Lie Algebras and Quantum Mechanics

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Basic Definitions

Definition

A **Group** $(G, *)$ is a set G with a binary operation $* : G \times G \to G$ such that

1. $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3) \quad \forall g_1, g_2, g_3 \in G$ (associativity)
2. $\exists e \in G$ such that $g * e = e * g = g \quad \forall g \in G$ (existence of identity)
3. $\forall g \in G \; \exists g^{-1} \in G$ such that $g^{-1} * g = g * g^{-1} = e$ (every element has an inverse)

The identity element of $(G, +)$ is 0.

A group is **Abelian** if $g_1 * g_2 = g_2 * g_1 \quad \forall g_1, g_2 \in G$ (commutativity)

Definition

We can define a **Field** $(\mathbb{K}, +, \cdot)$ to be a non-empty set $\mathbb{K}$ with two binary operations $+$ and $\cdot$ such that $(\mathbb{K}, +)$ and $(\mathbb{K} \setminus \{0\}, \cdot)$ are abelian groups.

Example

$(\mathbb{R}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$, where $\mathbb{R}$ is the set of Real Numbers and $\mathbb{C}$ is the set of Complex numbers are both Fields. We often denote these fields simply by $\mathbb{R}$ and $\mathbb{C}$.

Definition

A **Vector Space** over a field $(\mathbb{K}, +, \cdot)$ is a non-empty set $\mathcal{V}$ with two operations $\boldsymbol{+} : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and $\bullet : \mathbb{K} \times \mathcal{V} \to \mathcal{V}$ such that, $\forall \mathbf{v}, \mathbf{w} \in \mathcal{V}, a, b \in \mathbb{K}$

1. $(\mathcal{V}, \boldsymbol{+})$ is an abelian group.
2. $\exists 1 \in \mathbb{K}$ such that $1 \bullet \mathbf{v} = \mathbf{v}$ (existence of $\bullet$ identity)
3. $(a \cdot b) \bullet \mathbf{v} = a \cdot (b \bullet \mathbf{v})$ (associativity)
4. $a \bullet (\mathbf{u} \boldsymbol{+} \mathbf{v}) = (a \bullet \mathbf{u}) \boldsymbol{+} (a \bullet \mathbf{v})$ (left distributivity)
5. $(a + b) \bullet \mathbf{v} = (a \bullet \mathbf{v}) + (b \bullet \mathbf{v})$ (right distributivity)

Example

The set of all 2×2 matrices under the operations of matrix addition and matrix multiplication is a vector space over $\mathbb{C}$.

Definition

An **Algebra** $(\mathcal{A}, \boldsymbol{+}, \bullet, *)$ is a vector space over a field $(\mathbb{K}, +, \cdot)$ with an operation $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that $\forall X, Y, Z \in \mathcal{A}$ and $\forall a \in \mathbb{K}$

1. $a \bullet (X * Y) = (a \bullet X) * Y = X * (a \bullet Y)$
2. $\exists!\ I \in \mathcal{A}$ such that $X * I = I * X = X$ (A has a unique $*$ identity).
3. $X * (Y + Z) = (X * Y) \boldsymbol{+} (X * Z)$ (left distributivity)
4. $(X + Y) * Z = (X * Z) + (Y * Z)$ (right distributivity)

An algebra is **Associative** if $\forall X, Y, Z \in \mathcal{A}$

$$(X * Y) * Z = X * (Y * Z)$$

An algebra is **Commutative** if $\forall X, Y \in \mathcal{A}$

$$X * Y = Y * X$$

Definition

A **Lie Algebra** $\mathfrak{L}$ is a vector space over a field $\mathbb{K}$ on which we define a **Lie Bracket** $[,] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ that satisfies $\forall a, b \in \mathbb{K}$ and $\forall X, Y, Z \in \mathfrak{L}$:

1. $[X, aY + bZ] = a[X, Y] + b[X, Z]$ (bilinearity)
2. $[X, Y] = -[Y, X]$ (skew symmetry)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (Jacobi Identity)

If $\mathbb{K} = \mathbb{R}$ then $\mathfrak{L}$ is a **real** Lie Algebra, if $\mathbb{K} = \mathbb{C}$ then $\mathfrak{L}$ is a **complex** Lie Algebra.

Definition

A Matrix Lie Algebra is an algebra of matrices where the Lie Bracket is the **commutator** of X and Y:

$$[X, Y] = XY - YX$$

Definition

Two Lie Algebras $(\mathfrak{L}_1, [,]_1)$ and $(\mathfrak{L}_2, [,]_2)$ are **isomorphic** if $\exists \rho : \mathfrak{L}_1 \to \mathfrak{L}_2$ such that $\forall a, b \in \mathbb{K}$ and $\forall X, Y \in \mathfrak{L}_1$:

1. $\rho(aX + bY) = a\rho(X) + b\rho(Y)$ (ρ is linear)
2. $\exists \rho^{-1} : \mathfrak{L}_2 \to \mathfrak{L}_1$ such that $\rho^{-1} \circ \rho(X) = X$ (ρ is invertible)
3. $[\rho(X), \rho(Y)]_2 = \rho([X, Y])_1$

Theorem

(Ado's Theorem)[1] *Every (finite-dimensional) Lie Algebra is isomorphic to a Matrix Lie Algebra*

[1] The original Ado's Theorem imposes the restriction that the Lie Algebra be over a field of characteristic zero.

The Classical Harmonic Oscillator

Consider a point of mass m constrained to move along a straight line AB on a perfectly smooth horizontal floor. The point is attached to the end of an elastic spring with natural length l and modulus of elasticity λ whose other end is fixed to a vertical rod anchored to the floor. Initially the spring is compressed so that the particle is a distance x_0 from the equilibrium point.
By taking AB to be the x-axis with the equilibrium point at $x = 0$, we can determine the position of the particle by solving an ordinary differential equation.
Let $\omega^2 = \frac{\lambda}{lm}$. Using $F = -\frac{\lambda x}{l} = m\frac{d^2x}{dt^2}$ we obtain the O.D.E

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

with initial conditions $x(0) = -x_0$, $\dot{x}(0) = 0$.

The Hamiltonian

We can also obtain this differential equation by use of a **Classical Hamiltonian**, a function H depending on the position $x(t)$ and the **conjugate momentum** to x, $p(t)$ that satisfies the equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

The Classical Hamiltonian for the Classical Harmonic Oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

and we obtain

$$\frac{dx}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = -m\omega^2 x$$

and finally the O.D.E

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Operators

Definition

A **Linear Operator** is a function $f : \mathcal{V} \to \mathcal{V}$ such that $\forall \mathbf{v}, \mathbf{w} \in \mathcal{V}, k \in \mathbb{K}$

1. $f(\mathbf{v} + \mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w})$
2. $f(k\mathbf{v}) = kf(\mathbf{v})$

Example

Any 2×2 matrix is a linear operator on the vector space of 2-vectors.

The Quantum Harmonic Oscillator

The Classical Hamiltonian can be **Quantized** by replacing the variables x and p with the operators $\hat{x}$ and $\hat{p}$.

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

We now define the **Creation Operator** $a^\dagger$ and the **Annihilation Operator** a in terms of $\hat{x}$ and $\hat{p}$:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a), \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)$$

where $\hbar$ is a constant (the normalised Planck Constant).

If we consider the Hamiltonian to be a square matrix, we can find its normalised eigenvectors, also known as normalised eigenstates. We write the normalised eigenstates of the Hamiltonian in the following manner:

$$|n\rangle$$

Each eigenstate $|n\rangle, n \in \mathbb{Z}^+$ corresponds to an energy level of the system. The state $|0\rangle$ is the **Ground State** of the system, the state at which the system has the least energy. It satisfies:

$$a|0\rangle = 0$$

and the entire spectrum of eigenstates of H can be constructed from

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

The creation and annihilation operators act on a state $|n\rangle$ as follows:

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad a|n\rangle = \sqrt{n}|n-1\rangle$$

The Heisenberg Algebra

A finite dimensional Lie algebra $\mathcal{A}$ is a vector space of dimension $dim(\mathcal{A})$. $\mathcal{A}$ is generated by a basis of elements T^i, $i = 1, ..., dim(\mathcal{A})$.

Definition

The elements T^i are the **Generators** of the Lie Algebra

Note that the product $T^i T^j$ is not necessarilty in the Lie Algebra, it is part of another structure known as the **Universal Enveloping Algebra**.

Applying a commutator to any two elements of the basis of a Lie algebra results in a linear combination of the basis elements:

$$[T^i, T^j] = \sum_{k=1}^{dim(\mathcal{A})} C_k^{ij} T^k$$

Definition

The constants C_k^{ij} are called the **Structure Constants** [2] of the Lie Algebra.

[2] Some writers will give the linear combination as

$$[T^a, T^b] = i \sum_{c=1}^{dim(\mathcal{A})} f_c^{ab} T^c$$

where $i^2 = -1$ and state that the real constants f_c^{ab} are the structure constants

Let $\mathcal{F}$ be the set of all differentiable functions in $\mathbb{R}^n$.
Consider the operators $Q : \mathcal{F} \to \mathcal{F}$, $P : \mathcal{F} \to \mathcal{F}$ and $I : \mathcal{F} \to \mathcal{F}$ defined by

$$Q(f) = xf, \quad P(f) = \frac{\partial f}{\partial x}, \quad I(f) = f$$

Q,P and I form a basis for a Lie Algebra, $[P, Q] = I$, $[P, I] = [Q, I] = 0$. Taking $P = T^1$, $Q = T^2$ and $I = T^3$ the non-zero structure constants are $C_3^{12} = 1$, and $C_3^{21} = -1$. We have defined a Lie Algebra known as the **Heisenberg Algebra**.
Returning to the Quantum Harmonic oscillator, we can see a Heisenberg Algebra generated by $a^\dagger$, a, and the identity I [3]

$$[a, a^\dagger] = I, \quad [a, I] = [a^\dagger, I] = 0$$

[3] In order to generate an algebra compatible with the alternative definition of the structure constants, we select $\hat{x}$, $\hat{p}$ and I as generators.

The $\mathfrak{sl}(2)$ Algebra

Consider the set of all 2×2 traceless matrices. This set is spanned by the three matrices

$$a_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad a_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad a_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In fact, (a_+, a_-, a_0) is a basis for a Lie algebra called $\mathfrak{sl}(2)$. Returning to the Quantum Harmonic Oscillator, suppose we define the following operators:

$$h = a^\dagger a, \quad e = [\sqrt{-1 + a^\dagger a}]a^\dagger, \quad f = -a\sqrt{-1 + a^\dagger a}$$

We derive the action of these operators on the state of the tower:

$$h|n\rangle = n|n\rangle, \quad e|n\rangle = \sqrt{n(n+1)}|n{+}1\rangle, \quad f|n\rangle = -\sqrt{n(n-1)}|n{-}1\rangle$$

And then obtain the following commutation relations:

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = 2h$$

We then define:

$$T^1 = e, \quad T^2 = f, \quad T^3 = h$$

Then

$$[T^1, T^2] = 2T^3, \quad [T^3, T^1] = T^3, \quad [T^3, T^2] = -T^2$$

Which are the commutation relations of the $\mathfrak{sl}(2)$ Lie algebra.

The Pauli Matrices

Recall that a matrix A is **Hermitian** if

$$A = (\bar{A})^T$$

We write $A = A^\dagger$

Definition

The $\mathfrak{su}(2)$ Lie Algebra is the vector space of 2×2 traceless Hermitian matrices.

$\mathfrak{su}(2)$ can be generated by the **Pauli Matrices**. The Pauli Matrices in **Canonical Form** are given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Definition

Let $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ be a permutation. The **epsilon tensor** of three indices ϵ_{ijk} is defined by

$$\epsilon_{123} = 1, \quad \epsilon_{\pi(1)\pi(2)\pi(3)} = sgn(\pi), \quad \epsilon_{ijk} = 0 \textit{ otherwise}$$

The commutation relations of the Pauli spin matrices are given by

$$[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$$

Definition

The Quadratic Casimir operator on a Lie Algebra $\mathcal{L}$ is defined by

$$C_2 = \sum_{a,b=1}^{dim(\mathcal{L})} \kappa_{ab} T^a T^b$$

for a certain set of constants κ_{ab}

The (normalised) Quadratic Casimir operator in $\mathfrak{su}(2)$ is given by

$$C_2 = \frac{1}{4} \sum_{i=1}^{3} \sigma_i^2 = \frac{3}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Spin on the Electron

We can associate the electron with a two-dimensional vector $|v\rangle$ in the vector space $\mathbb{C}^2$, on which $\mathfrak{su}(2)$ can act. The vectors

$$|\uparrow\rangle_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

called the **spin up** and **spin down** states respectively, form a basis for this vector space. The Pauli Matrix σ_3 acts diagonally on these states:

$$\sigma_3|\uparrow\rangle_3 = |\uparrow\rangle_3, \ \sigma_3|\downarrow\rangle_3 = -|\downarrow\rangle_3$$

The observable associated to $s_z = \frac{1}{2}\sigma_3$ is called the z component of the spin.

The spin observable has three components (x, y, z) associated to the operators $s_x = \frac{1}{2}\sigma_1, s_y = \frac{1}{2}\sigma_2, s_z = \frac{1}{2}\sigma_3$
By computing the effect of the other Pauli matrices on the spin up and spin down states we find that

$$\sigma_1|\uparrow\rangle_3 = |\downarrow\rangle_3,\ \sigma_1|\downarrow\rangle_3 = |\uparrow\rangle_3,\ \sigma_2|\uparrow\rangle_3 = i|\downarrow\rangle_3,\ \sigma_2|\downarrow\rangle_3 = -i|\uparrow\rangle_3,$$

The electron is said to be a **spin half** particle. Now there exist infinitely many finite dimensional **representations** of $\mathfrak{su}(2)$, for which the Quadratic Casimir takes the value

$$C_2 = s(s+1)I, s = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$$

where s is the **Spin Quantum Number** and I is the identity matrix of the representation. The value $s = \frac{1}{2}$ gives us the representation relevant to the electron, hence the term 'spin half'.

Moving on to the Yangian

In the first section we defined the various algebraic structures in the following manner:

$$Group \rightarrow Field \rightarrow V.\ space \rightarrow Algebra \rightarrow L.\ Algebra \rightarrow M.\ L.\ Algebra$$

Other structures include the **Ring** and the **Module**
By defining a **counit** and a **coproduct** we can define:

$$Algebra \rightarrow Coalgebra \rightarrow Bialgebra$$

Equipping a Bialgebra with an extra map called an antipode gives us a Hopf Algebra.
Just as groups have subgroups and fields have subfields, so Lie Algebras have **Lie Subalgebras**:

$$Lie\ subalgebra \rightarrow Ideal\ subalgebra \rightarrow simple\ Lie\ algebra$$

And from this we can define the **Chevalley-Serre presentation** - in brief, a Chevalley-Serre presentation of a Lie Algebra selects a minimal set of generators which can be used to obtain all the remaining elements by repeated commutation.

If $\mathfrak{g}$ is a complex finite dimensional Lie Algebra, we can define a **Cartan-Weyl** basis for it in terms of its generators. We can then define a **Cartan Subalgebra** of $\mathfrak{g}$. If $\mathfrak{g}$ is a simple Lie Algebra (with simple root system) we can then find a **Cartan Matrix** of it. The structure constants of a Lie Algebra provide an adjoint representation of it. We define the **Killing Form** as the bilinear form associated to the adjoint representation of $\mathfrak{g}$.
A **Superalgebra** can be thought of as an algebra made up of *odd* and *even* components. Recall the commutator $[X, Y] = XY - YX$. The **Anticommutator** $\{X, Y\} = XY + YX$ furnishes us with the structure for the **Lie Superalgebra**.
We can represent the information about a Lie Algebra using a **Dynkin Diagram**
Let $\mathfrak{g}$ be a Lie Algebra. The **loop algebra** $\mathfrak{g}[u]$ associated to $\mathfrak{g}$ is the algebra of $\mathfrak{g}$ valued polynomials in the variable u. The **Yangian** $\mathcal{Y}(\mathfrak{g})$ is a *deformation* of the universal enveloping algebra of $\mathfrak{g}[u]$. Note, however that the Yangian is *not* a Lie algebra.

References

Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics, D.H. Sattinger, O.L.Weaver pp 22-25, pp 48-50

Notes for the course on Quantum Mechanics, Dr Alessandro Torrielli, http://www.surrey.ac.uk/maths/people/torrielli_allessandro/index.htm

Further Reading

• Books on Lie Algebras include: Naive Lie Theory by John Stillwell (Springer, Undergraduate Texts in Mathematics, 2008)
Lie Algebras by Nathan Jacobson (Dover Books on Mathematics, 1980)

• If you want to learn more about the project: Quantum Groups by Christian Kassel (Springer Verlag, Graduate Texts in Mathematics, 1995)
Dictionary on Lie Algebras and Superalgebras by Frappat, Sciarrino, Sorba [http://arxiv.org/abs/hep-th/9607161]

Further Reading

• For information on how the $\mathfrak{sl}(2)$ Lie Algebra appears in the Quantum Harmonic Oscillator see the research of Holstein and Primakoff ff:
Field dependence of the intrinsic domain magnetization of a ferromagnet, T.Holstein and H.Primakoff, Phys.Rev. 58, 1098(1940).
URL:http://prola.aps.org/abstract/PR/v58/i12/p1098_1.
• If you go onto the Surrey University website www.surrey.ac.uk you can find: Research papers and presentation notes related to the summer project on Dr Torrielli's Personal Webpage [accessible via the Research section of http://www.surrey.ac.uk/maths/people/torrielli_alessandro/index.htm].

Further Reading

- The news story of the project may be found on [http://www.surrey.ac.uk/maths/news/stories/2012/89914_summer_project_the_yangian_on_a_distinguished_dynkin_diagram_for_the_adscft_correspondence.htm]