On the Positivity of Solutions for a Generalized Diffusion Model

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Abstract

The objective of this paper aims to prove positivity of solutions for the following semi-linear dissipative partial differential equation

$$u_t = -\alpha u_{xxxx} - u_{xx} - uu_x + u(1 - u^2).$$

The equation can be considered as a model equation representing a class of dissipative partial differential equations containing differential operators of higher order than the Laplacian. It appears in various scientific contexts including population dynamics and chemical reactions, where the solutions of the model must be positive functions.

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1 Introduction

An important question in the study of the behaviour of solutions of dissipative partial differential equations is whether solutions with positive initial data will remain positive at any later time. In dissipative equations such as reaction-diffusion equations where the highest order differential operator is the Laplacian, it is often possible to establish positivity results by using the maximum-principle [16]. However, the maximum principle does not apply in the study of dissipative partial differential equations which contain differential operators of higher order than that of the Laplacian, and, consequently, it is a very difficult problem to establish results on positivity of solutions. In this paper we shall investigate a particular class of dissipative partial differential equations which possess a uniform steady state solution. For this class of equation, by using the sharpest interpolation inequalities to date, we shall prove under a certain condition that solutions are eventually (asymptotically) positive.

Positivity of solutions is important in many fields of applied mathematics such as mathematical biology, evolution of densities in chemical reactions and in particle dynamics. In such fields the solution of the equation may represent some physical quantity that cannot be negative, so a basic requirement of any realistic model is that its solutions should remain positive for all times. There are a number of important contributions to this problem in the literature (see for example Refs. [1, 8, 9, 11] and references therein). The present paper establishes *asymptotic* positivity of solutions of our equation, for all values of the parameter $\alpha > \frac{1}{4}$ (see below), and positivity of solutions for *all* values of the time t within an appropriate range of α. Achieving such results constitutes real progress in an equation where maximum principles are not available.

The equations which we study naturally extend the result on positivity of solutions already established for a particular class of equations [4], (see also Refs. [5, 6, 7]).

We shall consider the equation

$$u_t = -\alpha u_{xxxx} - u_{xx} - uu_x + u(1-u^2), \tag{1}$$

where α is a positive parameter, u is a scalar valued function of (x,t). The equation (1) defines a particular convection reaction diffusion system, where the $-\alpha u_{xxxx}$ and $-u_{xx}$ terms represent a positive and negative linear diffusion respectevely, with the $-\alpha u_{xxxx}$ having a stabilizing and the $-u_{xx}$ term having a destabilizing effect; this system has spatially uniform steady state solutions given by $u = \pm 1$, and it has been shown in cite10 that they are stable to small spatial disturbances as long as the term u_{xx} has a positive coefficient; however if one changes the sign from u_{xx} to $-u_{xx}$ then solutions in the form of spatially structured non-linear patterns may occur; this is the main reason why we shall take the minus sign in front of the Laplacian. The uu_x is the convective term and $u(1-u^2)$ represents the reaction term. For the above equation we shall use periodic boundary conditions on the spatial domain $\Omega = [0,1]$.

We define a set of quantities as follows:

$$H_n := \|\frac{\partial^n u}{\partial x^n}\|_2^2 = \int \left(\frac{\partial^n u}{\partial x^n}\right)^2 \mathrm{d}x, \quad \overline{\|u\|_\infty} := sup_{x\in[0,1]}|u(x)|, \tag{2}$$

where the integration is over the spatial domain $\Omega = [0,1]$.

Below we shall use Energy and Ladder methods [2, 3, 13, 15, 17], and interpolation inequalities [4, 13, 15], to obtain estimates for the above quantities, which we use for proving regularity and positivity of solutions for our equation. It is well known [13, 15] that the above quantities are fundamental for obtaining estimates on the behaviour of solutions of any partial differential equation (PDE); in particular in one spatial dimension it is sufficient to have control on the H_0 and the H_1 norm in order to be able to control the $\|u\|_\infty$ of the solutions, thereby giving a-priori estimates on any solution of the PDE. We begin with the analysis of H_0. We first differentiate it with respect to time and we then insert the right hand side of (1) obtaining,

$$\frac{1}{2}\dot{H}_0 \;=\; -\,\alpha\, H_2 + H_1 - \int (u^2 u_x)\,\mathrm{d}x + H_0 - \int u^4\,\mathrm{d}x, \tag{3}$$

where the dot denotes differentiation with respect to time. Note that on periodic boundary conditions the term $\int(u^2u_x)\,\mathrm{d}x = \int(\frac{1}{3}\frac{\partial u^3}{\partial x})\,\mathrm{d}x$ is zero; moreover the $-\alpha H_2$ and the H_1 terms have been obtained by integration by parts (notice that the integrated terms vanish on periodic boundary conditions).

Now we make some transformation of some of the terms in (3): we interpolate the second term on the right-hand side of (3) to get

$$H_1 \le H_2^{1/2} H_0^{1/2} \le \frac{1}{2}(\alpha H_2 + \frac{H_0}{\alpha});$$

we also notice that $-\int u^4\,\mathrm{d}x \le -H_0^2$. Therefore (3) becomes

$$\frac{1}{2}\dot{H}_0 \;\le\; -\,\frac{\alpha}{2}\,H_2 + (1+\frac{1}{2\alpha})H_0 - H_0^2. \tag{4}$$

By ignoring the $-\,\frac{\alpha}{2}\,H_2$ term and by using an absorbing ball argument we finally obtain that the long time asymptotic behaviour (denoted here and throughout by an overbar) of H_0 is given by

$$\overline{H_0} \le \frac{2\alpha+1}{2\alpha}. \tag{5}$$

As we are looking for regularity for the solutions of our model we need to have estimates on the H_1 seminorm. Similarly, differentiating it with respect to time and inserting the right hand side of (1) we obtain

$$\frac{1}{2}\dot{H}_1 \;=\; -\,\alpha\, H_3 + H_2 - \int u_x (u u_x)_x\,\mathrm{d}x + H_1 - \int u_x (u^3)_x\,\mathrm{d}x. \tag{6}$$

Computing the various derivatives one can see that (6) can be cast into

$$\frac{1}{2}\dot{H}_1 \;=\; -\,\alpha\, H_3 + H_2 \,+\, H_1 - 3\int u^2 u_x^2\,\mathrm{d}x - \int u_x^3\,\mathrm{d}x - \int u u_x u_{xx}\,\mathrm{d}x. \tag{7}$$

An integration by parts on the $\int u u_x u_{xx}\,\mathrm{d}x$ term shows that

$$\int u u_x u_{xx}\,\mathrm{d}x \;=\; -\int u u_x u_{xx}\,\mathrm{d}x \;-\; \int u_x^3\,\mathrm{d}x.$$

Therefore (7) becomes

$$\frac{1}{2}\dot{H}_1 \;=\; -\,\alpha\, H_3 + H_2 \,+\, H_1 - 3\int u^2 u_x^2\,\mathrm{d}x + \int u u_x u_{xx}\mathrm{d}x. \tag{8}$$

By using the Cauchy-Schwarz inequality followed by a Young's inequality, we split the term $\int u u_x u_{xx}\mathrm{d}x$ as follows:

$$\int u u_x u_{xx}\mathrm{d}x \,\le\, \left(6\int u^2 u_x^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\frac{1}{6}\int u_{xx}^2 \mathrm{d}x\right)^{\frac{1}{2}} \,\le\, 3\int u^2 u_x^2 \mathrm{d}x \,+\, \frac{1}{12}\int u_{xx}^2\mathrm{d}x,$$

and therefore (8) becomes

$$\frac{1}{2}\dot{H}_1 \;\le -\,\alpha\, H_3 + (1+\frac{1}{12})H_2 + H_1. \tag{9}$$

We now interpolate the second term on the right-hand side of (9) to get

$$H_2 \le H_3^{1/2} H_1^{1/2} \le \frac{1}{2}(\alpha H_3 + \frac{169}{144\alpha}H_1);$$

also we know that [2]

$$H_3 \;\le\; -\frac{H_1^3}{H_0^2},$$

and thus the asymptotic behaviour of H_1 is given by

$$\overline{H_1} \;\le\; \frac{\sqrt{169+288\alpha}}{12\alpha}\overline{H_0}. \tag{10}$$

By using the estimate we found in (5) for $\overline{H_0}$ we have

$$\overline{H_1} \;\le\; \frac{\sqrt{(2\alpha+1)(169+288\alpha)}}{24\alpha^2}. \tag{11}$$

Using the Gagliardo-Nirenberg inequality [12, 14] for the $\|u\|_\infty := sup_{x\in[0,1]}|u(x)|$ norm of the solution, namely $\|u\|_\infty^2 \leq cH_1^{\frac{1}{2}}H_0^{\frac{1}{2}} + H_0$ we can see that the asymptotic behaviour of the solutions of our model obeys the estimates

$$\overline{\|u\|}_\infty^2 \leq \frac{(2\alpha+1)^{3/4}(169+288\alpha)^{1/4}}{4\sqrt{3}\alpha^{3/2}} + \frac{2\alpha+1}{2\alpha}. \tag{12}$$

2 Convergence and Asymptotic Positivity of Solutions

In this section we are going to investigate the evolution of positive initial data, and we shall show that our model does indeed preserve positivity under certain conditions on the parameter α and on the structure of the L^2 norm of an appropriately scaled set of initial data. Our strategy first involves centering the equation on the uniform steady state solution $u \equiv 1$ and then showing that for all t sufficiently large (at least) solutions of the transformed equation are bounded (in absolute value) by unity. We therefore introduce $v(x,t)$, defined by

$$u(x,t) = 1 + v(x,t) \tag{13}$$

where u satisfies (1). If the function $v(x,t)$ satisfies

$$\|v\|_\infty \leq 1$$

then clearly u is a non-negative function. Substituting (13) into (1) we obtain for v the equation

$$v_t = -\,\alpha\, v_{xxxx} - v_{xx} + vv_x + v_x - 2v - 3v^2 - v^3. \tag{14}$$

We need to find conditions which will ensure that $\overline{\|v\|}_\infty \leq 1$. This is sufficient to ensure at least asymptotic positivity. Our strategy is to first obtain appropriate estimates for equation (14), and then to use these estimate to compute $\overline{\|v\|}_\infty$ such that $\overline{\|v\|}_\infty \leq 1$ is satisfied. We start by defining quantities similar to the H_0 and H_1, namely

$$J_0 := \|v\|_2^2 = \int v^2\,\mathrm{d}x,\;\; J_n := \|\frac{\partial^n v}{\partial x^n}\|_2^2 = \int \left(\frac{\partial^n v}{\partial x^n}\right)^2 \mathrm{d}x,\;\; \|v\|_\infty := sup_{x\in[0,1]}|v(x)|, \tag{15}$$

where, as above, the integration is over the spatial domain $\Omega := [0,1]$. Using the same strategy previously applied to H_0 and H_1, we obtain

$$\frac{1}{2}\,\dot{J}_0 = \int v\left[-\,\alpha\, v_{xxxx} - v_{xx} + vv_x + v_x - 2v - 3v^2 - v^3\right]\mathrm{d}x. \tag{16}$$

Performing similar arguments used above we finally obtain

$$\frac{1}{2}\,\dot{J}_0 \leq -\alpha J_2 + J_1 - 2J_0 - 3\int v^3\,\mathrm{d}x + \int v^2 v_x\,\mathrm{d}x + \int vv_x\,\mathrm{d}x - \int v^4\,\mathrm{d}x. \tag{17}$$

First we observe that the two integrals $\int v^2 v_x \,\mathrm{d}x$ and $\int v v_x \,\mathrm{d}x$ are zero on periodic boundary conditions; also, by using

$$J_1 \;\le\; J_2^{1/2} J_0^{1/2} \le \frac{1}{2}(\alpha J_2 + \frac{1}{\alpha} J_0)$$

and $-3\displaystyle\int v^3 \,\mathrm{d}x \le 3\|v\|_\infty J_0$ we arrive at

$$\frac{1}{2}\,\dot{J}_0 \;\le\; -\frac{\alpha}{2} J_2 - (2 - \frac{1}{2\alpha}) J_0 + 3\|v\|_\infty J_0 - \int v^4 \,\mathrm{d}x. \tag{18}$$

In order to manipulate the term $\|v\|_\infty$ we make use of the following sharp interpolation inequality [14]:

$$\|v\|_\infty \;\le\; \beta\, J_2(v)^{\frac{1}{8}} J_0(v)^{\frac{3}{8}} + J_0(v)^{\frac{1}{2}}, \qquad \beta = \left(\frac{4}{27}\right)^{\frac{1}{8}}. \tag{19}$$

Also, by the Cauchy-Schwarz inequality we have that $-\displaystyle\int v^4 \,\mathrm{d}x \;\le\; -J_0^2$. Substituting into (18) we obtain

$$\frac{1}{2}\,\dot{J}_0 \;\le\; -\frac{\alpha}{2} J_2 - (2 - \frac{1}{2\alpha}) J_0 + 3\beta J_2^{\frac{1}{8}} J_0^{\frac{11}{8}} + 3 J_0^{\frac{3}{2}} - J_0^2. \tag{20}$$

Now by using Young's inequality we split the $3\beta J_2^{\frac{1}{8}} J_0^{\frac{11}{8}}$ as follows;

$$3\beta J_2^{\frac{1}{8}} J_0^{\frac{11}{8}} = \alpha^{\frac{1}{8}} J_2^{\frac{1}{8}} \frac{3\beta}{\alpha^{\frac{1}{8}}} J_0^{\frac{11}{8}} \le \frac{1}{8}\alpha J_2 + \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}} J_0^{\frac{11}{7}}.$$

Therefore (18) becomes

$$\frac{1}{2}\,\dot{J}_0 \;\le\; \frac{-3\alpha}{8} J_2 - (2 - \frac{1}{2\alpha}) J_0 + \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}} J_0^{\frac{11}{7}} + 3 J_0^{\frac{3}{2}} - J_0^2. \tag{21}$$

By neglecting the negative definite term $\dfrac{-3\alpha}{8} J_2$, we finally have

$$\frac{1}{2}\,\dot{J}_0 \;\le\; -(2 - \frac{1}{2\alpha}) J_0 + \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}} J_0^{\frac{11}{7}} + 3 J_0^{\frac{3}{2}} - J_0^2 := f(J_0). \tag{22}$$

We need to study the solutions of the above differential inequality: because (22) is an autonomous differential inequality, its solutions are bounded above by the solutions of the one-dimensional ODE $\dot{J}_0 = 2f(J_0)$. Note that the solutions of (14) certainly cannot all satisfy $v \to 0$ since this equation has uniform steady states $v = -1$ and $v = -2$ in addition to $v = 0$. These other states correspond to the uniform steady states $u = 0$ and $u = -1$ of (1). At these states it is easily checked that f is positive. In fact, the function f is negative for J_0 small, and for J_0 large, and positive in some intermediate range. We may therefore conclude that if $J_0(t=0) < J^*$, where J^* is the smallest positive root of $f(J)=0$, then $J_0 \to 0$ as $t \to \infty$. A sufficient condition to ensure this is that

$$2 > \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}} J_0^{\frac{4}{7}} + 3J_0^{\frac{1}{2}}. \tag{23}$$

In order to estimate the $\|v\|_\infty$ of our perturbed equation (14) we need to have control on the J_1 norm as well. The evolution equation for J_1 is given by

$$\frac{1}{2}\dot{J}_1 \le -\alpha J_3 + J_2 - 2J_1 + \int v_x(vv_x)_x \,\mathrm{d}x + \int v_x v_{xx}\,\mathrm{d}x - 6\int v(v_x)^2\,\mathrm{d}x - 3\int v^2(v_x)^2\,\mathrm{d}x. \tag{24}$$

Analysing the various terms one can see that on periodic boundary conditions $\int v_x v_{xx}\,\mathrm{d}x = 0$ is equal to zero; also by integrating by parts the $\int v_x(vv_x)_x\,\mathrm{d}x$ and then splitting it by using a Young's inequality, one can see that $\int v_x(vv_x)_x\,\mathrm{d}x \le \frac{1}{2}J_2 + \frac{1}{2}\int v^2(v_x)^2\,\mathrm{d}x$; hence (24) can be cast into

$$\frac{1}{2}\dot{J}_1 \le -\alpha J_3 + \frac{3}{2}J_2 + -2J_1 - 6\int v(v_x)^2\,\mathrm{d}x - \frac{5}{2}\int (v_x)^2(v_x)^2\,\mathrm{d}x; \tag{25}$$

we now majorize the term $-6\int v(v_x)^2\,\mathrm{d}x$ by Cauchy-Scharwz inequality followed by an appropriate splitting by using a Young's inequality, adding then the terms we finally obtain

$$\frac{1}{2}\dot{J}_1 \le -\alpha J_3 + \frac{3}{2}J_2 + \frac{8}{5}J_1. \tag{26}$$

We now split

$$J_2 \le J_3^{1/2}J_1^{1/2} \le \frac{1}{2}(\alpha J_3 + \frac{1}{\alpha}J_1).$$

Thus one can see that the long time asymptotic behaviour of J_1 is given by

$$\overline{J_1} \le \frac{1}{\alpha}\sqrt{\frac{15+32\alpha}{5}}\overline{J_0}. \tag{27}$$

Taking into account that $\overline{J_0} \equiv \lim_{t\to\infty} J_0(t) = 0$, provided that $J_0(t=0) < 1$ and the condition (23) is satisfied, we find that

$$\overline{J_1} \equiv \lim_{t\to\infty} J_1(t) = 0. \tag{28}$$

Thus under the same restrictions, we can conclude that

$$\overline{\|v\|}_\infty \equiv \lim_{t\to\infty}\|v(t)\|_\infty \le \lim_{t\to\infty}\left(J_1^{\frac{1}{4}}J_0^{\frac{1}{4}} + J_0^{\frac{1}{2}}\right) = 0. \tag{29}$$

Hence from (13) we see that $\overline{\|u\|}_\infty \equiv \lim\limits_{t\to\infty} u(t) = 1$.

We may summarise our findings in the following theorem, which gives a restriction on the initial conditions, which is sufficient for convergence and hence for asymptotic positivity:

Theorem 1 *If $\alpha > \dfrac{1}{4}$ and the initial data satisfies*

$$J_0(t=0) = \int_0^1 (u(x,0)-1)^2\,dx < J^*,$$

where J^* is the smallest positive root of

$$f(J_0) := -J_0^2 + \frac{7}{8}\frac{(3\beta)^{\frac{8}{7}}}{\alpha^{\frac{1}{7}}} J_0^{\frac{11}{7}} + 3J_0^{\frac{3}{2}} - \left(2 - \frac{1}{2\alpha}\right) J_0 = 0, \tag{30}$$

then the solution $u(x,t)$ of (1) satisfies

$$\lim_{t\to\infty} u(x,t) = 1$$

uniformly for $x \in [0,1]$.

3 Positivity of Solutions

We have shown (theorem 1) that under certain restrictions on the L^2 norm of the initial datum, the solution of our equation converges uniformly to 1. The fact that the convergence is uniform enables us to further deduce that the solution must be positive for all t sufficiently large, but not necessarily for all t. Below we shall show that we can achieve positivity of solutions for all values of the time t provided some further resrtiction on the set of the initial data and the parameter α are met.

Let $\alpha > 1/4$ as in our previous section, and let

$$J_0(t=0) = \int_0^L (v(x,0))^2\,dx \le J^*,$$

where J^* is as previously defined. We then have

$$J_0(t) \le J^* \qquad \text{for all } t \ge 0. \tag{31}$$

We shall use this estimate in (27) to deduce that if

$$J_1(t=0) \le J^* \sqrt{\frac{15+32\alpha}{5\alpha^2}}$$

then

$$J_1(t) \le J^* \sqrt{\frac{15+32\alpha}{5\alpha^2}}, \tag{32}$$

for all $t \ge 0$. From $u - 1 = v$ and from the interpolation inequality

$$\|v(\cdot,t)\|_\infty \le J_1^{\frac{1}{4}} J_0^{\frac{1}{4}} + J_0^{\frac{1}{2}} \tag{33}$$

one can see that for positivity for all time it is sufficient to show that the right hand side of (33) is bounded above by 1 for all $t \geq 0$. In view of the bounds we have for J_0 and J_1, given by (31) and (32) respectively, it is therefore sufficient to show that

$$J^* \leq \left(\frac{5^{1/8}\alpha^{1/4}}{5^{1/8}\alpha^{1/4} + (15+32\alpha)^{1/8}} \right)^2 . \tag{34}$$

Recall that J^* is defined as the smallest positive root of $f(J) = 0$, where $f(J)$ is the function defined in the statement of theorem 1. What we have to show is that (34) is indeed satisfied for a range of values of the parameter α. From the structure of $f(J)$ one can see that the inequality is satisfied if

$$f\left(\left(\frac{5^{1/8}\alpha^{1/4}}{5^{1/8}\alpha^{1/4} + (15+32\alpha)^{1/8}} \right)^2 \right) \geq 0. \tag{35}$$

It is not difficult to analyze the inequality (35). One way is to use $MAPLE$ to plot it as a function of α; by doing that then one can see that there is an extensive range of the parameter α such that (35) is indeed satisfied: in fact the value of α at which the graph of (35) crosses the $\alpha-$axis is about 3.2×10^{14}!; of course we still must take $\alpha > 1/4$.

Our findings can be stated in the following theorem, formulated in terms of the original state variable u.

Theorem 2 *Let $\alpha > 1/4$ and let the initial data satisfy*

$$\int_0^L (u(x,0)-1)^2 \, dx \leq J^*,$$

where J^* is the smallest positive root of $f(J) = 0$ where $f(J)$ is as in the statement of theorem 1. Assume that the initial data also satisfies

$$\int_0^L \left(\frac{\partial u}{\partial x}(x,0) \right)^2 dx \leq J^* \sqrt{\frac{15+32\alpha}{5\alpha^2}}$$

and assume that α is such that

$$J^* \leq \left(\frac{5^{1/8}\alpha^{1/4}}{5^{1/8}\alpha^{1/4} + (15+32\alpha)^{1/8}} \right)^2 .$$

Then the solution $u(x,t)$ of (1) satisfies $u(x,t) \geq 0$ for all $t \geq 0$ and all $x \in [0,1]$.

References

[1] D.G. Aronson, L.A. Peletier, *Large Time Behaviour of Solutions of the Porous Medium Equation in Bounded Domains, Journal of Differential Equations* **39** (1981) 378-412.

[2] M.V. Bartuccelli, C.R. Doering, J.D. Gibbon and S.A.J. Malham, *Length Scales in Solutions of the Navier-Stokes Equations, Nonlinearity* **6** (1993) 549-568.

[3] M.V. Bartuccelli, J.D. Gibbon and M. Oliver, *Length Scales in Solutions of the Complex Ginzburg-Landau Equation, Physica D* **89** (1996) 267-286.

[4] M.V. Bartuccelli, S.A. Gourley and A.A. Ilyin, *Attractor Dimension, Convergence and Positivity of Solutions of a Fourth Order Reaction-Diffusion Equation.* In Press in the Proceedings of the Royal Society, Series **A**.

[5] M.V. Bartuccelli, S.A. Gourley and C.J. Woolcock, *Conditions For Positivity of Solutions of a Class of Dissipative Partial Differential Equations, Appl. Math. Lett.* Vol. **11** No. **6** (1998) 71-74.

[6] M.V. Bartuccelli, S.A. Gourley and C.J. Woolcock, *Length scales in Solutions of a Generalized Diffusion Model, Physica Scripta* **57** (1998) 9-19.

[7] M.V. Bartuccelli, *Ladder Theorems and Positivity of Solutions for Dissipative Partial Differential Equations, Nonlinear Phenomena in Complex Systems* **2** No.3 (1999) 19-22.

[8] F.Bernis and A. Friedman, *Higher Order Nonlinear Degenarate Parabolic Equations, Journal of Differential Equations* **83** (1990) 179-206.

[9] P.M. Bleher, J.L. Lebowitz and E.R. Speer, *Existence and Positivity of Solutions of a Fourth-Order Nonlinear PDE Describing Interface Fluctuation, Communications on Pure and Applied Mathematics* Vol. **XLVII** (1994) 923-942.

[10] D.S. Cohen and J.D. Murray, *A Generalized Diffusion Model for Growth and Dispersal in a Population, J. Math. Biol.* **12** (1981) 237.

[11] R. Dal Passo, H. Garcke and G. Grun, *On a Fourth-Order Degenarate Parabolic Equation: Global Entropy Estimates, Existence, and Qualitative Behaviour of Solutions, SIAM J. Math. Anal.* **29** 2 (1988) 321-342.

[12] C.R. Doering and J.D. Gibbon, **Applied Analysis of the Navier-Stokes Equations**, *Cambridge University Press*, (Cambridge, 1995).

[13] E. Gagliardo, *Ulteriori Proprieta' di Alcune Classi di Funzioni in piu Variabili, Ricerche di Matematica* **8** (1959) 24-51.

[14] A.A. Ilyin, *Best Constants in Multiplicative Inequalities for Sup-Norms, Journal of the London Mathematical Society* **58** (1998), 84-96.

[15] L. Nirenberg, *On Elliptic Partial Differential Equations, Annali della Scuola Normale Superiore di Pisa* **13** (1959).

[16] M.H. Protter and H.F. Weinberger, **Maximum Principles in Differential Equations**, (Prentice Hall New Jersey, 1967).

[17] B. Straughan, **The Energy Method, Stability and Nonlinear Convection**, (Springer-Verlag New York, 1992).