Bifurcation curves of subharmonic solutions and Melnikov theory under degeneracies

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Abstract

We study perturbations of a class of analytic two-dimensional autonomous systems with perturbations depending periodically on time; for instance one can imagine a periodically driven or forced system with one degree of freedom. In the first part of the paper, we revisit a problem considered by Chow and Hale on the existence of subharmonic solutions. In the analytic setting, under more general (weaker) conditions on the perturbation, we prove their results on the bifurcation curves dividing the region of nonexistence from the region of existence of subharmonic solutions. In particular our results apply also when one has degeneracy to first order — i.e. when the subharmonic Melnikov function is identically constant. Moreover we can deal as well with the case in which degeneracy persists to arbitrarily high orders, in the sense that suitable generalisations to higher orders of the subharmonic Melnikov function are also identically constant. The bifurcation curves consist in four branches joining continuously at the origin, where each of them can have a singularity (although generically they have not). The branches can form a cusp at the origin: we say in this case that the curves are degenerate as the corresponding tangent lines coincide. The method we use is completely different from that of Chow and Hale, and it is essentially based on the proof of convergence of the perturbation theory. It also allows us to treat the Melnikov theory in degenerate cases in which the subharmonic Melnikov function is either identically vanishing or has a zero which is not simple. This is investigated at length in the second part of the paper. When the subharmonic Melnikov function has a non-simple zero, we consider explicitly the case where there exist subharmonic solutions, which, although not analytic, still admit a convergent fractional series in the perturbation parameter.

1 Introduction

Subharmonic bifurcations have been extensively studied in the literature, and are by now a standard topic of many classical textbooks [20, 41]. The problem can be formulated as follows. Consider a two-dimensional autonomous system, and suppose that it has a periodic orbit of period $T = 2\pi p/q$, where p, q are relatively prime integers. Then one can be interested in studying whether, under the action of a periodic perturbation of period 2π, some periodic solutions with period T persist. Solutions with this property are called subharmonic solutions of order q/p.

Assume also that the perturbation depends on two parameters. A typical situation is when dissipation is present in a periodically driven or forced system with one degree of freedom [43, 51]; in this case two parameters naturally arise: the magnitude of the perturbation and the damping coefficient. An interesting problem is then to study the region in the space of parameters where subharmonic solutions can occur and to determine the bifurcation curves, which divide the regions of existence and non-existence of these solutions. Such a problem has been considered for instance by Chow and Hale [20], for systems of class C^r, $r \geq 2$. They found that, under suitable assumptions on the unperturbed system (essentially a local anisochronicity condition) and on the perturbation, the bifurcation curves exist, are smooth and have distinct tangent lines at the origin. The condition on the perturbation, if one takes the magnitude of the perturbation as one of the parameters, can be formulated in terms of the so-called subharmonic Melnikov function [52, 41]. It requires in particular that this function depends explicitly on the initial phase t_0 of the unperturbed periodic solutions which persist under perturbation.

In the first part of this paper we recover the same result by Chow and Hale, in the analytic setting, and we show that the condition on the perturbation can be weakened. In particular the subharmonic Melnikov function can be independent of t_0. As a consequence the bifurcation curves can be degenerate, in the sense that the bifurcation curves consist in two pair of branches with the same tangent at the origin, where they form a cusp. Moreover, in general, the branches are not analytic (they can even fail to be differentiable to arbitrary orders). Only if some further assumption is made do they turn out to be analytic.

In the case of dissipative systems in the presence of forcing, such as those studied by Hale and Táboas [43, 20], our result is significantly stronger as it requires no assumption at all on the periodic perturbation. In particular we find the following result in the analytic setting. Given any one-dimensional anisochronous mechanical system perturbed by a periodic forcing of magnitude ε and in the presence of dissipation, there can be analytic subharmonic solutions of order q/p only if the dissipation coefficient γ is below a threshold value $\gamma_0(q/p, \varepsilon)$. Here we show that for any rational value p/q there is an integer exponent $m = m(q/p) \in \mathbb{R}^*$ such that $\gamma_0(q/p, \varepsilon) = O(\varepsilon^m)$. This can be related, in a more general context, to a conjecture proposed in [1]. Moreover the case $m(p/q) = \infty$ corresponds to infinitely many cancellations, one at each perturbation order, which makes such a case very unlikely. Therefore, up to these exceptional cases, we can say that any resonant torus with frequency commensurate with the frequency of the forcing term admits subharmonic solutions of the corresponding order, provided the damping is small enough (below a threshold depending on the frequency). In other words, existence of any subharmonic solutions holds without making any assumption on the periodic perturbation, other than analyticity. Note that this is not a genericity result; we shall come back to this later on.

Our method is completely different from both that of Chow and Hale and the singularity theory approach [39, 40]. It is based on perturbation theory; in particular this requires for the system to be analytic. Chow and Hale's assumptions on the perturbation reflect a case in which a first order analysis is enough to deduce existence of subharmonic solutions. By contrast our results allow the analysis of cases in which it can be necessary to go beyond the first order, in principle to arbitrarily high orders. We also argue that in physical applications it can be essential to have such a stronger result. Indeed, in a concrete example in which, for instance, the perturbation is a trigonometric polynomial, Chow and Hale's assumptions on the perturbation, even if they are generic, fail to be satisfied for most values of the periods T. For those values a first order condition is not sufficient to detect the existence of the subharmonic solution, and one must go to higher orders. The numerical simulations performed in [1] for a driven quartic oscillator in the presence of dissipation show that this is necessary if one wants to explain the numerical findings for some values of the parameters. A more precise description of the method we use is as follows. We study the perturbation series of the subharmonic solutions: first we find conditions sufficient for these series to be well-defined to all orders, then we prove that if the perturbation is small enough convergence of the series holds. Technically, this is achieved by using the tree formalism, which was originally introduced in the context of KAM theory by Gallavotti [24], inspired by a pioneering paper by Eliasson [22], and thereafter has been applied in a long series of papers in the same or related fields [7, 26, 29, 30, 31, 32, 36, 33, 34, 35]; see also [25] for a review. We note that with respect to these papers in our case the analysis is much easier as we deal with periodic solutions instead of quasi-periodic solutions. In this respect our analysis could be considered as a propaedeutic introduction to the tree formalism, in a case in which there is no small divisors problem, so that no multiscale analysis has to be introduced; see also [8, 9] for a similar situation.

Existence of the subharmonic solutions could be proved also through other (nowadays more conventional) methods, for instance by a simple application of the implicit function theorem to the corresponding Poincaré map; see Appendix A for a possible implementation. We prefer to rely on the tree formalism for two reasons. First, it is very flexible, as it naturally extends to more general — and technically more difficult — problems, such as those with small divisors considered in the aforementioned papers. Already in the case of subharmonic solutions, it allows a natural generalisation of the Melnikov theory to the case in which the subharmonic Melnikov function vanishes identically to first order and higher orders have to be investigated; such an issue will be discussed explicitly in the second part of the paper (see below). Second, when performing analytical or numerical computations requiring arbitrarily high accuracy, high perturbation orders have to be reached, and the easiest most direct way to proceed is just through perturbation theory: so our approach allows a unified treatment for both theoretical investigations and computational ones.

In the second part of the paper we revisit the Melnikov theory on the existence of subharmonic solutions in one-parameter real analytic systems. We shall focus on periodic orbits, but in principle our method extends also to the study of homoclinic orbits. The standard Melnikov theory usually studies the case in which the subharmonic Melnikov function has a simple (i.e. first order) zero [41]. We shall consider degenerate cases in which the subharmonic Melnikov function either vanishes identically or has a zero which is of order higher than one. In the first case one has to go to higher orders, and if a suitable higher order generalisation of the subharmonic Melnikov function has a first order zero, then one can proceed very closely to the standard case, and existence of analytic subharmonic solutions is proved. The second case is more subtle: the subharmonic solutions (if they exist at all) are not expected to be analytic in the parameter. However, we shall see that, under some assumptions on the perturbation, subharmonic solutions exist and can be studied through perturbation theory notwithstanding their lack of analyticity in the perturbation parameter. In essence, the solutions are expressed as Puiseux series (i.e. fractional series) in the perturbation parameter.

We note that even if there are a lot of studies in the literature on the Melnikov theory in the degenerate case, both for subharmonic solutions and homoclinic orbits, most of them are confined to cases where either the Melnikov function (for homoclinic orbits) or the subharmonic Melnikov function (for periodic solutions) vanishes identically and a finite — often second — order analysis is enough to settle the problem; see for instance [42, 55, 54, 28]. This corresponds to a sub-case of Theorem 8. Of course there are exceptions, such as [23, 48, 49], dealing with analysis to arbitrarily high order. The situation in which the subharmonic Melnikov function has a zero which is not simple is a more intriguing problem, as new mathematical features arise in such a case. We shall discuss explicitly this situation, by making a simplifying non-degeneracy assumption to the second order contribution of the naive perturbation theory (we refer to Section 3.1 for a more precise formulation). It would be interesting to investigate how far the assumptions on the perturbation can be relaxed in order still to have subharmonic solutions. We also note that, under the aforementioned assumption, our result is stronger than that given in [56], for two reasons: first, it applies also to the case of zeroes of even order; second, it gives more information about the change of the phase of the unperturbed periodic solution which is continued under perturbation, by making precise its dependence on the perturbation parameter (again we refer to Section 3.1 for a more detailed comparison).

The results illustrated in this paper should also be compared with [18, 19], where a different scenario, such as the persistence of the whole invariant manifold corresponding to the resonant torus, arises in a case in which the subharmonic Melnikov function vanishes identically. Our analysis shows that a situation of this kind is highly non-generic.

Our method seems to be particularly suited for degenerate cases. These cases are non-generic (generically the first order is enough to settle the problem). It could be mentioned that genericity in the real-analytic setting is somewhat more involved than in the C^r Whitney topology (see for instance [14], where theorems by Kupka and Smale are extended from the smooth to the analytic case). On the contrary, our investigations aim rather to general — not generic — results, such as the existence of subharmonic solutions with no restriction on the perturbation. Results of this kind can be relevant, because in many physical applications the perturbation is just a given function, and often is taken to be (or approximated by) a trigonometric polynomial: hence, it can be of interest working in the analytic setting. In this setting, the curves of bifurcation from the nonexistence to the existence of subharmonic solutions generically are analytic and intersect transversally (this corresponds to a first order condition which is generically satisfied). However, in general they are not analytic at the origin.

Also the cases of the Melnikov theory that we study in the second part of the paper include non-generic cases. Our final aim would be to remove any assumption on the perturbation and characterise the analyticity properties of the subharmonic solutions in the perturbation parameter *for any perturbation*: what is proved here is only a partial step in this direction, and further investigations would be highly desirable. We note that results of this kind, that is results which hold for any perturbation, are usually non-trivial; see for instance [16, 17, 45, 33, 27] for other cases.

Finally we note that bifurcation phenomena, involving domains of existence and non-existence of periodic — and also quasi-periodic — solutions in the space of parameters, have been widely investigated in the literature. For instance we could mention the work by Broer *et al.*, based on the singularity theory method. In [10] resonance tongues (where periodic orbits exist) and their boundaries (consisting in parameter values where the periodic orbits disappear) have been studied for non-degenerate and degenerate Hopf bifurcations of maps using methods of equivariant contact equivalence. The method has been applied also to the study of stable (quasi-periodic) solutions for periodically and quasi-periodically forced systems, including Hill's equation with a quasi-periodic potential, especially in the conservative case [15, 12, 13, 11]. Again the analysis is based on the application of singularity theory, after a repeated averaging procedure [38]. This provides another method to study this kind of problem with a formalism which naturally allows consideration of cases where small divisor problems arise.

The paper is organised as follows. In the first part (Section 2) we shall study the bifurcation curves of the subharmonic solutions. In Section 2.1 we state our main results. These are summarised in Theorem 1, which deals with the general situation — that is when weaker conditions are assumed on the potential, — and Theorem 2, which reproduces Chow and Hale's result under the same assumptions on the perturbation. Sections 2.2 and 2.3 are devoted to the proof of Theorems 1 and 2. More precisely, in Section 2.2 we show the existence of a subharmonic solution in the form of a formal power series, while in Section 2.3 we prove the convergence of the series. In Section 2.4, Theorem 3 provides some simple extensions of Theorem 2, while Theorem 4 deals with the minimal number of subharmonic solutions of order q. In Section 2.5 we discuss, as an application of our results, the case of a forced one-dimensional system in the presence of dissipation: this will lead to Theorems 5 and 6 which extend the results of Hale and Táboas [43].

The second part of the paper (Section 3) is devoted to the Melnikov theory for subharmonic solutions. In Section 3.1 we shall make a comparison with the standard Melnikov theory [52, 41], and formulate some other results. More precisely, Theorem 7 corresponds to the Melnikov theory usually discussed in the literature [41], while Theorem 8 — to be proved in Section 3.2 — provides an extension of the results to degenerate situations in which the subharmonic Melnikov function vanishes identically but a suitable generalisation of it still has a simple zero. Finally, Theorems 9 and 10 show the existence of subharmonic solutions in certain cases in which the subharmonic Melnikov function (or some higher generalisation of it) has a zero which is not simple. The proof of Theorem 9 will be provided in Section 3.3, while that of Theorem 10 will be discussed in Section 3.4.

2 Bifurcation curves

2.1 Statement of the main results

Consider the ordinary differential equation

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, C, t), \\ \dot{A} = \varepsilon G(\alpha, A, C, t), \end{cases} \tag{2.1}$$

where $(\alpha, A) \in \mathcal{M} := \mathbb{T} \times W$, with $W \subset \mathbb{R}$ an open set, the map $A \to \omega(A)$ is real analytic in A, and the functions F and G depend analytically on their arguments and are 2π-periodic in α and t. Finally, ε, C are two real parameters.

The time periodicity in (2.1) might suggest to take a stroboscopic map (or Poincaré map) at time T when looking for solutions of period T. This would lead to a two-dimensional diffeomorphism on the annulus (cf. Appendix A).

One could also introduce a further (analytic) dependence on ε in the functions F and G, and the forthcoming analysis could be easily performed with some trivial adaptations. Therefore all the results and theorems stated below and in the next sections hold unchanged in that case too. Then, the formulation given in [20] is recovered, as a particular case, by introducing the parameter $\gamma = \varepsilon C$, and setting $\mu = (\mu_1, \mu_2)$, with $\mu_1 = \varepsilon$ and $\mu_2 = \gamma$.

For $\varepsilon = 0$ the variable A is kept fixed at some value A_0, while α rotates with constant angular velocity $\omega(A_0)$. Hence the motion of the variables (α, A, t) is quasi-periodic, and reduces to a periodic motion whenever $\omega(A_0)$ becomes commensurate with 1. Define $\alpha_0(t) = \omega(A_0)t$ and $A_0(t) = A_0$: in the *extended phase-space* $\mathcal{M} \times \mathbb{R}$ the solution

$(\alpha_0(t), A_0(t), t+t_0)$ describes an invariant torus, which is uniquely determined by the "energy" A_0. If $\omega(A_0)$ is rational we say that the torus is *resonant*. The parameter t_0 will be called the *initial phase*: it fixes the initial datum on the torus.

As a particular case we can consider that (A, α) are canonical coordinates (action-angle coordinates), but the formulation we are giving here is more general. In particular, it applies also to non-Hamiltonian systems, such as the electric circuit discussed in [4]. In general all non-resonant tori are completely destroyed under perturbation, if no further hypotheses are made on the perturbations F, G (such as that the full system is Hamiltonian). Also the resonant tori disappear, but some remnants are left: indeed usually a finite number of periodic orbits, called *subharmonic solutions*, lying on the unperturbed torus, can survive under perturbation.

Denote by $T_0(A) = 2\pi/\omega(A)$ the period of the trajectories on an unperturbed torus, and define $\omega'(A) := \mathrm{d}\omega(A)/\mathrm{d}A$. If $\omega(A_0) = p/q \in \mathbb{Q}$, call $T = T(A_0) = 2\pi q$ the period of the trajectories in the extended phase space. We shall call q/p the *order* of the corresponding subharmonic solutions. Define

$$M(t_0, C) = \frac{1}{T} \int_0^T \mathrm{d}t \, G(\alpha_0(t), A_0, C, t+t_0), \tag{2.2}$$

which is called the *subharmonic Melnikov function*. Here and in the following we do not write explicitly the dependence of the subharmonic Melnikov function on A_0, which is fixed once and for all. Note that $M(t_0, C)$ is 2π-periodic in t_0.

We make the following assumptions on the resonant torus with energy A_0.

Hypothesis 1 *One has $\omega'(A_0) \neq 0$.*

Hypothesis 2 *There exists an analytic curve $t \to C_0(t)$ from $[0, 2\pi)$ to $\mathbb{R}$ such that $M(t_0, C_0(t_0)) = 0$ and $\partial M(t_0, C_0(t_0))/\partial C \neq 0$ for all $t_0 \in [0, 2\pi)$.*

The function $C_0(t_0)$ is also 2π-periodic in t_0. We prove the following result. We prefer to state the result in terms of the parameter $\gamma = \varepsilon C$ — instead of C — to make more transparent the relation with [20].

Theorem 1 *Consider the system (2.1) and assume that Hypotheses 1 and 2 hold for the resonant torus with energy A_0 such that $\omega(A_0) = p/q$. There exist $\varepsilon_0 > 0$ and two continuous functions $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$, with $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(\varepsilon) \geq \gamma_2(\varepsilon)$ for $\varepsilon \geq 0$ and $\gamma_1(\varepsilon) \leq \gamma_2(\varepsilon)$ for $\varepsilon \leq 0$, such that (2.1) has at least one subharmonic solution of order q/p for $\gamma_2(\varepsilon) \leq \varepsilon C \leq \gamma_1(\varepsilon)$ when $\varepsilon \in (0, \varepsilon_0)$ and for $\gamma_1(\varepsilon) \leq \varepsilon C \leq \gamma_2(\varepsilon)$ when $\varepsilon \in (-\varepsilon_0, 0)$.*

The situation is depicted in Figure 1, in a case in which the two functions γ_1 and γ_2 are analytic and intersect transversally. The graphs described by the two functions are

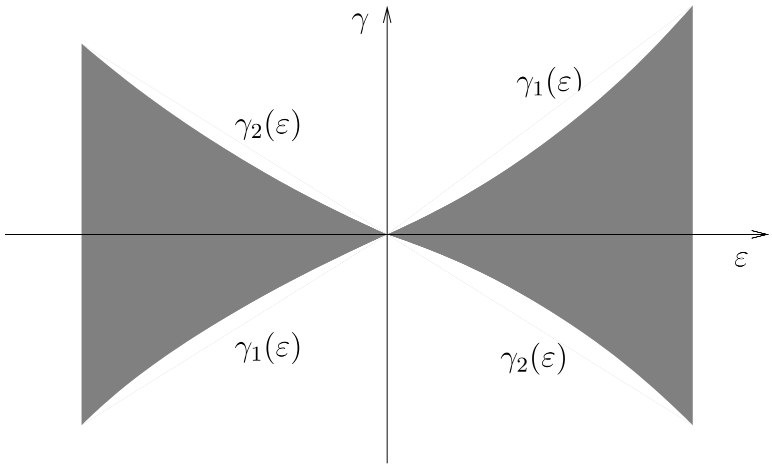

Figure 1: Set of existence (grey region) of subharmonic solutions in the plane (ε, γ), in a case in which the two bifurcation curves $\varepsilon \to \gamma_1(\varepsilon)$ and $\varepsilon \to \gamma_2(\varepsilon)$ are analytic and have different tangent lines at the origin.

called the *bifurcation curves* of the subharmonic solutions: they divide the plane into two disjoint sets such that only in one of them there are analytic subharmonic solutions.

The bifurcation curves consist in four branches joining at the origin. In general such branches are not analytic (at $\varepsilon = 0$): they are not even smooth, in the sense that they are not infinitely differentiable (at $\varepsilon = 0$). However, if some further assumptions are made on the subharmonic Melnikov function, smoothness (in fact analyticity) in ε can be obtained. Denote by $C_0'(t_0)$ and $C_0''(t_0)$ the first and second derivatives of the function $C_0(t_0)$ with respect to t_0.

Hypothesis 3 *If t_m and t_M are the values in $[0, 2\pi)$ for which the function $C_0(t_0)$ attains its minimum and its maximum, respectively, then $C_0''(t_m)C_0''(t_M) \neq 0$.*

The following result holds.

Theorem 2 *Consider the system (2.1) and assume that Hypotheses 1, 2 and 3 hold for the resonant torus with energy A_0 such that $\omega(A_0) = p/q$. There exist $\varepsilon_0 > 0$ and two functions $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$, analytic for $|\varepsilon| < \varepsilon_0$, with $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(\varepsilon) > \gamma_2(\varepsilon)$ for $\varepsilon > 0$ and $\gamma_1(\varepsilon) < \gamma_2(\varepsilon)$ for $\varepsilon < 0$, and with different tangent lines at the origin, such that (2.1) has at least one subharmonic solution of order q/p for $\gamma_2(\varepsilon) \leq \varepsilon C \leq \gamma_1(\varepsilon)$ when $\varepsilon \in (0, \varepsilon_0)$ and for $\gamma_1(\varepsilon) \leq \varepsilon C \leq \gamma_2(\varepsilon)$ when $\varepsilon \in (-\varepsilon_0, 0)$.*

Theorem 2 is analogous to Theorem 2.1 of [20], Section 11 — in the analytic setting instead of the differentiable one — while Theorem 1 requires fewer hypotheses. In particular it applies when Chow and Hale's $h_k(\alpha)$ function vanishes identically. In that case the graphs of the two functions γ_1 and γ_2 form a cusp at the origin: we refer to this situation as a case of *degenerate bifurcation curves*, see Figure 2. We shall also see in Section 2.3 that in fact, under weaker assumptions than those made in Hypothesis 3, we can find smoothness of the bifurcation curves, in the following sense: under suitable assumptions there exist two analytic functions $\widetilde{\gamma}_1(\varepsilon)$ and $\widetilde{\gamma}_2(\varepsilon)$ such that $\gamma_1(\varepsilon) = \max\{\widetilde{\gamma}_1(\varepsilon), \widetilde{\gamma}_2(\varepsilon)\}$ and $\gamma_2(\varepsilon) = \min\{\widetilde{\gamma}_1(\varepsilon), \widetilde{\gamma}_2(\varepsilon)\}$ for $\varepsilon > 0$, and $\gamma_1(\varepsilon) = \min\{\widetilde{\gamma}_1(\varepsilon), \widetilde{\gamma}_2(\varepsilon)\}$ and $\gamma_2(\varepsilon) = \max\{\widetilde{\gamma}_1(\varepsilon), \widetilde{\gamma}_2(\varepsilon)\}$ for $\varepsilon < 0$. We refer to Hypothesis 4 and Theorem 3 in Section 2.3 for a precise formulation of the results.

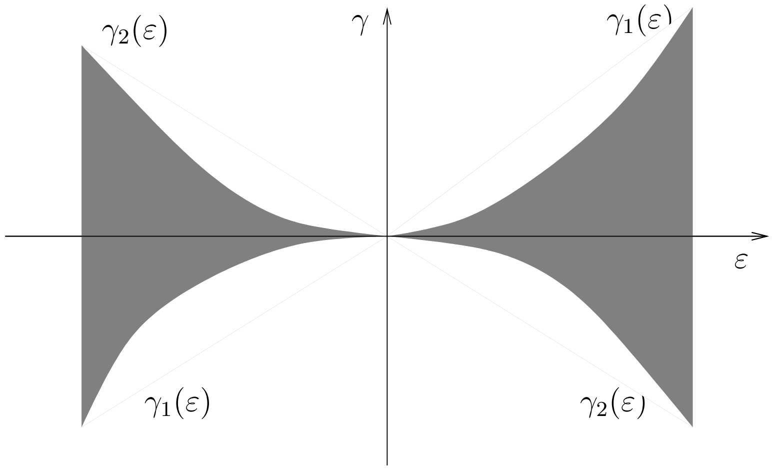

Figure 2: The bifurcation curves consist in four branches joining at the origin. In general the branches are not analytic at the origin. Furthermore, they can have the same tangent at the origin: in this case we say that the bifurcation curves are degenerate. The grey region in the figure represents a case in which the bifurcation curves have tangent lines parallel to the ε-axis.

We shall see in Section 2.3 — cf. Theorem 4 — that for $p = 1$ one has at least $2q$ subharmonic solutions of order q as far as $\min\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} < \gamma < \max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\}$ and at least q subharmonic solutions of order q when (ε, γ) belongs to one of the bifurcation curves, that is when either $\gamma = \gamma_1(\varepsilon)$ or $\gamma = \gamma_2(\varepsilon)$. This agrees with Chow and Hale's Theorem 2.1 in [20] in the cases in which the latter applies.

Possible extensions of Chow and Hale's results could be looked for in another direction, such as that of relaxing the hypotheses on the unperturbed system. This problem has been studied, for instance, in [44, 53].

The bifurcation curves studied here concern subharmonic solutions which are analytic in ε. In principle our results do not exclude existence of other subharmonic solutions which are not analytic. Indeed, one could speculate whether other periodic solutions with the same period exist for $\varepsilon \neq 0$. In the presence of dissipation, it is unlikely that solutions other than the attractive ones found with the method we have used, would be relevant for the dynamics — cf. for instance the problems investigated in [5, 2, 3, 1, 6]. In general the situation can be delicate; for instance when one investigates quasi-periodic solutions corresponding to lower-dimensional tori of quasi-integrable systems, where uniqueness becomes a subtle problem — cf. for instance [36, 27, 21]. Despite this, there are cases in which the problem can be settled — cf. [2, 33].

2.2 Existence of formal power series for the subharmonic solutions

We look for subharmonic solutions of (2.1) which are analytic in ε. First, we shall try to find solutions in the form of formal power series in ε

$$\alpha(t) = \alpha(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \alpha^{(k)}(t), \qquad A(t) = A(t;\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k A^{(k)}(t), \tag{2.3}$$

where $\alpha^{(0)}(t) = \omega(A_0)\, t$ and $A^{(0)}(t) = A_0$, with $\omega(A_0) = p/q$, and the functions $\alpha^{(k)}(t)$ and $A^{(k)}(t)$, periodic with period $T = 2\pi p$ for all $k \in \mathbb{N}$, are to be determined. We shall see that this will be possible provided the parameter C is chosen as a function of ε, again in the form of a formal power series in ε

$$C = C(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k C^{(k)}. \tag{2.4}$$

Moreover both the solution $(\alpha(t), A(t))$ and the constant C will be found to depend on the initial phase t_0: in particular one has $C(\varepsilon) = C(\varepsilon, t_0)$ such that $C(\varepsilon, t_0 + 2\pi) = C(\varepsilon, t_0)$ and $C^{(0)} = C_0(t_0)$, and, as we shall see, a sufficient condition for formal solvability to hold is that Hypotheses 1 and 2 are satisfied.

Note that this approach is typical of perturbation theory, and was followed, for instance in [50], where higher order corrections to the Melnikov function are computed, however without touching the issue of convergence of the perturbation series.

If we introduce the decompositions (2.3) and (2.4) into (2.1) and we denote with $W(t)$ the Wronskian matrix for the unperturbed linearised system, we obtain for $k \ge 1$ (cf. [1] for similar computations)

$$\begin{pmatrix} \alpha^{(k)}(t) \\ A^{(k)}(t) \end{pmatrix} = W(t) \begin{pmatrix} \bar\alpha^{(k)} \\ \bar A^{(k)} \end{pmatrix} + W(t) \int_0^t \mathrm{d}\tau\, W^{-1}(\tau) \begin{pmatrix} U^{(k)}(\tau) + F^{(k-1)}(\tau) \\ G^{(k-1)}(\tau) \end{pmatrix}, \tag{2.5}$$

where $(\bar\alpha^{(k)}, \bar A^{(k)})$ are corrections to the initial conditions, $U^{(1)}(t) = 0$,

$$\begin{aligned} U^{(k)}(t) &= [\omega(A) - \omega(A_0) - \omega'(A_0)\,(A - A_0)]^{(k)} \\ &:= \sum_{m=2}^{\infty} \frac{1}{m!} \frac{\partial^m}{\partial A^m} \omega(A_0) \sum_{\substack{k_1+\ldots+k_m=k \\ k_i \ge 1}} A^{(k_1)}(t) \ldots A^{(k_m)}(t), \end{aligned} \tag{2.6}$$

for $k \geq 2$ and

$$F^{(k)}(t) = [F(\alpha, A, C, t+t_0)]^{(k)} := \sum_{m=0}^{\infty} \sum_{\substack{r_1,r_2,r_3 \in \mathbb{Z}_+ \\ r_1+r_2+r_3=m}} \frac{\partial_1^{r_1}\partial_2^{r_2}\partial_3^{r_3}}{r_1! r_2! r_3!} F(\alpha_0(t), A_0, C_0, t+t_0)$$
$$\sum_{\substack{k_1+\ldots+k_m=k \\ k_i \geq 1}} \alpha^{(k_1)}(t) \ldots \alpha^{(k_{r_1})}(t)\, A^{(k_{r_1+1})}(t) \ldots A^{(k_{r_1+r_2})}(t)\, C^{(k_{r_1+r_2+1})} \ldots C^{(k_m)}, \quad (2.7)$$

with an analogous definition holding for $G^{(k)}(t)$, for $k \geq 1$. Here and henceforth, given a function of several arguments we are denoting by ∂_k the derivative with respect to the k-th argument; hence, given the function $F(\alpha, A, C, t+t_0)$ we have $\partial_1 F = \partial F/\partial \alpha$, $\partial_2 F = \partial F/\partial A$, and $\partial_3 F = \partial F/\partial C$. Note that by construction both $F^{(k)}(t)$ and $G^{(k)}(t)$ depend only on the coefficients $\alpha^{(k')}(t)$, $A^{(k')}(t)$ and $C^{(k')}$ with $k' \leq k$, while $U^{(k)}(t)$ depend only on the coefficients with $k' < k$.

The Wronskian matrix appearing in (2.5) can be written as

$$W(t) = \begin{pmatrix} 1 & \omega'(A_0)t \\ 0 & 1 \end{pmatrix}. \quad (2.8)$$

By using (2.8) in (2.5) we have

$$\begin{aligned} \alpha^{(k)}(t) &= \bar{\alpha}^{(k)} + t\,\omega'(A_0)\,\bar{A}^{(k)} + \int_0^t \mathrm{d}\tau\, \Phi^{(k-1)}(\tau) + \omega'(A_0) \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' G^{(k-1)}(\tau'), \\ A^{(k)}(t) &= \bar{A}^{(k)} + \int_0^t \mathrm{d}\tau\, G^{(k-1)}(\tau), \end{aligned} \quad (2.9)$$

where $\Phi^{(k-1)}(t) = U^{(k)}(t) + F^{(k-1)}(t)$ depends only on the coefficients $\alpha^{(k')}(t)$, $A^{(k')}(t)$ and $C^{(k')}$ with $k' \leq k-1$

We obtain a periodic solution of period T if, to any order $k \in \mathbb{N}$, one has

$$\langle G^{(k-1)} \rangle := \frac{1}{T} \int_0^T \mathrm{d}\tau\, G^{(k-1)}(\tau) = 0 \quad (2.10)$$

and

$$\omega'(A_0)\,\bar{A}^{(k)} + \langle \Phi^{(k-1)} \rangle + \omega'(A_0) \langle \mathcal{G}^{(k-1)} \rangle = 0, \qquad \mathcal{G}^{(k-1)}(t) = \int_0^t \mathrm{d}\tau\, G^{(k-1)}(\tau), \quad (2.11)$$

where, given any T-periodic function H we denote by $\langle H \rangle$ its mean, as done in (2.10).

The parameters $\bar{\alpha}^{(k)}$ are left undetermined, and we can fix them arbitrarily, as we have the initial phase t_0 which is still a free parameter. For instance we can set $\bar{\alpha}^{(k)} = 0$ for all $k \in \mathbb{N}$ or else we can define $\bar{\alpha}^{(k)} = \alpha_k(t_0)$ for $k \in \mathbb{N}$, with the constants $\alpha_k(t_0)$ to be fixed in the way which turns out to be more convenient for computations: we shall see a reasonable choice in the next Section.

Therefore, if equation (2.10) is satisfied, we have

$$\begin{aligned} \alpha^{(k)}(t) &= \bar{\alpha}^{(k)} + \int_0^t \mathrm{d}\tau \left(\Phi^{(k-1)}(\tau) - \langle \Phi^{(k-1)} \rangle\right) + \omega'(A_0) \int_0^t \mathrm{d}\tau \left(\mathcal{G}^{(k-1)}(\tau) - \langle \mathcal{G}^{(k-1)} \rangle\right), \\ A^{(k)}(t) &= \bar{A}^{(k)} + \mathcal{G}^{(k-1)}(t), \end{aligned} \tag{2.12}$$

with

$$\bar{A}^{(k)} = -\frac{\langle \Phi^{(k-1)} \rangle}{\omega'(A_0)} - \langle \mathcal{G}^{(k-1)} \rangle, \tag{2.13}$$

which is well-defined as $\omega'(A_0) \neq 0$ by Hypothesis 1.

So, in order to prove the formal solvability of (2.1) we have to check whether it is possible to fix the parameter C, as a function of ε and t_0, in such a way that (2.10) follows for all $k \geq 1$.

For $k = 1$ the condition (2.10) reads

$$\langle G^{(0)} \rangle = M(t_0, C) = 0, \tag{2.14}$$

and we can choose $C = C_0(t_0)$ so that this holds: this is assured by Hypothesis 2.

To higher order $k \geq 1$ we can write

$$G^{(k)}(\alpha(t), A(t), C, t+t_0) = \partial_3 G(\alpha_0(t), A_0, C_0, t+t_0)\, C^{(k)} + \Gamma^{(k)}(\alpha(t), A(t), C, t+t_0), \tag{2.15}$$

where the function $\Gamma^{(k)}(\alpha(t), A(t), C, t + t_0)$ depends on the coefficients $C^{(k')}$ of C with $k' < k$ (and on the functions $\alpha^{(k')}(t)$ and $A^{(k')}(t)$ with $k' \leq k$, of course). In other words, in (2.15) we have extracted explicitly the only term depending on $C^{(k)}$. Moreover one has

$$\langle \partial_3 G(\alpha_0(\cdot), A_0, C_0, \cdot + t_0) \rangle = \frac{1}{T} \int_0^T \mathrm{d}t\, \partial_3 G(\alpha_0(t), A_0, C_0, t + t_0) = \frac{\partial}{\partial C} M(t_0, C_0), \tag{2.16}$$

and by Hypothesis 2 one has $D(t_0) := \partial M(t_0, C_0(t_0))/\partial C \neq 0$, so that (2.10) is satisfied provided $C^{(k)}$ is chosen as

$$C^{(k)} = -\frac{1}{D(t_0)} \langle \Gamma^{(k)}(\alpha(\cdot), A(\cdot), C, \cdot + t_0) \rangle \equiv C_k(t_0). \tag{2.17}$$

Therefore we conclude that if we set $C_0 = C_0(t_0)$ and, for all $k \geq 1$, we choose $\bar{\alpha}^{(k)} = \alpha_k(t_0)$, $\bar{A}^{(k)}$ according to (2.13) and $C^{(k)} = C_k(t_0)$ according to (2.17), we obtain that in the expansions (2.3) the coefficients $\alpha^{(k)}(t)$ and $A^{(k)}(t)$ are well-defined periodic functions of period T. Of course this does not settle the problem of convergence of the series (2.3) and (2.4). This will be discussed in the next Section.

2.3 Convergence of the series for the subharmonic solutions

Here we shall prove that the formal power series found in Section 2.2 converge for ε small enough, say for $|\varepsilon| < \varepsilon_0$ for some $\varepsilon_0 > 0$. Then for fixed $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ we shall find the range allowed for C by computing the supremum and the infimum, for $t_0 \in [0, 2\pi)$ of the function $t_0 \to C(\varepsilon, t_0)$. The bifurcation curves will be defined in terms of the function $C(\varepsilon, t_0)$ — cf. (2.4) — as

$$\gamma_1(\varepsilon) = \varepsilon \sup_{t_0 \in [0,2\pi)} C(\varepsilon, t_0), \qquad \gamma_2(\varepsilon) = \varepsilon \inf_{t_0 \in [0,2\pi)} C(\varepsilon, t_0). \tag{2.18}$$

In general the functions (2.18) are not analytic in ε. We shall return to this at the end of the section.

To prove convergence of the series (2.3) and (2.4) it is more convenient to work in Fourier space. First of all let us define $\omega = 2\pi/T = 1/q$ (note that $\omega \neq \omega(A_0)$) and expand

$$F(\alpha, A, C, t + t_0) = \sum_{\nu,\sigma \in \mathbb{Z}} \mathrm{e}^{i\nu\alpha} \mathrm{e}^{i\sigma(t+t_0)} F_{\nu,\sigma}(A, C), \tag{2.19}$$

so that we can write

$$\begin{aligned} &\partial_1^{r_1} \partial_2^{r_2} \partial_3^{r_3} F(\omega(A_0)\, t, A_0, C_0(t_0), t + t_0) \\ &\qquad = \sum_{\nu \in \mathbb{Z}} \mathrm{e}^{i\nu\omega t} \sum_{\substack{\nu_0,\sigma_0 \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q = \nu}} \mathrm{e}^{i\sigma_0 t_0} \, (i\nu_0)^{r_1} \, \partial_2^{r_2} \partial_3^{r_3} F_{\nu_0,\sigma_0}(A_0, C_0(t_0)), \end{aligned} \tag{2.20}$$

and an analogous expression can be obtained with the function G replacing F. By the analyticity assumption on the functions F and G, we have the bounds

$$\begin{aligned} \left| \frac{\partial_2^{r_2} \partial_3^{r_3}}{r_2! r_3!} F_{\nu_0,\sigma_0}(A_0, C_0(t_0)) \right| &\leq P Q_1^{r_1} Q_2^{r_2} \mathrm{e}^{-\kappa(|\nu_0| + |\sigma_0|)}, \\ \left| \frac{\partial_2^{r_2} \partial_3^{r_3}}{r_2! r_3!} G_{\nu_0,\sigma_0}(A_0, C_0(t_0)) \right| &\leq P Q_1^{r_1} Q_2^{r_2} \mathrm{e}^{-\kappa(|\nu_0| + |\sigma_0|)}, \end{aligned} \tag{2.21}$$

for suitable positive constants P, Q_1, Q_2, κ.

We can also define $\partial_2^m U_{\nu,\sigma} = \delta_{\nu,0} \delta_{\sigma,0} \partial^m \omega(A_0)/\partial A^m$, and imagine, without loss of generality, that the constants P and Q_2 are such that $|\partial^m \omega(A_0)/\partial A^m| \leq m! P Q_2^m$.

Then, let us write in (2.3)

$$\alpha^{(k)}(t) = \sum_{\nu \in \mathbb{Z}} \mathrm{e}^{i\nu\omega t} \alpha_\nu^{(k)}, \qquad A^{(k)}(t) = \sum_{\nu \in \mathbb{Z}} \mathrm{e}^{i\nu\omega t} A_\nu^{(k)}, \tag{2.22}$$

so that (2.12) becomes

$$\alpha_\nu^{(k)} = \frac{\Phi_\nu^{(k-1)}}{i\omega\nu} + \omega'(A_0) \frac{G_\nu^{(k-1)}}{(i\omega\nu)^2}, \qquad A_\nu^{(k)} = \frac{G_\nu^{(k-1)}}{i\omega\nu}, \tag{2.23}$$

for all $\nu \neq 0$, whereas for $\nu = 0$ one has

$$\alpha_0^{(k)} = \alpha_k(t_0) - \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} \frac{\Phi_\nu^{(k-1)}}{i\omega\nu} - \omega'(A_0) \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} \frac{G_\nu^{(k-1)}}{(i\omega\nu)^2},$$

$$A_0^{(k)} = \bar{A}^{(k)} - \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} \frac{G_\nu^{(k-1)}}{i\omega\nu} = -\frac{\Phi_0^{(k-1)}}{\omega'(A_0)}, \tag{2.24}$$

with $\alpha_k(t_0)$ so far arbitrary and $\bar{A}^{(k)}$ given by (2.13). The Fourier coefficients $\Phi_\nu^{(k-1)}$ and $G_\nu^{(k-1)}$ can be read from (2.6), (2.7) and the analogous expression for $G^{(k)}(t)$. Hence one has $\Phi_\nu^{(k-1)} = U_\nu^{(k)} + F_\nu^{(k-1)}$, where $U_\nu^{(1)} = 0$,

$$\begin{aligned} U_\nu^{(k)} &= \left[\omega(A) - \omega(A_0) - \omega'(A_0)\,(A - A_0)\right]_\nu^{(k)} \\ &= \sum_{r_2=2}^{\infty} \sum_{\substack{\nu_1,\ldots,\nu_{r_2}\in\mathbb{Z}\\ \nu_1+\ldots+\nu_{r_2}=\nu}} \frac{\partial_2^{r_2}}{r_2!}\omega(A_0) \sum_{\substack{k_1+\ldots+k_{r_2}=k\\ k_i\geq 1}} A_{\nu_1}^{(k_1)} \ldots A_{\nu_{r_2}}^{(k_{r_2})}, \end{aligned} \tag{2.25}$$

where we have set $\partial_2 = \partial/\partial A^m$, for $k \geq 2$, and

$$F_\nu^{(k)} = \left[F(\alpha, A, C, t + t_0)\right]_\nu^{(k)} = \sum_{m=0}^{\infty} \sum_{\substack{r_1,r_2,r_3\in\mathbb{Z}_+\\ r_1+r_2+r_3=m}} \sum_{\substack{\nu_0,\sigma_0,\nu_1,\ldots,\nu_{r_1+r_2}\in\mathbb{Z}\\ \nu_0 p+\sigma_0 q+\nu_1+\ldots+\nu_{r_1+r_2}=\nu}} \frac{(i\nu_0)^{r_1}}{r_1!} \mathrm{e}^{i\sigma_0 t_0} \tag{2.26}$$

$$\frac{\partial_2^{r_2}\partial_3^{r_3}}{r_2! r_3!} F_{\nu_0,\sigma_0}(A_0, C_0(t_0)) \sum_{\substack{k_1+\ldots+k_m=k\\ k_i\geq 1}} \alpha_{\nu_1}^{(k_1)} \ldots \alpha_{\nu_{r_1}}^{(k_{r_1})}\, A_{\nu_{r_1+1}}^{(k_{r_1+1})} \ldots A_{\nu_{r_1+r_2}}^{(k_{r_1+r_2})}\, C^{(k_{r_1+r_2+1})} \ldots C^{(k_m)},$$

for $k \geq 1$, and an analogous definition holds for $G_\nu^{(k)}$, $k \geq 1$.

Furthermore one has

$$C^{(k)} = -\frac{1}{D(t_0)} \Gamma_0^{(k)} \tag{2.27}$$

where

$$\Gamma_0^{(k)} = \left[\Gamma(\alpha, A, C, t + t_0)\right]_0^{(k)} = \sum_{m=0}^{\infty} \sideset{}{^*}\sum_{\substack{r_1,r_2,r_3\in\mathbb{Z}_+\\ r_1+r_2+r_3=m}} \sum_{\substack{\nu_0,\sigma_0,\nu_1,\ldots,\nu_{r_1+r_2}\in\mathbb{Z}\\ \nu_0 p+\sigma_0 q+\nu_1+\ldots+\nu_{r_1+r_2}=0}} \frac{(i\nu_0)^{r_1}}{r_1!} \mathrm{e}^{i\sigma_0 t_0} \tag{2.28}$$

$$\frac{\partial_2^{r_2}\partial_3^{r_3}}{r_2! r_3!} G_{\nu_0,\sigma_0}(A_0, C_0(t_0)) \sum_{k_1+\ldots+k_m=k} \alpha_{\nu_1}^{(k_1)} \ldots \alpha_{\nu_{r_1}}^{(k_{r_1})}\, A_{\nu_{r_1+1}}^{(k_{r_1+1})} \ldots A_{\nu_{r_1+r_2}}^{(k_{r_1+r_2})}\, C^{(k_{r_1+r_2+1})} \ldots C^{(k_m)},$$

where $*$ means that the term with $r_1 = r_2 = 0$ and $r_3 = 1$ has to be discarded — cf. (2.15).

Therefore we see from the first equation in (2.24) that it is convenient to fix

$$\alpha_k(t_0) = \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} \frac{\Phi_\nu^{(k-1)}}{i\omega\nu} + \omega'(A_0) \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} \frac{G_\nu^{(k-1)}}{(i\omega\nu)^2} \qquad \Longrightarrow \qquad \alpha_0^{(k)} = 0, \tag{2.29}$$

so that only the functions $A^{(k)}(t)$ have the zeroth Fourier coefficient.

In particular for $k = 1$ we find

$$\begin{aligned} \alpha_\nu^{(1)} &= \frac{1}{i\omega\nu} \sum_{\substack{\nu_0,\sigma_0\in\mathbb{Z}\\ \nu_0 p+\sigma_0 q=\nu}} \mathrm{e}^{i\sigma_0 t_0} F_{\nu_0,\sigma_0}(A_0, C_0(t_0)) + \frac{\omega'(A_0)}{(i\omega\nu)^2} \sum_{\substack{\nu_0,\sigma_0\in\mathbb{Z}\\ \nu_0 p+\sigma_0 q=\nu}} \mathrm{e}^{i\sigma_0 t_0} G_{\nu_0,\sigma_0}(A_0, C_0(t_0)), \\ A_\nu^{(1)} &= \frac{1}{i\omega\nu} \sum_{\substack{\nu_0,\sigma_0\in\mathbb{Z}\\ \nu_0 p+\sigma_0 q=\nu}} \mathrm{e}^{i\sigma_0 t_0} G_{\nu_0,\sigma_0}(A_0, C_0(t_0)), \end{aligned} \tag{2.30}$$

for $\nu \neq 0$, and

$$A_0^{(1)} = -\frac{1}{\omega'(A_0)} \sum_{\substack{\nu_0,\sigma_0\in\mathbb{Z}\\ \nu_0 p+\sigma_0 q=0}} \mathrm{e}^{i\sigma_0 t_0} F_{\nu_0,\sigma_0}(A_0, C_0(t_0)), \tag{2.31}$$

for $\nu = 0$, while by writing

$$\begin{aligned} C^{(1)} = -\frac{1}{D(t_0)} \Big(& \sum_{\substack{\nu_1,\nu_2\in\mathbb{Z}\\ \nu_1+\nu_2=0}} \sum_{\substack{\nu_0,\sigma_0\in\mathbb{Z}\\ \nu_0 p+\sigma_0 q=\nu_1}} \mathrm{e}^{i\sigma_0 t_0} i\nu_0\, G_{\nu_0,\sigma_0}(A_0, C_0(t_0))\, \alpha_{\nu_2}^{(1)} \\ & + \sum_{\substack{\nu_1,\nu_2\in\mathbb{Z}\\ \nu_1+\nu_2=0}} \sum_{\substack{\nu_0,\sigma_0\in\mathbb{Z}\\ \nu_0 p+\sigma_0 q=\nu_1}} \mathrm{e}^{i\sigma_0 t_0} \partial_2 G_{\nu_0,\sigma_0}(A_0, C_0(t_0))\, A_{\nu_2}^{(1)} \Big) \equiv C_1(t_0), \end{aligned} \tag{2.32}$$

we can express $C^{(1)}$ in terms of the quantities in (2.30).

In order to study the convergence of the series it is convenient to express all quantities in terms of trees. The strategy is very simple: one iterates the relations (2.23), (2.24) and (2.26), which express the coefficients of order k in terms of the coefficients of lower order, until we are left only with the coefficients of first order, for which the explicit expressions (2.30), (2.31) and (2.32) are at our disposal. The analysis, although very easy, is rather technical, so it will be deferred to Appendix B.

Now, we come back to the problem of determining the boundary of the set in the plane (ε, γ), with $\gamma = \varepsilon C$, in which there are subharmonic solutions of order q/p.

We have to find the solutions of (2.18), that is, solve the equation

$$0 = \frac{\partial}{\partial t_0} C(\varepsilon, t_0) = C_0'(t_0) + \varepsilon C_1'(t_0) + \varepsilon^2 C_2'(t_0) + \ldots, \tag{2.33}$$

where $C_k'(t_0) = \mathrm{d}C_k(t_0)/\mathrm{d}t_0$.

The function $t_0 \to C(\varepsilon, t_0)$ is analytic in t_0 for all $|\varepsilon| < \varepsilon_0$ (for which it is defined and analytic in ε), so that for fixed ε the equation (2.33) can always be solved. It has at least the two solutions $t_0 = \tau_1(\varepsilon)$ and $t_0 = \tau_2(\varepsilon)$ corresponding to the absolute minimum and to the absolute maximum, respectively, of the function $C(\varepsilon, t_0)$. In general these solutions are not smooth in ε. This proves Theorem 1.

Suppose now that at the value t_0 such that $C_0'(t_0) = 0$ one has furthermore $C_0''(t_0) \neq 0$. Note that generically this condition is satisfied. In that case, if $\tau_0 = \tau_0(\varepsilon)$ is a solution of (2.33) — τ_0 is a point of minimum or maximum for $C(\varepsilon, t_0)$ — then τ_0 must be analytically close to t_0 (by the implicit function theorem). Hence $\varepsilon \to \tau_0(\varepsilon)$ is an analytic function of ε, so that also $\varepsilon \to C_1(\varepsilon)$ and $\varepsilon \to C_2(\varepsilon)$ are smooth (in fact analytic) in ε. Therefore we can conclude that *in general* the bifurcation curves are not analytic, although *generically* they are. Therefore Theorem 2 also follows.

2.4 Some extensions

The last observation of Section 2.3 suggests how to extend Theorem 2 to obtain smooth bifurcation curves when Hypothesis 3 fails to be satisfied.

Hypothesis 4 *There exists $k \geq 1$ such that the functions $C_p(t_0)$ are identically constant in t_0 for all $p = 0, \ldots, k-1$. If t_m and t_M are the values in $[0, 2\pi)$ for which the function $C_k(t_0)$ attains its minimum and its maximum, respectively, then $C_k''(t_m)C_k''(t_M) \neq 0$.*

The following result extends Theorem 2, as it deals with the case in which the subharmonic Melnikov function does not depend explicitly on t_0, that is $C_0'(t_0) \equiv 0$.

Theorem 3 *Consider the system (2.1) and assume that Hypotheses 1, 2 and 4 hold for the resonant torus with energy A_0 such that $\omega(A_0) = p/q$. There exist $\varepsilon_0 > 0$ and two functions $\widetilde{\gamma}_1(\varepsilon)$ and $\widetilde{\gamma}_2(\varepsilon)$, analytic for $|\varepsilon| < \varepsilon_0$, with $\widetilde{\gamma}_1(0) = \widetilde{\gamma}_2(0)$ and $\widetilde{\gamma}_1(\varepsilon) \neq \widetilde{\gamma}_2(0)$ for all $\varepsilon \neq 0$, such that the two functions*

$$\gamma_1(\varepsilon) = \begin{cases} \max\{\widetilde{\gamma}_1(\varepsilon), \widetilde{\gamma}_2(\varepsilon)\}, & \varepsilon > 0, \\ \min\{\widetilde{\gamma}_1(\varepsilon), \widetilde{\gamma}_2(\varepsilon)\}, & \varepsilon < 0, \end{cases} \qquad \gamma_2(\varepsilon) = \begin{cases} \min\{\widetilde{\gamma}_1(\varepsilon), \widetilde{\gamma}_2(\varepsilon)\}, & \varepsilon > 0, \\ \max\{\widetilde{\gamma}_1(\varepsilon), \widetilde{\gamma}_2(\varepsilon)\}, & \varepsilon < 0, \end{cases} \tag{2.34}$$

have the same tangent lines at the origin, and (2.1) has at least one subharmonic solution of order q/p for $\gamma_2(\varepsilon) \leq \varepsilon C \leq \gamma_1(\varepsilon)$ when $\varepsilon \in (0, \varepsilon_0)$ and for $\gamma_1(\varepsilon) \leq \varepsilon C \leq \gamma_2(\varepsilon)$ when $\varepsilon \in (-\varepsilon_0, 0)$.

The proof follows the same lines as that of Theorem 2. The only difference is that up to order $k-1$ the initial phase is left undetermined. In fact to first order one has $M(t_0, C) = M(C) = 0$ which fixes $C = C_0$ (by Hypothesis 2). Also to orders $k' = 2, \ldots, k-1$ the constants C_k are fixed and are independent of t_0 by Hypothesis 4. Then we can write $C(\varepsilon, t_0) = \mathfrak{C}_1(\varepsilon) + \mathfrak{C}_2(\varepsilon, t_0)$, with $\mathfrak{C}_1(\varepsilon) = C_0 + \varepsilon C_1 + \ldots + \varepsilon^{k-1} C_{k-1}$ and $\mathfrak{C}_2(\varepsilon, t_0) = \varepsilon^k (C_k(t_0) + O(\varepsilon))$, and from order k on the constants C_k are fixed as functions of t_0. Moreover equation (2.33) reduces to $0 = C_k'(t_0) + \varepsilon C_{k+1}'(t_0) + \ldots$. Therefore we can reason as in the previous case ($k = 0$) and we find that $C_k(t_0)$ has at least two stationary points $t_0 = t_1$ and $t_0 = t_2$, corresponding to the minimum point and to the maximum point, respectively. By Hypothesis 4 also $\mathfrak{C}_2(\varepsilon, t_0)$ has two stationary points at $\tau_1(\varepsilon) = t_1 + O(\varepsilon)$ and $\tau_2(\varepsilon) = t_2 + O(\varepsilon)$, with $\tau_1(\varepsilon)$ and $\tau_2(\varepsilon)$ analytic in ε for ε small enough. Then we can define $\widetilde{\gamma}_1(\varepsilon) = C(\varepsilon, \tau_1(\varepsilon))$ and $\widetilde{\gamma}_2(\varepsilon) = C(\varepsilon, \tau_2(\varepsilon))$: by construction, both $\widetilde{\gamma}_1(\varepsilon)$ and $\widetilde{\gamma}_1(\varepsilon)$ are analytic in ε for ε small enough. If we define $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ according to (2.34) then the proof of the theorem is achieved.

Note that in this case the definition (2.34) coincides with the general definition (2.18) for the bifurcation curves. Furthermore, if we assume Hypothesis 3 instead of Hypothesis 4, then one has $\widetilde{\gamma}_1(\varepsilon) = \gamma_1(\varepsilon)$ and $\widetilde{\gamma}_1(\varepsilon) = \gamma_1(\varepsilon)$, so that also $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ are analytic, as stated in Theorem 2.

Finally we note that if the functions $C_k(t_0)$ are identically constant in t_0 for all $k \in \mathbb{Z}_+$ then one has $C(\varepsilon, t_0) = C(\varepsilon)$. In this case the two curves $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ coincide, and all values of t_0 are allowed. This means that the whole manifold corresponding to the resonant torus persists. On the other hand the parameter C must be fixed in a very precise way, as a function of ε, and any small deviation from that value destroys the torus. This result can be compared with [18, 19], where a similar situation is discussed.

For (ε, γ) inside the set of existence of subharmonic solutions one can investigate how many of them exist. For $p = 1$ the initial phase t_0 varies in the interval $[0, 2\pi q]$, where $T_0 = 2\pi q$ is the period of the unperturbed periodic solution. The function $C(\varepsilon, t_0)$ has period 2π in t_0, so that it is repeated q times in the interval $[0, 2\pi q]$. Hence for any fixed value $|\varepsilon| < \varepsilon_0$ and any C strictly between the maximum and the minimum value attained by the function $t_0 \to C(\varepsilon, t_0)$ there are at least $2q$ values t_i, $i = 1, \ldots, 2q$, such that $C = C(\varepsilon, t_i)$. If C coincides with either its maximum or its minimum then there are at least q values t_i, $i = 1, \ldots, q$, such that $C = C(\varepsilon, t_i)$. Therefore we can conclude that, for $p = 1$, inside the set of existence of subharmonic solutions there are at least $2q$ such solutions, as found in [20], while on the boundary of that set there are q of them.

We can summarise the discussion above in the following statement.

Theorem 4 *Under the same assumptions of Theorem 1 assume $p = 1$. Take $|\varepsilon| < \varepsilon_0$, and for such values of ε let $\varepsilon \to \gamma_1(\varepsilon)$ and $\varepsilon \to \gamma_2(\varepsilon)$ be the two bifurcation curves whose existence is assured by Theorem 1. For $\min\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} < \gamma < \max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\}$ there at least $2q$ subharmonic solutions of order q. If either $\gamma = \gamma_1(\varepsilon)$ or $\gamma = \gamma_2(\varepsilon)$ one has at least q subharmonic solutions of order q.*

Theorem 4 should be compared with Theorem 2.1 in [20].

2.5 Application to dissipative systems with forcing

Let us consider a one-dimensional system, subject to a conservative force $g(x)$, in the presence of dissipation and of a periodic forcing. If the periodic forcing and the dissipation coefficient are both small we can write the equations for the system as

$$\ddot{x} + g(x) + \gamma \dot{x} = \varepsilon f(x,t), \qquad \gamma = \varepsilon\, C, \tag{2.35}$$

where $\varepsilon f(x,t)$ is the forcing of period 2π and C is a parameter. Assume that both g and f are analytic in their arguments. If f depends only on t, equation (2.35) reduces to the equation studied in [43].

Let us assume that the unperturbed system ($\varepsilon = 0$) is Liouville-integrable and anisochronous. This means that, in action-angle variables, the equations (2.35) can be written in the form (2.1), and, furthermore, that Hypothesis 1 is satisfied.

We define the subharmonic Melnikov function in terms of the action-angle variable as in (2.2). To check that Hypothesis 2 is also satisfied we use the following result.

Lemma 1 *The subharmonic Melnikov function is invariant under a transformation of coordinates.*

Proof. Consider a system of differential equations in $\mathbb{R}^2$

$$\dot{x} = f(x) + \varepsilon g(x,t), \tag{2.36}$$

and define the subharmonic Melnikov function [52, 41, 20] for a subharmonic solution $x_0(t)$ of period T as

$$M(t_0) = \frac{1}{T}\int_0^T \mathrm{d}t \big(f_1(x_0(t))\, g_2(x_0(t), t+t_0) - f_2(x_0(t))\, g_1(x_0(t), t+t_0) \big). \tag{2.37}$$

Take the transformation of coordinates $\xi \to x = h(\xi)$. In the new coordinates the system reads

$$\dot{\xi} = \phi(\xi) + \varepsilon \gamma(\xi, t), \tag{2.38}$$

where $\phi(\xi) = \partial h^{-1}(h(\xi))\, f(h(\xi))$ and $\gamma(\xi) = \partial h^{-1}(h(\xi))\, g(h(\xi))$, and the subharmonic Melnikov function becomes

$$\mathcal{M}(t_0) = \frac{1}{T}\int_0^T \mathrm{d}t \big(\phi_1(\xi_0(t))\, \gamma_2(\xi_0(t), t+t_0) - \phi_2(\xi_0(t))\, \gamma_1(\xi_0(t), t+t_0) \big), \tag{2.39}$$

where $\xi_0(t)$ is the subharmonic solution expressed in the new variables.

By noting that

$$\partial h^{-1}(h(\xi)) = (\partial h(\xi))^{-1} = \frac{1}{J} \begin{pmatrix} \partial_2 h_2(\xi) & -\partial_2 h_1(\xi) \\ -\partial_1 h_2(\xi) & \partial_1 h_1(\xi) \end{pmatrix}, \tag{2.40}$$

where $J = \det \partial h = \partial_1 h_1 \partial_2 h_2 - \partial_1 h_2 \partial_2 h_1$ is the Jacobian of the transformation, one obtains

$$\begin{aligned}
\mathcal{M}(t_0) &= \frac{1}{T} \int_0^T \mathrm{d}t \frac{1}{J} \Big((\partial_2 h_2 f_1 - \partial_2 h_1 f_2)(-\partial_1 h_2 g_1 + \partial_1 h_1 g_2) - \\
&\qquad (-\partial_1 h_2 f_1 + \partial_1 h_1 f_2)(\partial_2 h_2 g_1 - \partial_2 h_1 g_2) \Big) \\
&= \frac{1}{T} \int_0^T \mathrm{d}t \frac{1}{J} (\partial_1 h_1 \partial_2 h_2 - \partial_1 h_2 \partial_2 h_1)(f_1 g_2 - f_2 g_1), \qquad (2.41)
\end{aligned}$$

where the function h is computed in $\xi_0(t)$ and the functions f, g are computed in $x_0(t) = h(\xi_0(t))$. Hence (2.37) yields $\mathcal{M}(t_0) = M(t_0)$, so that the assertion follows. ■

This means that we can compute the subharmonic Melnikov function for the system (2.35) in the coordinates $(x, y) = (x, \dot{x})$. In that case the unperturbed vector field is $(y, -g(x))$ and the perturbation reads $(0, -\varepsilon C y + \varepsilon f(x, t))$, so that the subharmonic Melnikov function becomes

$$\begin{aligned}
M(t_0, C) &= \frac{1}{T} \int_0^T \mathrm{d}t \, y_0(t) \big(- C y_0(t) + f(x_0(t), t + t_0) \big) \\
&= -C \langle y_0^2 \rangle + \langle y_0 f(x_0(\cdot), \cdot + t_0) \rangle. \qquad (2.42)
\end{aligned}$$

Therefore the subharmonic Melnikov function vanishes provided $C = C_0(t_0)$, where $C_0(t_0) = (\langle y_0^2 \rangle)^{-1} \langle y_0 f(x_0(\cdot), \cdot + t_0) \rangle$, which is well-defined because $\langle y_0^2 \rangle > 0$. Moreover one has $\partial M(t_0, C)/\partial C = -\langle y_0^2 \rangle \neq 0$. Therefore Hypothesis 2 is also satisfied, and Theorem 2 applies to the system (2.35).

We can state our result as follows.

Theorem 5 *Consider the system (2.35) and assume that Hypothesis 1 holds for the invariant torus with energy A_0 such that $\omega(A_0) = p/q$. There exist $\varepsilon_0 > 0$ and two continuous functions $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$, with $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(\varepsilon) \geq \gamma_2(\varepsilon)$ for $\varepsilon \geq 0$ and $\gamma_1(\varepsilon) \leq \gamma_2(\varepsilon)$ for $\varepsilon \leq 0$, such that (2.35) has at least one subharmonic solution of period $2\pi p$ for $\gamma_2(\varepsilon) \leq \varepsilon C \leq \gamma_1(\varepsilon)$ when $\varepsilon \in (0, \varepsilon_0)$ and for $\gamma_1(\varepsilon) \leq \varepsilon C \leq \gamma_2(\varepsilon)$ when $\varepsilon \in (-\varepsilon_0, 0)$.*

Of course Theorem 5 is a corollary of Theorem 1. It should be compared with Corollary 2.3 in [20] (cf. also [43]). Our result is stronger as it requires, in Chow and Hale's notations, only Hypothesis (H_1), which corresponds to our Hypothesis 1. If one assumes also Hypothesis (H_4) of [20], which corresponds to our Hypothesis 3, then Theorem 2 applies, and the result of [20] is recovered.

One expects that, in the case of system (2.35), the two bifurcation curves $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ contain the real axis, that is $\min\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} \leq 0 \leq \max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\}$. Indeed for

$\gamma = 0$ the equation (2.35) describes a quasi-integrable Hamiltonian system, and existence of periodic solutions is well known in this case, at least under some non-degeneracy condition on the unperturbed system, such as Hypothesis 1. If $C_0(t_0)$ is not zero then it is easy to check that the set of existence of subharmonic solutions includes the real axis. This follows from the following result.

Lemma 2 *The function $C_0(t_0)$ has zero mean.*

Proof. Call

$$F(x_0(t)) = \int_0^{2\pi} \frac{\mathrm{d}t_0}{2\pi} \, f(x_0(t), t + t_0) = \int_0^{2\pi} \frac{\mathrm{d}t_0}{2\pi} \, f(x_0(t), t_0). \tag{2.43}$$

By (2.42) the mean (with respect to t_0) of $C_0(t_0)$ is

$$\begin{aligned} \int_0^{2\pi} \frac{\mathrm{d}t_0}{2\pi} \, C_0(t_0) &= \frac{1}{\langle y_0^2 \rangle} \int_0^{2\pi} \frac{\mathrm{d}t_0}{2\pi} \int_0^T \frac{\mathrm{d}t}{T} \, \dot{x}_0(t) f(x_0(t), t + t_0) \\ &= \int_0^T \frac{\mathrm{d}t}{T} \, \dot{x}_0(t) F(x_0(t)), \end{aligned} \tag{2.44}$$

which vanishes, as the integrand can be written as a total derivative with respect to t. ∎

In particular Lemma 2 implies that if $C_0(t_0)$ is not identically constant then its maximum is strictly positive and its minimum is strictly negative, hence $\max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} > 0$ and $\min\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} < 0$.

To extend the same result to the case in which the functions $C_{k'}(t_0)$ are identically constant in t_0 for all $k' \le k - 1$, with $k \ge 1$ arbitrarily high, is more delicate, and it requires some work. The result is the following one.

Lemma 3 *Assume that for some $\bar{k} \in \mathbb{Z}_+$ the coefficients $C_{k'}(t_0)$ vanish identically for all $k' = 0, \ldots, \bar{k} - 1$. Then $C_{\bar{k}}(t_0)$ has zero mean in t_0.*

The proof is given in Appendix C, and relies on the tree formalism introduced in Appendix B — which one should refer to for notations.

We shall also need the following result.

Lemma 4 *Assume that for some $\bar{k} \in \mathbb{Z}_+$ the coefficients $C_{k'}(t_0)$ are identically constant for all $k' = 0, \ldots, \bar{k} - 1$. Then $C_{k'}(t_0) \equiv 0$ for all $k' = 0, \ldots, \bar{k} - 1$.*

Proof. The proof is by induction. Fix $0 \le k < \bar{k}$, and assume that $C_{k'}(t_0) \equiv 0$ for all $k' \le k - 1$. Then by Lemma 3 the function $C_k(t_0)$ has zero mean. Since it is constant by hypothesis then $C_k(t_0) \equiv 0$. ∎

Let $k \in \mathbb{Z}_+$ be such that $C_{k'}(t_0)$ is identically constant in t_0 for $k' = 0, \ldots, k-1$ whereas $C_k(t_0)$ depends explicitly on t_0. If $k = 0$ this simply means that $C_0(t_0)$ depends explicitly on t_0. By Lemma 4 one has $C_{k'}(t_0) \equiv 0$ for all $k' \leq k-1$, and by Lemma 4 the function $C_k(t_0)$ has zero mean in t_0. Since $C_k(t_0)$ is not identically constant then $\sup_{t_0 \in [0,2\pi)} C_k(t_0) > 0$ and $\inf_{t_0 \in [0,2\pi)} C_k(t_0) < 0$. Furthermore, in such a case $C(\varepsilon, t_0) = \varepsilon^k (C_k(t_0) + O(\varepsilon))$, so that also

$$\sup_{t_0 \in [0,2\pi)} C(\varepsilon, t_0) > 0, \qquad \inf_{t_0 \in [0,2\pi)} C(\varepsilon, t_0) < 0, \tag{2.45}$$

for ε small enough. If we recall the definition (2.18) of the bifurcation curves we can formulate the following result.

Theorem 6 *Under the same assumptions of Theorem 5 let $\varepsilon \to \gamma_1(\varepsilon)$ and $\varepsilon \to \gamma_2(\varepsilon)$ be the two bifurcation curves whose existence is assured by Theorem 5. One has $\gamma_1(\varepsilon) \geq 0 \geq \gamma_2(\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0)$ and $\gamma_1(\varepsilon) \leq 0 \leq \gamma_2(\varepsilon)$ for $\varepsilon \in (-\varepsilon_0, 0)$.*

As (2.45) shows, if there is $k \geq 0$ such that $C_{k'}(t_0) \equiv 0$ for $k' = 0, \ldots, k-1$ and $C_k(t_0) \neq 0$, then one has the strict inequalities $\gamma_1(\varepsilon) > 0 > \gamma_2(\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0)$ and $\gamma_1(\varepsilon) < 0 < \gamma_2(\varepsilon)$ for $\varepsilon \in (-\varepsilon_0, 0)$. On the contrary if all C_k vanish identically, so that the full function $C(\varepsilon, t_0)$ has to be zero, then $\gamma_1(\varepsilon) = \gamma_2(\varepsilon) = 0$.

Therefore Theorems 5 and 6 show that any one-dimensional anisochronous mechanical system, when perturbed by a periodic forcing and in the presence of dissipation, up to the exceptional cases in which the functions $C_k(t_0)$ are constant — and hence vanish, by Lemma 4 — in t_0 for all $k \in \mathbb{Z}_+$, admits subharmonic solutions of all orders, without any assumption on the perturbation, — a result which does not follow from the analysis of [43, 20].

The case that all the functions $C_k(t_0)$ are identically constant in t_0 is really exceptional. This can be appreciated by the following argument. If the function $C(\varepsilon, t_0)$ does not depend on t_0 then not only, by Lemma 4, it must vanish identically, i.e. $C(\varepsilon, t_0) = C(\varepsilon) \equiv 0$, but we find also that t_0 is left undetermined. In other words the periodic solution persists for all values of t_0. This means that if we take the system (2.35) with $\gamma = 0$, so that it becomes an autonomous quasi-integrable Hamiltonian system, with no dissipation left, the full resonant torus with frequency $\omega = p/q$ persists under perturbation. This situation is certainly unlikely, even if not impossible in principle. For instance one can take the system described by the Hamiltonian

$$H(x, y, t) = \frac{1}{2} y^2 + \frac{1}{4} x^4 + \varepsilon f(t) \left(\frac{1}{2} y^2 + \frac{1}{4} x^4 - E \right)^2, \tag{2.46}$$

with E corresponding to the unperturbed solution $(x_0(t), y_0(t))$ with frequency ω. Then such a solution still satisfies the corresponding Hamilton equations for all values of ε and for all values of the initial phase t_0: that is the full resonant torus with frequency ω persists. In particular if $\omega = p/q$ is rational, so that the frequency of the unperturbed solution becomes commensurate with the frequency 1 of the perturbing potential f, the corresponding torus is resonant.

It is important to stress that if we look for a subharmonic solution which continues some unperturbed periodic solution with a given period $T = 2\pi q/p$ it is not so rare that the corresponding integral $\langle y_0 f(x_0(\cdot), \cdot + t_0)\rangle$ identically vanishes. In fact, if f is a trigonometric polynomial (which is often the case in physical applications) this happens for all p/q but a finite set of values. An explicit example has been considered in [1]. In these cases the subharmonic Melnikov function does not depend on t_0 and it is linear in C: hence (2.42) can be satisfied only by taking $C_0(t_0) \equiv 0$. Then, it becomes essential to go to higher orders of perturbation theory to study for which values of C a subharmonic solution of order q/p appears. Again, we refer to [1] for a situation in which one must perform a higher order analysis to explain the numerical findings.

3 Melnikov theory in degenerate cases

3.1 Statement of the results

The Melnikov theory [41] considers systems which, in suitable coordinates, can be written as in (2.1), without the parameter C:

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, t), \\ \dot{A} = \varepsilon G(\alpha, A, t), \end{cases} \tag{3.1}$$

where all notations are as explained after (2.1). Define the subharmonic Melnikov function as

$$M(t_0) = \frac{1}{T}\int_0^T \mathrm{d}t\, G(\alpha_0(t), A_0(t), t + t_0), \tag{3.2}$$

and set $M'(t_0) = \mathrm{d}M(t_0)/\mathrm{d}t_0$. Here and henceforth $A_0(t) = A_0$ and $\alpha_0(t) = \omega(A_0)t$.

We can repeat the analysis of formal solvability in Section 2.2, with some adaptations due to the fact that no extra parameters $C^{(k)}$ are at our disposal to any perturbation orders.

In particular to first order one needs $M(t_0) = 0$, so that t_0 must be a zero for the subharmonic Melnikov function. This suggests, as done in the first part of the paper, that we write the system (3.1) in the form

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, t + t_0), \\ \dot{A} = \varepsilon G(\alpha, A, t + t_0), \end{cases} \tag{3.3}$$

in such a way that we can set equal to zero the initial angle of the unperturbed solution to be continued, as done after (3.2).

To higher orders we can write

$$G^{(k)}(\alpha(t), A(t), C, t+t_0) = \partial_1 G(\alpha_0(t), A_0, t+t_0)\, \bar{\alpha}^{(k)} + \Gamma^{(k)}(\alpha(t), A(t), t+t_0), \tag{3.4}$$

where the function $\Gamma^{(k)}(\alpha(t), A(t), t+t_0)$ depends on the corrections $\bar{\alpha}^{(k')}$ to the initial phase, only with $k' < k$.

To any perturbation order k the constant $\bar{\alpha}^{(k)}$ is left undetermined. Anyway we are no longer free to fix it equal to some arbitrary value, for instance zero, as we no longer have the constants $C^{(k)}$ as free parameters. Hence we shall need the corrections $\bar{\alpha}^{(k)}$ to assure solvability of the equations of motion to any order. This will be possible in the light of the following result.

Lemma 5 *One has* $\omega(A_0)\langle \partial_1 G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle = -M'(t_0)$.

Proof. One has

$$\frac{\mathrm{d}}{\mathrm{d}t} G(\alpha_0(t), A_0, t+t_0) = \omega(A_0)\, \partial_1 G(\alpha_0(t), A_0, t+t_0) + \frac{\partial}{\partial t_0} G(\alpha_0(t), A_0, t+t_0), \tag{3.5}$$

where we have used the fact that $\dot{A}_0(t) = 0$ and $\dot{\alpha}_0(t) = \omega(A_0)$. If we integrate (3.5) over a period we obtain

$$\begin{aligned} 0 &= \frac{1}{T}\int_0^T \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t} G(\alpha_0(t), A_0, t+t_0) \\ &= \omega(A_0)\langle \partial_1 G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle + \frac{\partial}{\partial t_0}\langle G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle, \end{aligned} \tag{3.6}$$

so that

$$\omega(A_0)\langle \partial_1 G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle = -\frac{\partial}{\partial t_0}\langle G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle = -M'(t_0). \tag{3.7}$$

Hence the assertion follows. ∎

We shall call, slightly improperly, the constants $\bar{\alpha}^{(k)}$ the *corrections to the initial phase*. Indeed, we can either fix the initial phase t_0 and change of the initial value $\alpha(0)$ of the angle variable $\alpha(t)$ or leave t_0 as a free parameter to be modified at each order and set $\bar{\alpha}^{(k)} = 0$ for all $k \geq 1$. The two procedures are completely equivalent, and we shall find more convenient to choose the first one.

Thus, if we impose the condition that t_0 be a simple zero for the subharmonic Melnikov function we find that in (3.4) the mean of the derivative $\partial_1 G(\alpha_0(t), A_0, t+t_0)$ is different from zero, and this allows us to fix $\bar{\alpha}^{(k)}$ in such a way as to make the mean of $G^{(k)}(\alpha(t), A(t), t+t_0)$ vanish. Hence, by fixing the constants $\bar{A}^{(k)}$ as explained in Section 2.2 and the constants $\bar{\alpha}^{(k)}$ as stated above, we find that a solution in the form of a formal power series in ε exists. The convergence of the series, hence the existence of an analytic solution, can be proved by reasoning as in Section 2.3. We do not repeat the analysis, which would essentially be a word for word copy of what was done in Section 2.3.

In conclusion, we have proved the following result — well-known in the literature [41].

Theorem 7 *Consider the resonant torus with frequency $\omega = p/q$ for the system (3.1) with $\varepsilon = 0$, and assume that t_0 is a simple zero for the subharmonic Melnikov function (3.2) corresponding to such a frequency. There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the system (3.1) has at least one subharmonic solution of order q/p. Such a solution reduces to one on the unperturbed torus as $\varepsilon \to 0$.*

However, our analysis permits us to generalise the result above. Define

$$M_0(t_0) = M(t_0), \qquad M_k(t_0) = \langle G^{(k)}(\alpha(\cdot), A(\cdot), \cdot + t_0)\rangle, \quad k \in \mathbb{N}, \tag{3.8}$$

where the notations of (3.4) have been used. We note since now that if $M_{k'}(t_0)$ vanishes identically for all $k' = 0, 1, \ldots, k-1$, then also $M_k(t_0)$ is well-defined; this will be explicitly proved in Section 3.2. We refer to the functions $M_k(t_0)$ as the *higher order subharmonic Melnikov functions*. The following result follows.

Theorem 8 *Consider the resonant torus with frequency $\omega = p/q$ for the system (3.1) with $\varepsilon = 0$. Assume that the functions $M_{k'}$ are identically zero for all $k' = 0, 1, \ldots, k-1$, and assume that t_0 is a simple zero for the function M_k. There exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the system (3.1) has at least one subharmonic solution of order q/p. Such a solution reduces to one on the unperturbed torus as $\varepsilon \to 0$.*

The proof is given in Section 3.2.

Of course, the system (2.1) can also be studied as illustrated in this section. One simply treats the parameter C as fixed, and one fixes the initial phase t_0 in such a way that Theorem 7 or Theorem 8 can be applied — of course, provided the corresponding hypotheses are satisfied. This has been done in [1] to study the subharmonic solutions of a forced cubic oscillator in the presence of dissipation.

We also note that, as a particular case of Theorem 8, it can happen that $M_k \equiv 0$ for all $k \in \mathbb{Z}_+$. In that case formal solvability of the equations holds to all orders, and the convergence of the series requires no condition on t_0, and it can be proved by proceeding as in Section 3.2. In particular in such a case the full resonant torus persists under perturbation. Of course, the identical vanishing of all functions M_k is a very unlikely situation, and, without any further parameter at our disposal, we cannot realistically expect this to happen. This shows that the persistence of the full torus when the subharmonic Melnikov function is identically zero is a very rare event.

Theorems 7 and 8 deal with the case in which $M(t_0)$ either vanishes identically or has a simple zero in some value of t_0. Another possibility is that $M(t_0) = 0$ for some t_0, and yet

t_0 is not a simple zero. In that case the vanishing of the subharmonic Melnikov function allows pushing perturbation theory in ε up to first order (included), as the following result shows.

Lemma 6 *Set $(\alpha_0(t), A_0(t)) = (\omega(A_0)t, A_0)$, with $\omega(A_0) = p/q$. Assume $\omega'(A_0) \neq 0$. Consider the subharmonic Melnikov function $M(t_0)$ in (3.2), and assume that t_0 is a zero for $M(t_0)$. Then there exist two periodic functions $\alpha_1(t)$ and $A_1(t)$, with $\langle \alpha_1 \rangle = 0$ and $\omega'(A_0)\langle A_1 \rangle + \langle F \rangle = 0$, such that $(\alpha_0(t)+\varepsilon\alpha_1(t), A_0+\varepsilon A_1(t))$ solves (3.3), up to first order in ε.*

Proof. By substituting $(\alpha_0(t)+\varepsilon\alpha_1(t), A_0+\varepsilon A_1(t))$ into the equations of motion one finds, to first order in ε,

$$\dot{\alpha}_1 = \omega'(A_0)\, A_1 + F, \qquad \dot{A}_1 = G, \tag{3.9}$$

where $F = F(\alpha_0(t), A_0, t+t_0)$ and $G = G(\alpha_0(t), A_0, t+t_0)$. Then $M(t_0) = 0$ implies $\langle G \rangle = 0$, so that the second equation in (3.9) can be solved. We write its solution as $A_1(t) = \langle A_1 \rangle + \widetilde{A}_1(t)$, with $\langle \widetilde{A}_1 \rangle = 0$, and fix $\langle A_1 \rangle$ in such a way that $\omega'(A_0)\langle A_1 \rangle + \langle F \rangle = 0$, so that also the first equation becomes soluble. Call $\alpha_1(t)$ the corresponding zero-mean solution. Then the assertion follows. ■

However, if the zero of the subharmonic Melnikov function is not simple, perturbation theory cannot be pursued further in general, as is easy to check. More generally if the higher order subharmonic Melnikov functions M_k defined in (3.8) vanish identically for all k up to some order $\bar{k}-1$, perturbation theory can be worked out up to order $\bar{k}-1$, but then it can be continued to higher order along the lines outlined before (cf. Theorem 8) only if $M_{\bar{k}}$ has a simple zero. Then, when the zero is not simple, one can ask whether some kind of perturbation theory is still possible or even, more generally, whether a subharmonic solution exists at all. We shall see that, at least with some extra assumptions, a positive answer can be given to both questions. We start with the case in which $M(t_0)$ has a non-simple zero t_0. Introduce the constant

$$a_1 := \langle \partial_1 G(\alpha(\cdot), A_0, \cdot + t_0)\, \alpha_1(\cdot) + \partial_2 G(\alpha(\cdot), A_0, \cdot + t_0)\, A_1(\cdot) \rangle, \tag{3.10}$$

expressed in terms of the functions α_1 and A_1 introduced in the statement of Lemma 6. Then we can formulate the following result.

Theorem 9 *Consider the system (3.1) and assume that A_0 be such that $\omega := \omega(A_0) = p/q$ and $\omega'(A_0) \neq 0$. Define the subharmonic Melnikov function according to (3.2), with $(\alpha_0(t), A_0(t)) = (\omega t, A_0)$. Assume that*
(i) there exists $k_0 \geq 0$ such that the derivatives $\mathrm{d}^k M(t_0)/\mathrm{d}t_0^k$ are identically zero for all $k = 0, 1, \ldots, k_0 - 1$, while $D := \mathrm{d}^{k_0} M(t_0)/\mathrm{d}t_0^{k_0} \neq 0$;
(ii) one has $a_1 \neq 0$.
Then there exists $\varepsilon_0 > 0$ such that the following assertions hold.

(1) If k_0 is odd, then for $|\varepsilon| < \varepsilon_0$ the system (3.1) has at least one subharmonic solution of order q/p, which is analytic in $|\varepsilon|^{1/k_0}$.
(2) If k_0 is even and $\varepsilon a_1 D < 0$, then for $|\varepsilon| < \varepsilon_0$ the system (3.1) has at least one subharmonic solution of order q/p, which is analytic in $|\varepsilon|^{1/k_0}$.

The proof of Theorem 9 will be given in Section 3.3. An example in which the conditions (i) and (ii) in Theorem 9 are satisfied is provided by (3.1) with $\omega(A) = A$, $F(\alpha, A, t) = 8 \sin \alpha \sin t$ and $G(\alpha, A, t) = \sin^2 \alpha \, (4 \cos^2 t - 1)$. An easy computation gives $M(t_0) = \sin^2 t_0$ and $a_1 = -1$; see Appendix D. In that case one needs $\varepsilon > 0$ in (3.1) in order to have a subharmonic solution which reduces to one of the unperturbed ones as $\varepsilon \to 0$; again see Appendix D.

The condition $a_1 \neq 0$ aims to fix the first correction to the initial phase by the second order analysis; cf. the analogous condition in [27], where fractional Lindstedt series were proved to exist for lower-dimensional tori. Therefore, it is a simplifying hypothesis, which certainly can be relaxed. We leave as an open problem to find the most general assumption on the perturbation in order to have a subharmonic solution. Note that, in general, some condition is expected to be necessary: for instance in the aforementioned example no subharmonic solution can exist for $\varepsilon < 0$. However the example corresponds to a case in which the subharmonic Melnikov function has a zero of even order. We expect that no condition at all is required on the perturbation when the zero is of odd order; cf. [56].

On the other hand Theorem 9 is stronger than Theorem 4 of [56], when applied to a model for which $a_1 \neq 0$. Indeed, it deals also with the case in which t_0 is a zero of even order. Moreover it makes precise the dependence on the perturbation parameter of the change of phase of the persisting unperturbed periodic solution: it indicates that this is analytic in $|\varepsilon|^{1/k_0}$, i.e. in a fractional power of ε (from [56] we can only deduce that for a suitable change of phase, tending to 0 as ε tends to 0, a subharmonic solution exists).

As already noted, we can imagine cases in which the functions $M_{k'}$ are identically zero for all $k' = 0, 1, \ldots, \bar{k} - 1$, while $M_{\bar{k}}$ has a zero t_0 which is not simple. In that case we define $a_{\bar{k}}$ analogously to what was done in (3.10), by considering the contributions to $G(\alpha, A, t + t_0)$ to order $\bar{k}$ which do not depend on the mean of $\alpha(t)$, i.e. which can be obtained by imposing $\alpha_0^{(k')} = 0$ for all $k' = 1, \ldots, \bar{k} - 1$. Then the following generalization of Theorem 9 holds.

Theorem 10 *Consider the system (3.1) and assume that A_0 be such that $\omega := \omega(A_0) = p/q$ and $\omega'(A_0) \neq 0$. Define the higher order subharmonic Melnikov functions according to (3.8), with $(\alpha_0(t), A_0(t)) = (\omega t, A_0)$ and $(\alpha^{(k')}(t), A^{(k')}(t))$ recursively defined for $k' = 1, \ldots, \bar{k} - 1$. Assume that*
(i) the functions $M_{k'}$ are identically zero for all $k' = 0, 1, \ldots, \bar{k} - 1$,
(ii) there is $k_0 \geq 0$ such that the derivatives $\mathrm{d}^k M_{\bar{k}}(t_0)/\mathrm{d}t_0^k$ are identically zero for all $k = 0, 1, \ldots, k_0 - 1$, while $D := \mathrm{d}^{k_0} M_{\bar{k}}(t_0)/\mathrm{d}t_0^{k_0} \neq 0$;
(ii) one has $a_{\bar{k}} \neq 0$.

Then there exists $\varepsilon_0 > 0$ such that the following assertions hold.
(1) If k_0 is odd, then for $|\varepsilon| < \varepsilon_0$ the system (3.1) has at least one subharmonic solution of order q/p, which is analytic in $|\varepsilon|^{1/k_0}$.
(2) If k_0 is even and $\varepsilon a_{\bar{k}} D < 0$, then for $|\varepsilon| < \varepsilon_0$ the system (3.1) has at least one subharmonic solution of order q/p, which is analytic in $|\varepsilon|^{1/k_0}$.

The proof can be obtained with a little effort by combining the proof of Theorem 8 with that of Theorem 9, up to some preliminary considerations which will be discussed in Section 3.4.

3.2 Higher order subharmonic Melnikov functions

To prove Theorem 8 we start by showing that the perturbation series for the subharmonic solutions, hence also for the higher order subharmonic Melnikov functions, are well defined. First of all, we need the following preliminary result.

Lemma 7 *Consider a T-periodic function F of the form $F(\omega t, A_0, t+t_0)$, with ω and A_0 constants, and denote with $\langle F\rangle$ the mean over a period T. If $\langle F\rangle$ vanishes identically in t_0 then also $\langle \partial_1^n \partial_2^m F\rangle = 0$ for all t_0 and for all $n, m \in \mathbb{Z}_+$.*

Proof. The proof is by induction on n. For $n = 0$ one has $\langle \partial_2^m F\rangle = \partial_2^m \langle F\rangle = 0$. Next, assume that $\langle \partial_1^{n-1} \partial_2^m F\rangle = 0$. Then $\omega\langle \partial_1^n \partial_2^m F\rangle = \omega\langle \partial_1(\partial_1^{n-1}\partial_2^m F)\rangle = \langle \mathrm{d}(\partial_1^{n-1}\partial_2^m F)/\mathrm{d}t\rangle - \langle \partial_3(\partial_1^{n-1}\partial_2^m F)\rangle = -\langle \partial_3(\partial_1^{n-1}\partial_2^m F)\rangle = \partial_3\langle \partial_1^{n-1}\partial_2^m F\rangle = 0$, by the inductive hypothesis. ∎

We use Lemma 7 to show that, if $\omega = \omega(A_0) = p/q$ and $\omega'(A_0) \neq 0$, the perturbation series for the subharmonic solutions of (3.3) of order q/p are well defined.

We look for a solution of (3.3) in the form (2.3) of a power series in ε. To any order k the functions $\alpha^{(k)}(t)$ and $A^{(k)}(t)$ are well defined, and given by (2.12), provided the compatibility conditions (2.10) and (2.11) are satisfied. We can rewrite (2.11), hence the second line of (2.24), as

$$\omega'(A_0)\, A_0^{(k)} + \Phi_0^{(k-1)} = 0, \tag{3.11}$$

where, by construction, the function $\Phi_0^{(k)}$ depends on the constants $\alpha_0^{(k')}$ and $A_0^{(k')}$ only for $k' < k$. Hence, we can use (3.11) to deduce $A_0^{(k)}$ in terms of the constants of lower order.

To any order k the function $G^{(k)}(t)$ can be expressed in terms of the functions $\alpha^{(k')}(t)$ and $A^{(k')}(t)$, with $k' < k$, hence it will depend on the constants $\alpha_0^{(k')}$ and $A_0^{(k')}$, with $k' < k$. For $k \geq 1$ call $\Omega^{(k)}(t)$ the function obtained from $G^{(k)}(t)$ by setting $\alpha_0^{(k')} = A_0^{(k')} = 0$ for all $k' = 1, \ldots, k$, and set $\Omega^{(0)}(t) = G^{(0)}(t)$. The following result holds.

Lemma 8 *For any $k \geq 1$ we can write $\langle G^{(k)} \rangle$ as*

$$\langle G^{(k)} \rangle = \sum_{n,m=0}^{\infty} \sum_{k_0=0}^{k} \sum_{\substack{k_1,\ldots,k_{n+m}\geq 1 \\ k_1+\ldots+k_{n+m}=k}} Z_{n,m}^{(k)} \frac{1}{n!}\frac{1}{m!} \langle \partial_1^n \partial_2^m \Omega^{(k_0)} \rangle \, \alpha_0^{(k_1)} \ldots \alpha_0^{(k_n)} A_0^{(k_{n+1})} \ldots A_0^{(k_{n+m})}, \tag{3.12}$$

where $Z_{n,m}^{(k)}$ are suitable combinatorial factors, and the term with $n = m = 0$ (which forces $k_0 = 0$) has to be interpreted as $\langle \Omega^{(k)} \rangle = \Omega_0^{(k)}$.

The proof is given in Appendix E. Note that in fact th sum over n and m in (3.12) contains only a finite number of summands. Hence for any $k \geq 2$ we can rewrite $\langle G^{(k-1)} \rangle$ in (2.10) according to (3.12).

Suppose that $\langle G^{(0)} \rangle = M(t_0)$ vanishes identically. Then the initial phase t_0 remains arbitrary and, by fixing $A_0^{(1)}$ according to (3.11) in terms of t_0, the functions $\alpha^{(1)}(t)$ and $A^{(1)}(t)$ are well defined. Moreover the constant $\alpha_0^{(1)}$ is left undetermined. In particular, $M_1(t_0) = \langle G^{(1)} \rangle$ is also well defined as it is expressed in terms of the functions $\alpha^{(1)}(t)$ and $A^{(1)}(t)$. Suppose that also $M_1(t_0)$ is identically zero; note that by Lemma 7 one has $M_1(t_0) = \Omega_0^{(1)}$, so that the vanishing of $M_1(t_0)$ does not depend on the value of the constant $\alpha_0^{(1)}$. Then we can fix $A_0^{(2)}$ from (3.11) in terms of t_0 and the parameter $\alpha_0^{(1)}$, and solve the equations of motion to second order to obtain $\alpha^{(2)}(t)$ and $A^{(2)}(t)$. Note that so far t_0, $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ are still arbitrary.

In the same way, for all k' as far as $M_{k'}(t_0) = \langle G^{(k')} \rangle$ is identically zero, the equations of motion can be solved, independently of the value of t_0 and of the constants $\alpha_0^{(1)}, \ldots, \alpha_0^{(k'-1)}$, which all remain arbitrary. Again, for this to be possible each constant $A_0^{(k')}$ has to be fixed from (3.11) in terms of t_0 and of the constants $\alpha_0^{(1)}, \ldots, \alpha_0^{(k'-1)}$, while $\alpha_0^{(k')}$ is left undetermined. Note that, again by Lemma 7, one has $M_{k'}(t_0) = \Omega_0^{(k')}$ for all such k': this makes the property that the functions $M_{k'}$ vanish identically to be independent of the values of the constants $\alpha_0^{(1)}, \ldots, \alpha_0^{(k'-1)}$.

Now, suppose that $M_{k'}(t_0)$ vanishes identically for k' up to $k-1$, and that instead $M_k(t_0)$ has a simple zero — i.e. $D := M_k'(t_0) \neq 0$. Then $M_k(t_0) = \Omega_0^{(k)}$ (by Lemma 7), and again the equations of motion to order k can be solved as in the previous cases. The only difference is that now t_0 must be fixed to be the simple zero of $M_k(t_0)$; the constants $\alpha_0^{(1)}, \ldots, \alpha_0^{(k)}$ are still arbitrary parameters, while for $k' = 1, \ldots, k$ each constant $A_0^{(k')}$ is fixed in terms of t_0 and of the parameters $\alpha_0^{(1)}, \ldots, \alpha_0^{(k'-1)}$.

Now we pass to the next order $k+1$. By Lemma 7 we can write $\langle G^{(k+1)} \rangle = \langle \partial_1 \Omega^{(k)} \rangle \alpha_0^{(1)} + \langle \partial_2 \Omega^{(k)} \rangle A_0^{(1)} + \Omega_0^{(k+1)} = 0$, where we recall that $A_0^{(1)}$ depends only on t_0. Moreover one has $\omega(A_0) \langle \partial_1 \Omega^{(k)} \rangle = \langle \mathrm{d}\Omega^{(k)}/\mathrm{d}t \rangle - \langle \partial \Omega^{(k)}/\partial t_0 \rangle = -\partial \langle \Omega^{(k)} \rangle / \partial t_0 = -\partial \langle G^{(k)} \rangle / \partial t_0 = -M_k'(t_0) = -D \neq 0$. Hence we can fix $\alpha_0^{(1)}$ in terms of t_0.

And so on: for all higher orders $k' \geq k+2$ one can write $\langle G^{(k')}\rangle$ as in (3.12). By using Lemma 7 once more we see that the sum contains a term $\langle \partial_1 \Omega^{(k)}\rangle \alpha_0^{(k'-k)}$ plus other terms which depend on the parameters $\alpha_0^{(k'')}$ only for $k'' < k'$ (hence which have been fixed in terms of t_0 at some previous step). Hence we can fix also $\alpha_0^{(k'-k)}$ in terms of t_0.

This shows that the formal series (2.3) are well defined. In particular this yields that the higher order subharmonic Melnikov functions are also well defined. Of course, we have still to prove convergence of the series. But this can be done as in the proof of Theorem 7, by using trees, and we shall not repeat the analysis.

3.3 Fractional series for subharmonic solutions

In this Section we prove Theorem 9. Set $\varepsilon = \sigma \eta^{k_0}$, with $\eta > 0$ and $\sigma \in \{\pm 1\}$ to be fixed. We shall look for solutions of (3.3) of the form

$$\alpha(t) = \alpha_0(t) + \sum_{k=1}^{\infty} \eta^k \alpha_0^{[k]} + \sum_{k=k_0}^{\infty} \eta^k \sum_{\substack{\nu\in\mathbb{Z}\\ \nu\neq 0}} \mathrm{e}^{i\nu\omega t}\alpha_\nu^{[k]}, \qquad A(t) = A_0 + \sum_{k=k_0}^{\infty} \eta^k \sum_{\nu\in\mathbb{Z}} \mathrm{e}^{i\nu\omega t} A_\nu^{[k]}, \quad (3.13)$$

where a different notation for the Taylor label is used with respect to (2.3) to stress that we are expanding in η and not in ε. Then (3.3) becomes, for all $k \geq 1$ and $\nu \neq 0$,

$$(i\omega\nu)^2 \alpha_\nu^{[k]} = (i\omega\nu)\Phi_\nu^{[k]} + \omega'(A_0)\,\Gamma_{\boldsymbol{\nu}}^{[k]}, \qquad (i\omega\nu)A_\nu^{[k]} = \Gamma_\nu^{[k]}, \quad (3.14)$$

provided one has

$$\omega'(A_0)\,A_0^{[k]} + \Phi_0^{[k]} = 0, \qquad \Gamma_0^{[k]} = 0, \quad (3.15)$$

for all $k \geq 1$ and for $\nu = 0$. In (3.14) and (3.15) we have defined

$$\Phi = \omega(A) - \omega(A_0) - \omega'(A_0)\,(A - A_0) + \varepsilon F(\alpha, A, t + t_0), \qquad \Gamma = \varepsilon G(\alpha, A, t + t_0), \quad (3.16)$$

and denoted by $\Phi_\nu^{[k]}$ and $\Gamma_\nu^{[k]}$ the Fourier component with label ν of the contribution of order k in η of the function Φ and Γ, respectively; cf. for instance (2.25) for analogous notations. Note that in (3.16) the parameter ε must be expressed in terms of η as $\varepsilon = \sigma\eta^{k_0}$.

In (3.15) it is convenient to write

$$\widetilde{\Phi}_0^{[k]} := \omega'(A_0)\,A_0^{[k]} + \Phi_0^{[k]} = 0, \quad (3.17)$$

where by construction $\Phi_0^{[k]}$ can depend on the constants $A_0^{[k']}$ only for $k' < k$.

Before proving that a solution of the form (3.11) really exists, we need a preliminary result which generalises Lemma 5.

Lemma 9 *One has* $(-\omega(A_0))^j \langle \partial_1^j G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle = \mathrm{d}^j M(t_0)/\mathrm{d}t_0^j$ *for all* $j \in \mathbb{Z}_+$.

The proof is omitted as it can be easily obtained by induction on j, by reasoning as in the proof of Lemma 5.

The identities (3.15) are satisfied for $k < k_0$ because Φ and Γ are of order $\varepsilon = O(\eta^{k_0})$; see (3.16). In particular for all $k < k_0$ one has $A_0^{[k]} = 0$, while the parameters $\alpha_0^{[k]}$ can assume any value.

For $k = k_0$ the identities (3.15) can be obtained by fixing t_0 so that $M(t_0) = 0$ and choosing $A_0^{[k_0]}$ according to (3.17). The parameter $\alpha_0^{[k]}$, so far, remains arbitrary.

For $k_0 < k < 2k_0$ the identities $\Gamma_0^{[k]} = 0$ are still satisfied by the assumption (i) on $M(t_0)$, by Lemma 9 and by the observation that the first constant $A_0^{[k]}$ to be non-vanishing is that with $k = k_0$. The identities $\widetilde{\Phi}_0^{[k]} = 0$ can be made to hold by fixing recursively the constants $A_0^{[k]}$ when equating to zero the right hand side of (3.17).

For $k = 2k_0$ again the identity $\widetilde{\Phi}_0^{[2k_0]} = 0$ can be easily imposed by suitably choosing $A_0^{[2k_0]}$. The identity $\Gamma_0^{[2k_0]} = 0$ can be dealt with as follows. By using the assumptions on $M(t_0)$ and Lemma 9 we can write

$$\Gamma_0^{[2k_0]} = \frac{1}{k_0!} D(\alpha_0^{[1]})^{k_0} + \sigma a_1 = 0, \tag{3.18}$$

where a_1 is defined according to (3.10). Note that up to order $2k_0 - 1$ the solution (3.13) equals the solution obtained from the naive perturbation theory in ε up to first order, up to the values of the parameters $\alpha_0^{[k]}$, which now are arbitrary and which, in any case, do not appear in a_1. Therefore, since $D \neq 0$ by assumption (i), we can use (3.18) to fix $\alpha_0^{[1]}$ as

$$\alpha_0^{[1]} = \left(-\frac{k_0! \sigma a_1}{D} \right)^{1/k_0}, \tag{3.19}$$

provided this expression makes sense. If k_0 is odd then we can fix both $\sigma = 1$ and $\sigma = -1$: as $\varepsilon = \sigma\eta^{k_0}$, this means that both the cases $\varepsilon > 0$ and $\varepsilon < 0$ can be taken into account. On the contrary if k_0 is even we are forced to fix σ in such a way that $\sigma a_1 D < 0$, and, as a consequence, only either positive or negative values of ε can be considered. This justifies the different assertions for odd k_0 and even k_0 in the statement of the theorem.

To go to higher orders $k > 2k_0$ simply note that the identities $\widetilde{\Phi}_0^{[k]} = 0$ can be obtained once more by suitably fixing $A_0^{[k]}$. On the other hand the identities $\Gamma_0^{[k]} = 0$ can be obtained by writing

$$\Gamma_0^{[k]} = \frac{1}{(k_0 - 1)!} D(\alpha_0^{[1]})^{k_0 - 1} \alpha_0^{[k - 2k_0 + 1]} + \widetilde{\Gamma}_0^{[k]}, \tag{3.20}$$

where $\widetilde{\Gamma}_0^{[k}$, by construction, can depend on the constants $\alpha_0^{[k']}$ only for $k' < k - 2k_0 + 1$. Hence we can use (3.20) to fix $\alpha_0^{[k-2k_0+1]}$ for $k > 2k_0$ as

$$\alpha_0^{[k-2k_0+1]} = -\frac{(k_0 - 1)!}{D(\alpha_0^{[1]})^{k_0 - 1}} \widetilde{\Gamma}_0^{[k]}, \tag{3.21}$$

provided, of course, $\alpha_0^{[1]} \neq 0$. But this follows from assumption (ii) — indeed such an assumption was made exactly with this aim.

We can summarise the discussion as follows. If we fix $\alpha_0^{[1]}$ according to (3.19) and set

$$A_0^{[k]} = -\frac{1}{\omega'(A_0)}\Phi_0^{[k]}, \quad k \geq k_0, \qquad \alpha_0^{[k]} = -\frac{(k_0-1)!}{D(\alpha_0^{[1]})^{k_0-1}}\widetilde{\Gamma}_0^{[k+2k_0-1]}, \quad k \geq 2, \tag{3.22}$$

then we can find a $2\pi/\omega$-periodic solution of (3.3) in the form of a formal power series in η. The convergence of the series can be discussed exactly as in Section 2.3 and Appendix B, and no further difficulties arise.

3.4 Fractional series when the subharmonic Melnikov function is zero

Theorem 10 can be proved essentially by reasoning as for Theorem 8. The main difference is that we have to take into account the recursion scheme envisaged in Section 3.3 to deal with the case in which the subharmonic Melnikov function has a zero with is not simple.

So, we look for solutions of the form (3.13) instead of (2.3). As in Section 3.2, the equations of motion can be solved to any order provided the compatibility conditions (3.15) are satisfied.

Note that the higher order subharmonic Melnikov functions can be expressed as $M_k(t_0) = \langle G^{[kk_0]}\rangle$. For all $k \geq k_0+1$ one can write $\langle G^{[k]}\rangle$ as

$$\langle G^{[k]}\rangle = \sum_{n,m=0}\sum_{\tilde{k}=0}^{k}\sum_{\substack{k_1,\ldots,k_{n+m}\geq 1\\ k_1+\ldots+k_{n+m}=k-\tilde{k}}} Z_{n,m}^{(k)} \frac{1}{n!}\frac{1}{m!}\langle\partial_1^n\partial_2^m\Omega^{[\tilde{k}]}\rangle\,\alpha_0^{[k_1]}\ldots\alpha_0^{[k_n]}A_0^{[k_{n+1}]}\ldots A_0^{[k_{n+m}]} \tag{3.23}$$

where the function $\Omega^{[k]}$ is obtained from $G^{[k]}$ by setting $\alpha_0^{[k']} = A_0^{[k']} = 0$ for all $1 \leq k' < k$ — see (3.12) in Section 3.2 for analogous notations — and the term with $n = m = 0$ has to be interpreted as $\Omega_0^{[k]}$. The proof of (3.23) proceeds as that of Lemma 9 given in Appendix E.

By assumption one has $\langle G^{[kk_0]}\rangle = 0$ for all $k = 1,\ldots,\bar{k}-1$, hence, because of Lemma 7, also $\langle\partial_1^n\partial_2^m G^{[kk_0]}\rangle = 0$ for all $k = 1,\ldots,\bar{k}-1$ and all $n,m \in \mathbb{Z}_+$. Note that in principle, the property that all functions $M_{k'}$ vanish up to order $k' = \bar{k}-1$ could be ill-posed, because of the presence of the arbitrary constants $\alpha_0^{[1]},\ldots,\alpha^{[k]})_0$, but we shall see, thanks to (3.23), that in fact such a property holds independently of the values of these constants.

For $1 \leq k < k_0$ the functions $\widetilde{\Phi}^{[k]}$ and $\Gamma^{[k]}$ are identically zero, so that for all $k = 1,\ldots,k_0-1$ one obtains $\alpha_\nu^{[k]} = 0$ for $\nu \neq 0$ and $A_\nu^{[k]} = 0$ for all ν, according to (3.13). Moreover the constants $\alpha_0^{[1]},\ldots,\alpha_0^{[k_0-1]}$ remain arbitrary, as well as t_0.

To order k_0 one has $\langle G^{[k_0]}\rangle = \langle G^{(1)}\rangle = 0$ for all t_0, by assumption. Again the equations of motion can be solved, provided $A_0^{[k_0]}$ is suitably fixed in terms of t_0 according to (3.17). On the contrary $\alpha_0^{[k_0]}$ is not fixed, and gives another arbitrary constant.

When considering the following orders $k_0 < k < \bar{k}k_0 - 1$ we can reason essentially in the same way. The equations of motion can be solved to any order, independently of the value of the constants $\alpha_0^{[1]}, \dots, \alpha^{[k-k_0]}$, because for $1 \le k \le \bar{k} - 1$ one has $\langle \Omega^{[kk_0]}\rangle = \langle G^{[kk_0]}\rangle = 0$ by assumption, hence $\langle \partial_1^n \partial_2^m \Omega^{[kk_0]}\rangle = 0$ for all $n, m \in \mathbb{Z}_+$ by Lemma 7.

Therefore up to order $\bar{k}k_0 - 1$ the initial phase t_0 and the constants $\alpha^{[1]}, \dots, \alpha^{[\bar{k}k_0-1]}$ are still arbitrary parameters, while each constant $A_0^{[k]}$, for $k = k_0 + 1, \dots, \bar{k}k_0 - 1$, must be fixed in terms of t_0 and $\alpha_0^{[1]}, \dots, \alpha_0^{[k-k_0]}$.

To order $\bar{k}k_0$, the same argument applies. The only difference is that now t_0 is to be fixed as the non-simple zero of $M(t_0)$, which exists by assumption. This, together with a suitable choice of the constant $A_0^{[\bar{k}k_0]}$, allows to solve the equations of motion to order $\bar{k}k_0$.

To orders $\bar{k}k_0 < k < (\bar{k}+1)k_0 - 1$ again we can rely on Lemma 7 to deduce that $\langle G^{[k]}\rangle$ vanishes for any choice of the parameters $\alpha_0^{[1]}, \dots, \alpha_0^{[\bar{k}k_0]}$.

The first non-trivial contribution arises for $k = (\bar{k}+1)k_0$. In that case one has, from (3.22),

$$\langle G^{[(\bar{k}+1)k_0]}\rangle = \langle \partial_1^{k_0} \Omega^{[\bar{k}k_0]}\rangle (\alpha_0^{[1]})^{k_0} + \sigma a_{\bar{k}} = 0, \tag{3.24}$$

where $a_{\bar{k}}$ takes into account all the other contributions, and depends only on t_0. Since, by assumption (ii), one has $(-\omega(A_0))^{k_0} \langle \partial_1^{k_0} G^{[\bar{k}k_0]}\rangle = \partial_3^{k_0} \langle G^{[\bar{k}k_0]}\rangle = \mathrm{d}^{k_0} M_{\bar{k}}(t_0)/\mathrm{d}t_0^{k_0} = D \neq 0$, then $\langle \partial_1^{k_0} \Omega^{[\bar{k}k_0]}\rangle = \langle \partial_1^{k_0} G^{[\bar{k}k_0]}\rangle$ is also different from zero, so that we can use (3.24) to fix $\alpha_0^{[1]}$ in terms of t_0.

For all $k > (\bar{k}+1)k_0$ we can reason as in Section 3.2, and can fix the constants $\alpha^{[k-(\bar{k}+1)k_0+1]}$ in terms of t_0, by using that $\langle \partial_3^{k_0} G^{[\bar{k}k_0]}\rangle$ and $\alpha_0^{[1]}$ are both non-zero.

Hence, eventually we find recursion relations for all the constants $\alpha_0^{[k]}$ and $A_0^{[k]}$ in terms of t_0. Once more the convergence the series can be discussed as was done in Section 2.3 and Appendix B.

A Proof based on the implicit function theorem

Here we sketch a proof of existence of subharmonic solutions, based on the application of the implicit function theorem.

Without loss of generality we can assume $A_0 = 0$. Set $\omega(A_0) = \omega$ and $\omega'(A_0) = k$, and rescale $A = \varepsilon\xi$. Then in terms of (α, ξ) the equation (2.1) becomes

$$\begin{cases} \dot{\alpha} = \omega + \varepsilon\left(k\xi + F(\alpha, 0, C, t) + \varepsilon\, f(\alpha, \xi, C, t, \varepsilon)\right), \\ \dot{\xi} = G(\alpha, 0, C, t) + \varepsilon\, g(\alpha, \xi, C, t, \varepsilon), \end{cases} \tag{A.1}$$

for suitable analytic functions f and g. The corresponding Poincaré map, that is the stroboscopic map at time $T = 2\pi/\omega$, reads

$$\begin{cases} \alpha \to \alpha + \omega T + \varepsilon \displaystyle\int_0^T \mathrm{d}t \left(k\xi(t) + F(\alpha(t), 0, C, t) + \varepsilon\, f(\alpha(t), \xi(t), C, t, \varepsilon) \right), \\ \xi \to \xi + \displaystyle\int_0^T \mathrm{d}t \left(G(\alpha(t), 0, C, t) + \varepsilon\, g(\alpha(t), \xi(t), C, t, \varepsilon) \right), \end{cases} \tag{A.2}$$

which can be rewritten as

$$\begin{cases} \alpha \to \alpha + \omega T + \varepsilon \left(k\xi + N_1(\alpha, C) + \varepsilon\, f_1(\alpha, \xi, C, \varepsilon) \right), \\ \xi \to \xi + M_1(\alpha, C) + \varepsilon\, g_1(\alpha, \xi, C, \varepsilon), \end{cases} \tag{A.3}$$

for suitable analytic functions M_1, N_1, f_1 and g_1. Here the origin of time is fixed as $t = 0$, so that $\alpha = \alpha(0)$ becomes the free parameter: up to this difference in notation, $M_1(\alpha, C)$ is the subharmonic Melnikov function (2.2). Therefore existence of a fixed point for the Poincaré map, hence of a periodic solution with period T for the system (A.1), requires

$$\begin{cases} k\xi + N_1(\alpha, C) + \varepsilon\, f_1(\alpha, \xi, C, \varepsilon) = 0, \\ M_1(\alpha, C) + \varepsilon\, g_1(\alpha, \xi, C, \varepsilon) = 0, \end{cases} \tag{A.4}$$

which, under the further Hypothesis 2, entails an analytic solution $C = C(\alpha, \varepsilon)$. In turn, from this we can deduce the assertions of Theorems 1 to 5.

The proof is only apparently simpler. First, we have not given an explicit expression of all the functions involved. Second, obtaining a formula for them to within any given order essentially requires going through the calculations of perturbation theory described in the text. Of course, if only an existence result is required, the implicit function theorem method would be more direct.

B Tree formalism

Trees are defined in the standard way. We briefly recall the basic notations, by referring to [25] for an introductory review and further details, and also to [26, 34] for a discussion in similar contexts.

A tree θ is defined as a partially ordered set of points, connected by oriented *lines*. The lines are consistently oriented toward a unique point $\mathfrak{r}$ called the *root*. The root admits only one entering line called the *root line*. All points except the root are called *nodes*. Denote with $V(\theta)$ and $L(\theta)$ the set of nodes and lines in θ, respectively, and with $|L(\theta)|$ and $|V(\theta)|$ the number of lines and nodes of θ, respectively.

If a line ℓ connects two points $\mathfrak{v}_1, \mathfrak{v}_2$ and is oriented from $\mathfrak{v}_2$ to $\mathfrak{v}_1$, we say that $\mathfrak{v}_2 \prec \mathfrak{v}_1$ and we shall write $\ell_{\mathfrak{v}_2} = \ell$. We shall say also that ℓ exits $\mathfrak{v}_2$ and enters $\mathfrak{v}_1$. It can be convenient to imagine that the line ℓ carries an arrow pointing toward the node $\mathfrak{v}_1$: the arrow will be thought of as superimposed on the line itself.

More generally we write $\mathfrak{v}_2 \prec \mathfrak{v}_1$ if $\mathfrak{v}_1$ is on the path of lines connecting $\mathfrak{v}_2$ to the root: hence the orientation of the lines is opposite to the partial ordering relation $\prec$. Along the path from $\mathfrak{v}_2$ to $\mathfrak{v}_1$ all arrows point toward $\mathfrak{v}_1$. In particular all arrows point toward the root.

Each line ℓ carries a pair of labels (h_ℓ, δ_ℓ), with $h_\ell \in \{\alpha, A, C\}$ and $\delta_\ell \in \{1, 2\}$ such that $\delta_\ell = 1$ for $h_\ell \neq \alpha$. We call h_ℓ and δ_ℓ the *component label* and the *degree label* of the line ℓ, respectively. Given a node $\mathfrak{v}$ call $r_{\mathfrak{v}1}$, $r_{\mathfrak{v}2}$, and $r_{\mathfrak{v}3}$ the number of lines entering $\mathfrak{v}$ carrying a component label $h = \alpha$, $h = A$, and $h = C$, respectively. Hence, the values of $r_{\mathfrak{v}1}, r_{\mathfrak{v}2}, r_{\mathfrak{v}3}$ are uniquely determined by the component labels of the lines entering $\mathfrak{v}$.

We associate with each node $\mathfrak{v}$ two *mode labels* $\nu_\mathfrak{v}, \sigma_\mathfrak{v} \in \mathbb{Z}$ and we also set for convenience $h_\mathfrak{v} = h_{\ell_\mathfrak{v}}$ and $\delta_\mathfrak{v} = \delta_{\ell_\mathfrak{v}}$. We also introduce a further *badge label* $\beta_\mathfrak{v}$ by setting $\beta_\mathfrak{v} \in \{0, 1\}$ when $h_\mathfrak{v} = h$ and $\delta_\mathfrak{v} = 1$ and $\beta_\mathfrak{v} = 1$ in all the other cases.

With each line ℓ we associate a further label $\nu_\ell \in \mathbb{Z}$, called the *momentum* of the line, such that

$$\nu_\ell = \nu_{\ell_\mathfrak{v}} = \sum_{\substack{\mathfrak{w} \in V(\theta) \\ \mathfrak{w} \preceq \mathfrak{v}}} (\nu_\mathfrak{w} + \sigma_\mathfrak{w}) , \tag{B.1}$$

with the constraints that $\nu_\ell = 0$ if $h_\ell = C$ and $\nu_\ell \neq 0$ if $h_\ell = \alpha$. The relation (B.1) expresses a conservation law at each node: the momentum of the line exiting $\mathfrak{v}$ is the sum of the momenta of the lines entering $\mathfrak{v}$ plus the mode labels of the node $\mathfrak{v}$ itself. Note that the momentum "flows" through each line in the sense of the arrow superimposed on the line.

The trees with all the labels listed above are called *labelled trees.* Then given a labelled tree θ we associate with each line ℓ a *propagator*

$$g_\ell = \begin{cases} \dfrac{\omega'(A_0)^{\delta_\ell - 1}}{(i\omega\nu_\ell)^{\delta_\ell}}, & h_\ell = \alpha, A, \quad \nu_\ell \neq 0, \\[3ex] -\dfrac{1}{\omega'(A_0)}, & h_\ell = A, \quad \nu_\ell = 0, \\[3ex] -\dfrac{1}{D(t_0)}, & h_\ell = C, \quad \nu_\ell = 0, \end{cases} \tag{B.2}$$

and with each node $\mathfrak{v}$ a *node factor*

$$
N_{\mathfrak{v}} = \begin{cases}
\dfrac{(i\nu_0)^{r_{\mathfrak{v}1}}\partial_2^{r_{\mathfrak{v}2}}\partial_3^{r_{\mathfrak{v}3}}}{r_{\mathfrak{v}1}!r_{\mathfrak{v}2}!r_{\mathfrak{v}3}!}\mathrm{e}^{i\sigma_{\mathfrak{v}}t_0}F_{\nu_{\mathfrak{v}},\sigma_{\mathfrak{v}}}(A_0,C_0(t_0)), & h_{\mathfrak{v}}=\alpha, \quad \delta_{\mathfrak{v}}=1, \quad \beta_{\mathfrak{v}}=1 \\[2ex]
\dfrac{\partial_2^{r_{\mathfrak{v}2}}}{r_{\mathfrak{v}2}!}\omega(A_0), & h_{\mathfrak{v}}=\alpha, \quad \delta_{\mathfrak{v}}=1, \quad \beta_{\mathfrak{v}}=0 \\[2ex]
\dfrac{(i\nu_0)^{r_{\mathfrak{v}1}}\partial_2^{r_{\mathfrak{v}2}}\partial_3^{r_{\mathfrak{v}3}}}{r_{\mathfrak{v}1}!r_{\mathfrak{v}2}!r_{\mathfrak{v}3}!}\mathrm{e}^{i\sigma_{\mathfrak{v}}t_0}G_{\nu_{\mathfrak{v}},\sigma_{\mathfrak{v}}}(A_0,C_0(t_0)), & h_{\mathfrak{v}}=\alpha, \quad \delta_{\mathfrak{v}}=2, \quad \beta_{\mathfrak{v}}=1 \\[2ex]
\dfrac{(i\nu_0)^{r_{\mathfrak{v}1}}\partial_2^{r_{\mathfrak{v}2}}\partial_3^{r_{\mathfrak{v}3}}}{r_{\mathfrak{v}1}!r_{\mathfrak{v}2}!r_{\mathfrak{v}3}!}\mathrm{e}^{i\sigma_{\mathfrak{v}}t_0}G_{\nu_{\mathfrak{v}},\sigma_{\mathfrak{v}}}(A_0,C_0(t_0)), & h_{\mathfrak{v}}=A, \quad \delta_{\mathfrak{v}}=1, \quad \beta_{\mathfrak{v}}=1 \\[2ex]
\dfrac{(i\nu_0)^{r_{\mathfrak{v}1}}\partial_2^{r_{\mathfrak{v}2}}\partial_3^{r_{\mathfrak{v}3}}}{r_{\mathfrak{v}1}!r_{\mathfrak{v}2}!r_{\mathfrak{v}3}!}\mathrm{e}^{i\sigma_{\mathfrak{v}}t_0}G_{\nu_{\mathfrak{v}},\sigma_{\mathfrak{v}}}(A_0,C_0(t_0)), & h_{\mathfrak{v}}=C, \quad \delta_{\mathfrak{v}}=1, \quad \beta_{\mathfrak{v}}=1
\end{cases} \tag{B.3}
$$

with the constraints that when $h_{\mathfrak{v}} = C$ (and $\delta_{\mathfrak{v}} = 1$) one has either $r_{\mathfrak{v}3} \geq 2$ or $r_{\mathfrak{v}1}+r_{\mathfrak{v}2} \geq 1$, and when $\beta_{\mathfrak{v}} = 0$ (and $h_{\mathfrak{v}} = h$, $\delta_{\mathfrak{v}} = 1$) one has $r_{\mathfrak{v}1} = r_{\mathfrak{v}3} = 0$ and $r_{\mathfrak{v}2} \geq 2$. These constraints reflect the condition $*$ in (2.28) and, respectively, the fact that only derivatives with respect to A appear in (2.25).

Finally we define the *value* of a tree θ the number

$$
\mathrm{Val}(\theta) = \Big(\prod_{\ell\in L(\theta)} g_\ell\Big)\Big(\prod_{\mathfrak{v}\in V(\theta)} N_{\mathfrak{v}}\Big), \tag{B.4}
$$

which is a well-defined quantity: indeed all propagators and node factors are bounded quantities.

Call the *order* of the tree θ the number

$$
k(\theta) = \{\ell \in L(\theta) : h_\ell \neq C, \quad \beta_\ell \neq 0\}, \tag{B.5}
$$

the *total momentum* of θ the momentum $\nu(\theta)$ of the root line, and the *total component label* of θ the component label $h(\theta)$ associated to the root line. The number of nodes (and lines) of any tree θ is related to its order $k(\theta)$ as follows.

Lemma 10 *For any tree θ one has $|L(\theta)| = |V(\theta)| \leq 3k(\theta)$.*

Proof. The equality $|L(\theta)| = |V(\theta)|$ is obvious by construction. We prove by induction on k the bounds

$$
|V(\theta)| \leq \begin{cases} 3k(\theta)-2, & h(\theta)=\alpha, A, \\ 3k(\theta)-1, & h(\theta)=C. \end{cases} \tag{B.6}
$$

For $k = 1$ the bound (B.6) is trivially satisfied, as a direct check shows: simply compare (2.30) to (2.32) with the definition of trees in that case. Assume that the bound holds for all $k' < k$, and let us show that then it holds also for k. Call ℓ_0 the root line of θ and $\mathfrak{v}_0$ the node which the root line exits. Call r_1, r_2, and r_3 the number of lines entering $\mathfrak{v}_0$ with component labels α, A, and C, respectively, and denote with $\theta_1, \ldots, \theta_{r_1+r_2+r_3}$ the subtrees which have those lines as root lines. Then

$$|V(\theta)| = 1 + \sum_{r=j}^{r_1+r_2+r_3} |V(\theta_j)|. \tag{B.7}$$

If ℓ_0 has component label $h_{\ell_0} \in \{\alpha, A\}$ and badge label $\beta_{\ell_0} = 1$ we have

$$|V(\theta)| \leq 1 + 3\,(k-1) - r_3 - 2\,(r_1 + r_2) \leq 3k - 3 < 3k - 2, \tag{B.8}$$

by the inductive hypothesis and by the fact that $k(\theta_1) + \ldots + k(\theta_{r_1+r_2+r_3}) = k - 1$. If ℓ_0 has component label $h_{\ell_0} = \alpha$ and badge label $\beta_{\ell_0} = 0$ we have

$$|V(\theta)| \leq 1 + 3k - r_2 \leq 3k - 3 < 3k - 2, \tag{B.9}$$

by the inductive hypothesis, by the fact that $k(\theta_1) + \ldots + k(\theta_{r_1+r_2+r_3}) = k$, and by the constraint that $r_2 \geq 2$ and $r_1 = r_3 = 0$. Finally if ℓ_0 has component label $h_{\ell_0} = C$ we have

$$|V(\theta)| \leq 1 + 3k - r_3 - 2\,(r_1 + r_2) \leq 3k - 1, \tag{B.10}$$

by the inductive hypothesis, by the fact that $k(\theta_1) + \ldots + k(\theta_{r_1+r_2+r_3}) = k$, and by the constraint that either $r_3 \geq 2$ or $r_1 + r_2 \geq 1$ in such a case — cf. the comment after (B.3). Therefore the assertion is proved. ∎

Define $\Theta_{k,\nu,h}$ as the set of all trees of order $k(\theta) = k$, total momentum $\nu(\theta) = \nu$, and total component label $h(\theta) = h$. By collecting together all the definitions given above, one obtains the following result.

Lemma 11 *The Fourier coefficients $\alpha_\nu^{(k)}$ and $A_\nu^{(k)}$ and the constants C_k can be written in terms of trees as*

$$\begin{aligned} \alpha_\nu^{(k)} &= \sum_{\theta \in \Theta_{k,\nu,\alpha}} \mathrm{Val}(\theta), \qquad \nu \neq 0, \qquad\qquad \alpha_0^{(k)} = 0 \\ A_\nu^{(k)} &= \sum_{\theta \in \Theta_{k,\nu,A}} \mathrm{Val}(\theta), \qquad C^{(k)} = \sum_{\theta \in \Theta_{k,0,C}} \mathrm{Val}(\theta). \end{aligned} \tag{B.11}$$

for all $k \geq 1$.

The proof of (B.11) can be performed by induction; cf. [25] for details.

The number of unlabelled trees of order k is bounded by the number of random walks of $2k$ steps, hence by 2^{2k} [37]. The sum over all labels except the mode labels and the momenta is bounded again by a constant to the power k — simply because all such labels can assume only a finite number of values. Finally the sum over the mode labels — which uniquely determine the momenta through the relation (B.1) — can be performed by using for each node half the exponential decay factor $\mathrm{e}^{-\kappa(|\nu_\mathfrak{v}|+|\sigma_\mathfrak{v}|)}$ provided by the bounds (2.21). The conclusion is that we obtain eventually the following result.

Lemma 12 *The Fourier coefficients and constants in (B.11) satisfy the bounds*

$$\left|\alpha_\nu^{(k)}\right| \le B_1 B_2^k \mathrm{e}^{-\kappa|\nu|/2}, \qquad \left|A_\nu^{(k)}\right| \le B_1 B_2^k \mathrm{e}^{-\kappa|\nu|/2}, \qquad \left|C^{(k)}\right| \le B_1 B_2^k, \tag{B.12}$$

for suitable constants B_1 and B_2.

The bounds of Lemma 12 prove the convergence of the series (2.3) and (2.4) for $|\varepsilon| < \varepsilon_0$, with ε_0 small enough. Note that with respect to [25] here the analysis is much easier as there is no small divisors problem.

The construction described above also provides a useful algorithm which can be implemented numerically in order to compute the solution to any prescribed accuracy (provided ε is small enough).

C Proof of Lemma 2

Write the system (2.35) in action-angle variables. Then there exists a Hamiltonian function $H(\alpha, A, t, \varepsilon) = H_0(A) + \varepsilon H_1(\alpha, A, t)$ such that $\omega(A) = \partial_A H_0(A)$ and

$$\begin{cases} \dot\alpha = \omega(A) + \varepsilon \partial_A H_1(\alpha, A, C, t) + \varepsilon C\,\Phi(\alpha, A), \\ \dot A = -\varepsilon \partial_\alpha H_1(\alpha, A, C, t) + \varepsilon C\,\Psi(\alpha, A), \end{cases} \tag{C.1}$$

where $\Phi = -y\,\partial\alpha/\partial y$ and $\Psi = y\,\partial A/\partial y$. Then (2.23) become

$$\begin{aligned} \alpha_\nu^{(k)} &= \frac{1}{i\omega\nu}\left(U^{(k)} + \partial_A H_1^{(k-1)}\right)_\nu + \omega'(A_0)\frac{1}{(i\omega\nu)^2}\left(-\partial_\alpha H_1^{(k-1)}\right)_\nu + (C\Phi)_\nu^{(k-1)}, \\ A_\nu^{(k)} &= \frac{1}{i\omega\nu}\left(-\partial_\alpha H_1^{(k-1)}\right)_\nu + (C\Psi)_\nu^{(k-1)}, \end{aligned} \tag{C.2}$$

for all $k \in \mathbb{N}$ and all $\nu \neq 0$, with $U^{(1)} = 0$ and $U^{(k)} = [\omega(A) - \omega(A_0) - \omega'(A_0)\,(A - A_0)]^{(k)}$ for $k \ge 2$. Moreover (2.27) reads

$$\sum_{k'=0}^{k} C_{k'}\Psi_0^{(k')} + \bar\Gamma_0^{(k)} = 0, \qquad \bar\Gamma_0^{(k)} = \left(-\partial_\alpha H_1^{(k-1)}\right)_0, \tag{C.3}$$

which, for $k=\bar{k}$, gives $\Gamma_0^{(\bar{k})}=\bar{\Gamma}_0^{(\bar{k})}$ and $C_{\bar{k}}\Psi_0^{(0)}+\bar{\Gamma}_0^{(\bar{k})}=0$ because $C_1=\ldots=C_{\bar{k}-1}=0$ by assumption. Moreover $\Psi^{(0)}=-\langle y_0^2\rangle\neq 0$, by Lemma 1 and Hypothesis 2.

Therefore $C_k=C^{(k)}$, with $C^{(k)}$ given by the sum (B.11) of tree values. We can split the set $\Theta_{k,0,C}$ into the union of disjoint families $\mathcal{F}$ as follows. Given a tree $\theta\in\Theta_{k,0,C}$ call $\mathfrak{v}_0$ the node which is connected to the root through the root line, and define $V_0(\theta)$ as the subset of nodes $\mathfrak{v}\in V(\theta)$ such that all the lines ℓ along the path connecting $\mathfrak{v}$ to $\mathfrak{v}_0$ have $\nu_\ell\neq 0$. Then define $\mathcal{F}=\mathcal{F}(\theta)$ as the set of trees obtained from θ by "shifting" the root line to any node in $V_0(\theta)$, i.e. by attaching the root line to any node $\mathfrak{v}\in V_0(\theta)$. Of course, as a consequence of the shift of the root line from $\mathfrak{v}_0$ to $\mathfrak{v}$, the arrows of all lines along the path between the two nodes are reversed. If one recalls the diagrammatic rules introduced in Section 2.3 to associate with any tree θ a value $\mathrm{Val}(\theta)$, this means that all lines with labels $(h,\delta)=(\alpha,1)$ are transformed into lines with labels $(h,\delta)=(A,1)$. Moreover the momenta of all such lines change sign. The latter property can be seen as follows. The momentum is defined as the sum of all mode labels of the nodes preceding the lines — cf. (B.1) — and the sum of all the mode labels is zero for any tree $\theta\in\Theta_{k,0,C}$: then, when the arrow of a line ℓ is reversed the nodes preceding ℓ become the nodes following ℓ and vice versa, so that ν_ℓ becomes $-\nu_\ell$. Hence the propagators of the lines ℓ with $\delta_\ell=1$ change sign, whereas the propagators of the lines ℓ with $\delta_\ell=2$ are left unchanged. As a consequence, for each tree $\theta'\in\mathcal{F}(\theta)$ we can write $\mathrm{Val}(\theta)=i\nu_{\mathfrak{v}}\overline{\mathrm{Val}}(\theta)$, where $\mathfrak{v}$ is the node $\mathfrak{v}\in V_0(\theta)$ which the root line exits and $\overline{\mathrm{Val}}(\theta)$ is the same quantity for all $\theta'\in\mathcal{F}(\theta)$. Therefore

$$\sum_{\theta'\in\mathcal{F}(\theta)}\mathrm{Val}(\theta)=\overline{\mathrm{Val}}(\theta)\sum_{\mathfrak{v}\in V_0(\theta)}i\nu_{\mathfrak{v}}. \tag{C.4}$$

Moreover one has

$$\sum_{\mathfrak{v}\in V(\theta)}(\nu_{\mathfrak{v}}+\sigma_{\mathfrak{v}})=0 \quad\Longrightarrow\quad \sum_{\mathfrak{v}\in V_0(\theta)}(\nu_{\mathfrak{v}}+\sigma_{\mathfrak{v}})=0 \quad\Longrightarrow\quad \sum_{\mathfrak{v}\in V_0(\theta)}\nu_{\mathfrak{v}}=-\sum_{\mathfrak{v}\in V_0(\theta)}\sigma_{\mathfrak{v}}, \tag{C.5}$$

so that the mean in t_0 of (C.4) gives

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}t_0}{2\pi}\sum_{\theta'\in\mathcal{F}(\theta)}\mathrm{Val}(\theta) &= \int_0^{2\pi}\frac{\mathrm{d}t_0}{2\pi}\overline{\mathrm{Val}}(\theta)\sum_{\mathfrak{v}\in V_0(\theta)}i\nu_{\mathfrak{v}}\\ &= -\int_0^{2\pi}\frac{\mathrm{d}t_0}{2\pi}\overline{\mathrm{Val}}(\theta)\sum_{\mathfrak{v}\in V_0(\theta)}i\sigma_{\mathfrak{v}}=0,\end{aligned} \tag{C.6}$$

because the mean is the sum over all labels $\sigma_{\mathfrak{v}}\in V(\theta)$ such that $\sum_{v\in V(\theta)}\sigma_{\mathfrak{v}}=\sum_{v\in V_0(\theta)}\sigma_{\mathfrak{v}}=0$. By using the fact that the set $\Theta_{k,0,C}$ can be written as a disjoint union of the sets $\mathcal{F}$, we obtain that $\bar{\Gamma}_0^{(\bar{k})}$ has zero mean in t_0, so that the assertion follows. ∎

D An example

In this appendix we give an example where the conditions of Theorem 9 are satisfied. Consider the system

$$\begin{cases} \dot{\alpha} = A + 8\varepsilon \sin\alpha \sin(t+t_0), \\ \dot{A} = \varepsilon \sin^2\alpha \left(4\cos^2(t+t_0) - 1\right). \end{cases} \tag{D.1}$$

and consider the unperturbed solutions $(\alpha_0(t), A_0(t)) = (t, 1)$ with period 2π.

Set $s(t) = \sin t$ and $c(t) = \cos t$. As usually we denote by $\langle \cdot \rangle$ the mean of any 2π-periodic function; one has $\langle s^2 \rangle = 1/2$, $\langle s^2 c^2 \rangle = 1/8$, $\langle s^4 \rangle = 3/8$, and $\langle s^2 c^4 \rangle = 1/16$.

The subharmonic Melnikov function (3.2) becomes

$$M(t_0) = 4\langle s^2 c^2 \rangle \cos^2 t_0 + 4\langle s^4 \rangle \sin^2 t_0 - 8\langle c^3 s \rangle \sin t_0 \cos t_0 - \langle s^2 \rangle = \sin^2 t_0, \tag{D.2}$$

so that $M(0) = M'(0) = 0$ and $M''(0) = 2$.

To first order one has $\dot{A}_1 = \sin^2 t (4\cos^2 t - 1)$, so that $A_1(t) = \bar{A}_1 + \sin^3 t \cos t$, where $\bar{A}_1 = \langle A_1 \rangle$ has to be fixed by requiring $\bar{A}_1 + 8\langle s^2 \rangle = 0$; this gives $\bar{A}_1 = -4$. Then $\dot{\alpha}_1 = A_1 + 8\sin^2 t$ can be integrated, and gives $\alpha_1(t) = \bar{\alpha}_1 + \sin^4 t/4 - 4\sin t \cos t$, with $\bar{\alpha}_1$ such that $\langle \alpha_1 \rangle = 0$.

Therefore we have found that, by setting

$$\alpha_1(t) = \bar{\alpha}_1 + \frac{1}{4}\sin^4 t - 4\sin t \cos t, \qquad A_1(t) = -4 + \sin^3 t \cos t, \tag{D.3}$$

then $(t + \varepsilon\alpha_1(t), 1 + \varepsilon A_1(t))$ solve (3.3) with $t_0 = 0$ up to the first order. The constant a can be expressed in terms of such an approximate solution according to (3.10). By using that $\partial_1 G(\alpha, A, t) = 2\sin\alpha\cos\alpha(4\cos^2 t - 1)$ and $\partial_2 G(\alpha, A, t) = 0$, one has

$$\begin{aligned} a_1 &= 2\langle sc\left(4c^2 - 1\right)\left(s^4/4 - 4sc\right)\rangle \\ &= 2\langle s^5 c^3 \rangle - \frac{1}{2}\langle s^5 c \rangle - 32\langle s^2 c^4 \rangle + 8\langle s^2 c^2 \rangle = 0 + 0 - 2 + 1 = -1, \end{aligned} \tag{D.4}$$

hence $a_1 \neq 0$. Since $k_0 = 2$ one must require $\varepsilon a_1 D < 0$, which yields $\varepsilon = \eta^2 > 0$. Hence for ε positive and small enough there is a subharmonic solution of order 1.

It is not difficult to see that if $\varepsilon < 0$ there is no subharmonic solution of order 1 which reduces to one of the unperturbed ones as $\varepsilon \to 0$. This can be obtained by trying to write the solution in the form $\alpha = t + \eta + \beta$ and $A = 1 + B$, with $\langle \beta \rangle = 0$, and η, β and B all tending to 0 as $\varepsilon \to 0$, and explicitly checking that no solution of this form can exist. The discussion proceeds as in [27], Appendix B, which we refer to for details.

E Proof of Lemma 9

The expression $\langle G^{(k)} \rangle = G_0^{(k)}$ can be written as in (2.26), with G instead of F and $r_3 = 0$ (as there is no parameter C in the perturbation). Suppose now that we express all Fourier coefficients $\alpha_{\nu'}^{(k')}$ and $A_{\nu'}^{(k')}$ in terms of trees, except those with $\nu'=0$, which are kept as free parameters. We can iterate the construction, and every time a Fourier coefficient with label $\nu'=0$ appears, it is not further expanded. In this way we obtain eventually a tree with two kinds of end-points (i.e. of nodes with no entering lines), according to the value of the momentum of the exiting line. If the line ℓ exiting the end-point $\mathfrak{v}$ carries a momentum $\nu_\ell \neq 0$, then the end-point $\mathfrak{v}$ has the same labels and factors as the other points which have entering lines. If on the contrary one has $\nu_\ell = 0$, then the end-point $\mathfrak{v}$ carries the labels $k_\mathfrak{v} \in \mathbb{N}$ and $h_\mathfrak{v} \in \{\alpha, A\}$, and represents either $\alpha_0^{(k_\mathfrak{v})}$ (if $h_\mathfrak{v}=\alpha$) or $A_0^{(k_\mathfrak{v})}$ (if $h_\mathfrak{v}=A$). If the corresponding exiting line ℓ connect $\mathfrak{v}$ to a node $\mathfrak{w}$ then there is a further derivative acting on the node factor associated to the node $\mathfrak{w}$: such a derivative will be ∂_1 if $h_\mathfrak{v}=\alpha$ and ∂_2 if $h_\mathfrak{v}=A$; of course, if $(\nu_\mathfrak{w},\sigma_\mathfrak{w})$ are the mode labels associated to the node $\mathfrak{w}$ then ∂_1 yields a factor $i\nu_\mathfrak{w}$; see (B.3).

We can represent the tree as a tree with leaves: the leaves $\mathfrak{v}_1, \mathfrak{v}_2, \ldots$ represent the new kind of end-points, together with the corresponding exiting lines, while the rest of the tree, say θ_0, differs from those considered in Section 2.3 because of the extra possible derivatives acting on the node factors. Of course the order of θ_0 will be equal to k minus the sum of the labels $k_\mathfrak{v}$ associated to all the leaves $\mathfrak{v}$.

If we neglect these extra derivatives then the product of node factors and propagators in θ_0 gives a value $\mathrm{Val}(\theta_0)$, which would be a contribution to $G_0^{(k_0)}$. More precisely it is a contribution to $\Omega_0^{(k_0)}$ because it does not contain any coefficient $\alpha_0^{(k')}$ nor $A_0^{(k')}$.

Suppose now that we collect together all trees with the same leaves. Take, for instance, the case of trees with only one leaf representing $\alpha_0^{(1)}$ (for which $k_0=k-1$), and consider all trees have all the same θ_0. All of them are obtained by attaching the leaf to a node of θ_0 and applying an extra derivative ∂_1 to the node factor associated to that node. If we sum together all these contributions we obtain a quantity proportional to $\partial_1 \mathrm{Val}(\theta_0)$ times $\alpha_0^{(1)}$. If we sum over all possible choices of θ_0 we reconstruct $\langle \partial_1 G^{(k-1)} \rangle \alpha_0^{(1)}$.

The argument applies in general, independently on the number of leaves and their orders, so that, by grouping together all trees with the same θ_0 and with the same leaves, we reconstruct a contribution $\partial_1^n \partial_2^m \mathrm{Val}(\theta_0)$ times the constants represented by the $n+m$ leaves that we are considering. Summing all possible choices of θ_0 and of leaves, we arrive at (3.12), with suitable numbers $Z_{n,m}^{(k)}$ which takes into account the combinatorics.

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