Existence and construction of travelling wavefront solutions of Fisher equations with fourth order perturbations

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Abstract: We consider a generalised Fisher equation containing a fourth order spatial derivative term. Of interest is the question of the existence of travelling front solutions of the equation and their qualitative form. When the fourth order term has sufficiently small coefficient existence of such fronts is shown using invariant manifold theory. Asymptotic analysis is employed to construct an expression for the travelling front when its speed is large.

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1 Introduction

It is well known that Fisher's equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u) \tag{1.1}$$

admits travelling front solutions which connect the two uniform steady state solutions $u = 0$ and $u = 1$. Indeed, setting $u(x,t) = U(z)$ where $z = x - ct$, these fronts satisfy the ODE

$$U'' + cU' + U(1-U) = 0. \tag{1.2}$$

If the boundary conditions for the latter are taken as $U(-\infty) = 1$ and $U(\infty) = 0$ then the waves are necessarily rightward moving ($c > 0$) and, furthermore, if $c \geq 2$ the waves are strictly monotonically decreasing for all $z \in (-\infty, \infty)$.

It is the object of the present paper to discover to what extent these travelling fronts are preserved when one adds a fourth order spatial derivative term to (1.1), so that the equation becomes

$$\frac{\partial u}{\partial t} = -\varepsilon^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u(1-u). \tag{1.3}$$

Our approach, after converting (1.3) to travelling wave form, is first to show that an invariant manifold exists for the four-dimensional system of ODE's that the desired front must satisfy. The question of the existence of the front in this manifold is then resolved using Fredholm theory. When ε is small the front still looks like the Fisher front. We then compute how the front changes qualitatively when ε is not so small. Perturbation theory is used to obtain an asymptotic expression for the front when the wave speed c approaches infinity.

Equations like (1.3) are not entirely new. A slight generalisation of (1.3) has been studied by Cohen & Murray [4] as a possible model for pattern formation in a single species population. They carry out linear stability and bifurcation analyses about the uniform steady states. Their Laplacian term u_{xx} can be given a negative coefficient, when the term has a destabilising effect on a uniform steady state. On the other hand, a fourth order term with a negative coefficient always has a stabilising influence. They conclude that the interplay between these two terms can be a mechanism for spatial pattern formation. Biologically, fourth order derivative terms are considered to model *long range* effects. Murray [7] explains how equations with fourth (and even higher) order derivatives can be approximations of equations with *nonlocal* terms.

There are, in fact, various papers that study equations like (1.3) from the point of view of the steady solutions. For example, Peletier and Troy [8] show that fourth order diffusivity can give rise to chaotic steady spatial patterns. There are, however, fewer papers that consider the question of the existence of travelling fronts and their qualitative profiles. However, Dee & van Saarloos [5] consider equation (1.3) with a cubic nonlinearity. Akveld & Hulshof [1] consider a fourth order equation similar to (1.3) but again with a different nonlinearity. The latter authors, like us, use dynamical systems theory to construct an invariant manifold for the travelling wave equations. However, their approach to confirming the existence of a travelling front within the manifold is quite different from ours, involving the use of the implicit function theorem. In the present paper, we prove existence of the front by employing the Fredholm alternative. Further, the classical asymptotic analysis we use later in this paper is not restricted to ε small.

Methods similar to those of the present paper may also be used to prove existence of travelling front solutions to certain delay equations which can be reformulated as systems which are of higher order but without delay, and such that the ODE's for travelling front solutions will have order three or higher [2].

2 Preliminaries

If we convert (1.3) to travelling wave form, by writing $u(x,t) = U(z)$, $z = x - ct$, we obtain

$$-\varepsilon^2 U'''' + U'' + cU' + U(1-U) = 0 \tag{2.1}$$

and, defining $\phi = U'$, $p = \phi'$ and $q = \varepsilon p'$, we can recast (2.1) as a system:

$$\begin{aligned} U' &= \phi, \\ \phi' &= p, \\ p' &= \tfrac{1}{\varepsilon} q, \\ q' &= \tfrac{1}{\varepsilon}(p + c\phi + U(1-U)). \end{aligned} \tag{2.2}$$

The equilibrium states (steady states) of this system are $(U, \phi, p, q) = (0, 0, 0, 0)$ and $(1, 0, 0, 0)$ and these states are, of course, independent of ε.

Of course, the ODE for Fisher fronts, namely (2.1) with $\varepsilon = 0$, is only second order and thus can be analysed in the (U, U') phase plane in which the required travelling wave is simply a connection between a saddle point at $(1, 0)$ and (if $c \geq 2$) a stable node at the origin. With $\varepsilon > 0$ we are working in a four dimensional phase space. The plausibility of a heteroclinic connection between the two equilibrium states can be seen by calculating the dimensions of the local stable and unstable manifolds of these two equilibria. We then rigorously confirm the existence of a front in the subsequent sections of the paper.

The linearised (Jacobian) matrix J of the system (2.2) is

$$J(U, \phi, p, q) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\varepsilon \\ (1-2U)/\varepsilon & c/\varepsilon & 1/\varepsilon & 0 \end{pmatrix}$$

and, at the steady state $(0, 0, 0, 0)$ the eigenvalues λ of this matrix satisfy

$$\varepsilon^2 \lambda^4 - \lambda^2 - c\lambda - 1 = 0.$$

For ε sufficiently small it is easily seen that (i) if $c \geq 2$ this equation has three real negative roots and one real positive root; (ii) if $0 < c < 2$ it has two real roots of opposite sign, and two complex roots with negative real part. In either case the dimension of the stable manifold of the steady state $(0, 0, 0, 0)$ is three. Similarly it can be seen that for ε sufficiently small the eigenvalues at the steady state $(1, 0, 0, 0)$ are all real (two positive and two negative). The unstable manifold at this steady state therefore has dimension two.

The sum of the dimensions of the stable and unstable manifolds mentioned above is five, while the phase space has dimension four, so in principle they can intersect along a one-dimensional curve in $\mathbf{R}^4$. In the next sections we confirm the existence of a connection between the steady states $(1, 0, 0, 0)$ and $(0, 0, 0, 0)$, i.e., a travelling front solution for (1.3).

For the time being we shall fix $c \geq 2$ so that a monotone travelling front exists for the unperturbed ($\varepsilon = 0$) problem. Our aim is to show that such a front still exists for the perturbed problem when ε is sufficiently small. Later it will be argued that, in the perturbed problem, monotone fronts also exist for values of c just below 2.

3 Existence of an invariant manifold

Written slightly differently, system (2.2) is

$$\begin{aligned} dU/dz &= \phi, \\ d\phi/dz &= p, \\ \varepsilon dp/dz &= q, \\ \varepsilon dq/dz &= p + c\phi + U(1-U). \end{aligned} \tag{3.1}$$

Writing $\zeta = z/\varepsilon$, (3.1) becomes

$$\begin{aligned} dU/d\zeta &= \varepsilon\phi, \\ d\phi/d\zeta &= \varepsilon p, \\ dp/d\zeta &= q, \\ dq/d\zeta &= p + c\phi + U(1-U), \end{aligned} \tag{3.2}$$

which we call the dual "fast system" associated with (3.1).

If in (3.1) ε is set to zero, then U and ϕ are governed by

$$\frac{d^2U}{dz^2} + c\frac{dU}{dz} + U(1-U) = 0, \qquad \phi = \frac{dU}{dz}$$

(as for the Fisher front) while p and q must lie on the set

$$M_0 := \{(U, \phi, p, q) \in \mathbf{R}^4 : \quad q = 0 \text{ and } p + c\phi + U(1-U) = 0\} \tag{3.3}$$

which is a two-dimensional submanifold of $\mathbf{R}^4$. The system only makes sense on this manifold, and the manifold is invariant for the system.

What we claim is that, for ε sufficiently small, there exists a two-dimensional submanifold M_ε of $\mathbf{R}^4$ which is within distance ε of M_0 and which is invariant for the flow (3.1). By Fenichel's invariant manifold theory (Fenichel [6]) such a perturbed invariant manifold M_ε will exist if M_0 is *normally hyperbolic.* A manifold M_0 is said to be normally hyperbolic if the linearisation of the fast system, restricted to M_0, has exactly $\dim M_0$ eigenvalues on the imaginary axis, with the remainder of the system hyperbolic.

The linearisation of the fast system (3.2), restricted to M_0 (i.e. $\varepsilon = 0$) has matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-2U & c & 1 & 0 \end{pmatrix}$$

and the eigenvalues of this are $0, 0, 1, -1$. Thus, M_0 is normally hyperbolic and, by Fenichel's theory, for sufficiently small ε there exists a perturbed manifold M_ε with the properties listed above.

We should like to determine the dynamics on M_ε. We can write

$$M_\varepsilon = \{(U, \phi, p, q) \in \mathbf{R}^4 : \quad q = g(U, \phi, \varepsilon) \text{ and } p = -c\phi - U(1-U) + h(U, \phi, \varepsilon)\}$$

where the functions g and h (to be found) satisfy

$$g(U, \phi, 0) = h(U, \phi, 0) = 0.$$

Substitution into (3.1) yields that $g(U, \phi, \varepsilon)$ and $h(U, \phi, \varepsilon)$ satisfy the partial differential equations

$$\varepsilon\left((2U-1)\phi + cU(1-U) + c^2\phi - ch + \phi\frac{\partial h}{\partial U} + (-U(1-U) - c\phi + h)\frac{\partial h}{\partial \phi}\right) = g,$$

$$\varepsilon\left(\phi\frac{\partial g}{\partial U} + (-c\phi - U(1-U) + h)\frac{\partial g}{\partial \phi}\right) = h.$$

Next, we expand g and h in Taylor series in the variable ε:

$$g(U, \phi, \varepsilon) = g(U, \phi, 0) + \varepsilon g_\varepsilon(U, \phi, 0) + \tfrac{1}{2}\varepsilon^2 g_{\varepsilon\varepsilon}(U, \phi, 0) + \cdots,$$

and similarly for h and substitute into the above partial differential equations. Powers of ε^0 give

$$g(U, \phi, 0) = h(U, \phi, 0) = 0$$

as expected. Powers of ε give

$$\begin{aligned} g_\varepsilon(U, \phi, 0) &= (2U-1)\phi + c^2\phi + cU(1-U), \\ h_\varepsilon(U, \phi, 0) &= 0, \end{aligned}$$

and powers of ε^2 give

$$\tfrac{1}{2}g_{\varepsilon\varepsilon}(U,\phi,0) = 0,$$
$$\tfrac{1}{2}h_{\varepsilon\varepsilon}(U,\phi,0) = \phi(2\phi + c(1-2U)) + (-U(1-U) - c\phi)(2U - 1 + c^2).$$

Thus

$$h(U,\phi,\varepsilon) = \varepsilon^2 h_1(U,\phi,\varepsilon)$$

where

$$h_1(U,\phi,\varepsilon) = \phi(2\phi + c(1-2U)) + (-U(1-U) - c\phi)(2U - 1 + c^2) + O(\varepsilon)$$

and the system (3.1) becomes

$$\begin{aligned} dU/dz &= \phi, \\ d\phi/dz &= -U(1-U) - c\phi + \varepsilon^2(\phi(2\phi + c(1-2U)) + (-U(1-U) - c\phi)(2U - 1 + c^2)), \end{aligned} \tag{3.4}$$

which determines the dynamics on the slow manifold M_ε. We shall study this system in the next section.

4 The flow on the manifold M_ε

When $\varepsilon = 0$, system (3.4) reduces to the coupled first order ODEs for the classical Fisher front. We let (U_0, ϕ_0) be the solution of (3.4) when $\varepsilon = 0$. In the (U,ϕ) phase plane, this front is a connection between a saddle point at $(1,0)$ and (if $c \geq 2$) a stable node at $(0,0)$. Note that when $\varepsilon > 0$, the system (3.4) still has these same critical points, together with another at $(\frac{1}{2}(1-c^2-1/\varepsilon^2), 0)$ that is not part of the region of phase space of interest for us. We shall now use Fredholm theory to show that when $\varepsilon > 0$ is sufficiently small a heteroclinic connection exists between the critical points $(1,0)$ and $(0,0)$ of (3.4). The connection corresponds to a travelling front solution of (1.3).

To seek such a connection set

$$U = U_0 + \varepsilon^2 \tilde{U}, \qquad \phi = \phi_0 + \varepsilon^2 \tilde{\phi},$$

and substitute into (3.4). To lowest order the differential equation system governing $\tilde{U}, \tilde{\phi}$ is

$$\frac{d}{dz}\begin{pmatrix} \tilde{U} \\ \tilde{\phi} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 2U_0(z) - 1 & -c \end{pmatrix}\begin{pmatrix} \tilde{U} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ h_1(U_0, \phi_0, 0) \end{pmatrix} \tag{4.1}$$

and we want to prove this system has a solution satisfying

$$\tilde{U}, \tilde{\phi} \to 0 \quad \text{as} \quad z \to \pm\infty.$$

Working in the space L^2 of square integrable functions, with the inner product

$$\int_{-\infty}^{\infty} (\mathbf{x}(z), \mathbf{y}(z))\, dz,$$

$(\cdot,\cdot)$ being the Euclidean inner product on $\mathbf{R}^2$, Fredholm theory says that (4.1) will have a solution if and only if

$$\int_{-\infty}^{\infty} \left(\mathbf{x}(z), \begin{pmatrix} 0 \\ h_1(U_0(z), \phi_0(z), 0) \end{pmatrix} \right) dz = 0$$

for all functions $\mathbf{x}(z)$ in the kernel of the adjoint of the operator defined by the left hand side of (4.1).

With the above inner product the adjoint equation is

$$\frac{d\mathbf{x}}{dz} = \begin{pmatrix} 0 & 1 - 2U_0(z) \\ -1 & c \end{pmatrix} \mathbf{x}. \tag{4.2}$$

Although the matrix in (4.2) is non-constant, U_0 is the Fisher front about which we know a great deal. Letting $z \to \infty$ we have $U_0 \to 0$ and the matrix is then a constant matrix with eigenvalues λ satisfying

$$\lambda^2 - c\lambda + 1 = 0.$$

Since $c \geq 2$ the eigenvalues are both real and positive. Thus as $z \to \infty$ any solution of (4.2), other than the zero solution, must grow exponentially. The only solution in L^2 is therefore the zero solution $\mathbf{x}(z) = \mathbf{0}$ and consequently the Fredholm orthogonality condition trivially holds. We have proved existence of the desired connection in the manifold M_ε. We summarise our result in the following theorem.

Theorem 1 *If $c \geq 2$ there exists ε_0 such that, for every $\varepsilon \in (0, \varepsilon_0]$, equation (1.3) admits a monotone travelling front solution $u(x,t) = U(z)$ satisfying $U(-\infty) = 1$ and $U(\infty) = 0$, where $z = x - ct$.*

Although our analysis has required $c \geq 2$ the following formal analysis, using the differential equations (3.4), suggests that for a fixed (small) value of $\varepsilon > 0$ monotone travelling fronts also exist for c just below 2. Recall that when $\varepsilon = 0$ the minimum wave propagation speed, for waves that are everywhere positive, is 2. We wish to see how this minimum speed changes when $\varepsilon > 0$ is small. We linearise far ahead of the front where $U \to 0$, i.e., about the critical point $(0, 0)$ of (3.4). The linearised matrix about this critical point has eigenvalues λ satisfying

$$\lambda^2 + (c - \varepsilon^2 c(2 - c^2))\lambda + 1 - \varepsilon^2(1 - c^2) = 0.$$

To have solutions decaying to the origin with U remaining positive the origin must be a node rather than a focus. Thus, we need

$$(c - \varepsilon^2 c(2 - c^2))^2 \geq 4(1 - \varepsilon^2(1 - c^2))$$

and equality will give the minimum speed $c_{\min}$. Since ε is small we attempt the ansatz

$$c_{\min} = 2 - \sum_{n=1}^{\infty} c_n \varepsilon^{2n}$$

which yields $c_{\min} = 2 - \varepsilon^2 + \frac{15}{4}\varepsilon^4 + \cdots$. Thus the minimum propagation speed is slightly lower than for the classical Fisher front.

5 The wave profile when ε is large

We have shown in previous sections that if ε is small, equation (1.3) admits a rightward moving travelling front solution that connects the uniform states $u = 0$ and $u = 1$. Our aim in this section is to obtain some further information about this front when ε is no longer small (assuming, of course, that the front still persists). We rewrite (1.3) as

$$\frac{\partial u}{\partial t} = -\alpha \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u(1 - u) \tag{5.1}$$

where $\alpha > 0$, or, in travelling wave form,

$$-\alpha U'''' + U'' + cU' + U(1 - U) = 0 \tag{5.2}$$

with $U(-\infty) = 1$ and $U(\infty) = 0$.

We first show that for α sufficiently large, $U(z)$ no longer decays *monotonically* to 1 as $z \to -\infty$. Linearising about $U = 1$ and seeking solutions of the linearised equation proportional to $\exp(\lambda z)$ gives

$$f(\lambda) := \alpha\lambda^4 - \lambda^2 - c\lambda + 1 = 0. \tag{5.3}$$

We can prove that for α sufficiently large (5.3) has at least two complex roots, by showing that $f'(\lambda)$ has only one real root. Indeed, $f'(\lambda) = 4\alpha\lambda^3 - 2\lambda - c$ and, bearing in mind the signs of the coefficients of the latter, a simple graphical argument shows that $f'(\lambda) = 0$ will have only one real root if $f'(\lambda_0) < 0$ where λ_0 is the smaller of the two roots of $f''(\lambda) = 0$. Calculating λ_0 we find that $\lambda_0 = -1/\sqrt{6\alpha}$ and

$$f'(\lambda_0) = \frac{4}{3\sqrt{6\alpha}} - c$$

which indeed is negative for α sufficiently large.

In fact when α is sufficiently large $f(\lambda) = 0$ has no real roots at all. Indeed, for large α we have already established $f'(\lambda) = 0$ has only one real root, which will give a global minimum for f. The root of $f' = 0$ is easily seen to be $O(\alpha^{-1/3})$ for large α and, at this root, $f \sim 1 > 0$ so that $f = 0$ has no real roots.

Next we will construct an asymptotic expression for the travelling front $U(z)$ satisfying (5.2) in the limit when the wave speed $c \to \infty$. There is no requirement that α be small. This analysis will reveal the main qualitative differences between travelling front solutions of (5.1) and those of Fisher's equation. The analysis is based on ideas of Canosa [3] who constructed an asymptotic solution to Fisher's equation when $c \to \infty$. In fact Canosa's solution is known to be highly accurate for any $c \geq 2$ (Murray [7]).

For c large we define $\varepsilon = 1/c^2$ and seek a solution of (5.2) of the form

$$U(z) = g(\zeta) \qquad \text{where} \qquad \zeta = z/c = \sqrt{\varepsilon}z.$$

Equation (5.2) becomes

$$-\alpha\varepsilon^2 g'''' + \varepsilon g'' + g' + g(1-g) = 0 \tag{5.4}$$

with boundary conditions $g(-\infty) = 1$ and $g(\infty) = 0$. Since this problem is invariant to translations in ζ we may also take

$$g(0) = \tfrac{1}{2}.$$

This avoids arbitrary constants appearing in the solution. We seek a solution to (5.4) in the form

$$g(\zeta) = g_0(\zeta) + \varepsilon g_1(\zeta) + \varepsilon^2 g_2(\zeta) + \cdots$$

where $g_0(0) = \frac{1}{2}$ and $g_n(0) = 0$ for $n \geq 1$. Powers of ε^0 give

$$g_0' + g_0(1 - g_0) = 0, \qquad\qquad g_0(0) = \tfrac{1}{2}$$

which has solution $g_0(\zeta) = (\exp(\zeta) + 1)^{-1}$. To next order,

$$g_1' + (1 - 2g_0)g_1 = -g_0''$$

which has solution

$$g_1(\zeta) = \frac{e^\zeta}{(1+e^\zeta)^2} \ln\left(\frac{4e^\zeta}{(e^\zeta+1)^2}\right).$$

Powers of ε^2 give

$$g_2' + (1 - 2g_0)g_2 = \alpha g_0'''' - g_1'' + g_1^2$$

the solution of which is

$$\begin{aligned}
g_2(\zeta) = \tfrac{1}{2}e^\zeta(e^\zeta+1)^{-4}\big(&-6 - 4\zeta + 4\zeta e^{2\zeta}\ln 2 + 4\alpha e^{2\zeta}\ln(e^\zeta+1) - 2\alpha\zeta e^{2\zeta} \\
&8\alpha e^\zeta \ln(e^\zeta+1) + 4\ln(e^\zeta+1) + 12\alpha e^\zeta - 4\zeta e^{2\zeta}\ln(e^\zeta+1) - 4\alpha\zeta e^\zeta - 4(\ln 2)^2 \\
&-4\ln 2 + 6e^{2\zeta} - 6\alpha - 6\alpha e^{2\zeta} + 4e^{2\zeta}\ln 2 + 4e^{2\zeta}(\ln 2)^2 - 8e^{2\zeta}\ln 2\ln(e^\zeta+1) - 2\alpha\zeta \\
&+4\alpha\ln(e^\zeta+1) + 8\ln 2\ln(e^\zeta+1) - 4\zeta\ln 2 - 4\alpha\ln 2 - 4(\ln(e^\zeta+1))^2 - \zeta^2 - 4\zeta e^\zeta \\
&+\zeta^2 e^{2\zeta} + 4e^{2\zeta}(\ln(e^\zeta+1))^2 - 4e^{2\zeta}\ln(e^\zeta+1) + 4\zeta\ln(e^\zeta+1) - 8\alpha e^\zeta\ln 2 - 4\alpha e^{2\zeta}\ln 2\big).
\end{aligned}$$

In terms of the original variables, our expression for the travelling front is therefore

$$U(z) = g_0(z/c) + \frac{1}{c^2}g_1(z/c) + \frac{1}{c^4}g_2(z/c) + \cdots \tag{5.5}$$

with g_0, g_1 and g_2 given as above. Figure 1 shows the graph of this asymptotic solution.

Figure 1 here

Note that the effect of the fourth order term in (1.3) is only seen at second order; the g_0 and g_1 terms in (5.5) do not involve α and are precisely the same as are obtained when applying this solution procedure to Fisher's equation. This suggests that travelling fronts are very robust to the addition of fourth order terms not only in terms of their persistence but also in that there is little change in their qualitative form. Note however that Fig. 1 is for a wave speed of 1.6 yet the front is still positive everywhere. In Fisher's equation the minimum speed for a positive front is 2.

6 Discussion

We have shown that if ε in (1.3) is sufficiently small then (1.3) admits travelling front solutions connecting the uniform states $u = 0$ and $u = 1$. We have also constructed an asymptotic expression for the front when $c \to \infty$ and this does not require the fourth order term to be small. The fact that α (in (5.1)) only appears in the second order term of the asymptotic solution, which has a coefficient of c^{-4}, with the first two terms being the same as in the corresponding asymptotic solution of Fisher's equation, suggests that the introduction of the fourth order term does not have a particularly significant effect on the profile unless α is large when oscillations will develop on its leeward side. Otherwise the main effect of this term seems to be to reduce the front's minimum speed.

We have not addressed the question of the stability of the fronts which is, of course, a nontrivial matter even for Fisher's equation. An interesting and worthwhile further investigation would be to determine what classes of initial data evolve to travelling fronts that are everywhere positive. Positivity of solutions in fourth order equations is a genuinely difficult problem since little is known about the sign of the fourth order term in the evolution and thus maximum-principle and related theorems do not apply. At present we are working on positivity preservation theorems for (1.3), but on a finite spatial domain.

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Figure Caption

Fig. 1: Plot of the travelling front solution of (1.3) given by expression (5.5). Parameter values were $c = 1.6$ and $\alpha = 2.5$.