\,\mathrm{d}x = -3\int v^2v_{xx}^2\,\mathrm{d}x - 6\int vv_x^2v_{xx}\,\mathrm{d}x$; neglecting the negative definite term and integrating by parts the remaining term we get $-6\int vv_x^2v_{xx}\,\mathrm{d}x = 2\int v_x^4\,\mathrm{d}x$; by interpolating we finally obtain

$$2\int v_x^4\,\mathrm{d}x \le cJ_3^{\frac{5}{6}}J_0^{\frac{7}{6}} \le \frac{5}{6}\alpha J_3 + \frac{c}{6\alpha^5}J_0^7.$$

The last term in (26) can be handled as follows: first we integrate by parts obtaining $6\int vv_xv_{xxx}\,\mathrm{d}x = -6\int v_x^2v_{xx}\,\mathrm{d}x - 6\int vv_{xx}^2\,\mathrm{d}x$; the first term is zero on periodic boundary conditions; on the second term we use first a Cauchy-Schwarz inequality and an interpolation to obtain: $-6\int vv_{xx}^2\,\mathrm{d}x \le 6\left(\int v_{xx}^4\,\mathrm{d}x\right)^{\frac{1}{2}}J_0^{\frac{1}{2}} \le cJ_3^{\frac{5}{8}}J_1^{\frac{3}{8}}J_0^{\frac{1}{2}}$; by using Young's inequality we finally get:

$$-6\int vv_{xx}^2\,\mathrm{d}x \le \frac{\alpha}{8}J_3 + \frac{3c}{8}\left(\frac{5}{\alpha}\right)^{\frac{5}{3}}J_1J_0^{\frac{4}{3}}.$$

Thus (26) becomes (note that $J_1 \le J_2^{\frac{1}{2}}J_0^{\frac{1}{2}} \le 2J_2 + \frac{J_0}{8}$)

$$\frac{1}{2}\dot{J}_1 \le -\frac{\alpha}{24}J_3 - 2J_2 + 2J_2 + \frac{J_0}{8} + \frac{c\alpha^5}{6}J_0^7 + \frac{3c}{8}\left(\frac{5}{\alpha}\right)^{\frac{5}{3}}J_1J_0^{\frac{4}{3}}. \tag{27}$$

Thus to leading orders one can see that the long time asymptotic behaviour of J_1 is given by

$$\overline{J_1} \le \left(\frac{c}{\alpha}\right)^{\frac{4}{3}}\overline{J_0}^{\frac{5}{3}}. \tag{28}$$

Taking into account that $\overline{J_0} \equiv \lim_{t\to\infty} J_0(t) = 0$ we find that

$$\overline{J_1} \equiv \lim_{t\to\infty} J_1(t) = 0. \tag{29}$$

Thus we can conclude that

$$\overline{\|v\|_\infty} \equiv \lim_{t\to\infty}\|v(t)\|_\infty \le \lim_{t\to\infty}\left(J_1^{\frac{1}{4}}J_0^{\frac{1}{4}} + J_0^{\frac{1}{2}}\right) = 0. \tag{30}$$

From (15) we see that $\overline{\|u\|_\infty} \equiv \lim_{t\to\infty} u(t) = 1$ provided that $J_0(t=0) < 1$ and the condition (25) is satisfied.

We may summarise our findings in the following theorem, which gives a restriction on the initial conditions, which is sufficient for convergence and hence for asymptotic positivity:

Theorem 1 *If $\alpha > \frac{9}{2}$ and the initial data satisfies*

$$J_0(t=0) = \int_0^1 (u(x,0)-1)^2\,dx < J^*,$$

where J^* is the smallest positive root of

$$f(J_0) := -J_0^2 + \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}} J_0^{\frac{11}{7}} + 3J_0^{\frac{3}{2}} - \left(1 - \frac{9}{2\alpha}\right) J_0 = 0, \tag{31}$$

then the solution $u(x,t)$ of (1) satisfies

$$\lim_{t\to\infty} u(x,t) = 1$$

uniformly for $x \in [0,1]$.

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