# Loop Integrands from the Riemann Sphere

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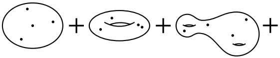
# Motivation

## Feynman diagrams

$$\mathcal{M} = \sum_{\text{graphs } \Gamma}$$

- graph combinatorics
- combinatorical problem

## Worldsheet models

$\mathcal{M} =$  $+ \ \ldots$

- integration over moduli space
- geometric problem

## CHY formulae

$$\mathcal{M} = \sum_{z_i \,|\, k_i \cdot P(z_i)=0} \frac{\mathcal{I}_L(k_i, \epsilon_i, z_i)\mathcal{I}_R(k_i, \tilde{\epsilon}_i, z_i)}{J}$$

- localized on scattering equations
- algebraic problem

# Scattering Equations and CHY formulae

## The Scattering Equations

[Cachazo-He-Yuan, Mason-Skinner]

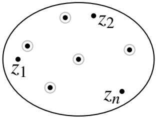
The scattering equations

- underpin CHY formulae for tree-level scattering amplitudes
- arising from Ambitwistor worldsheet models
- determine $n$ points $z_i$ on a Riemann surface

To define the scattering equations, construct $P(z, z_i) \in \Omega^0(\Sigma, K_\Sigma)$

$$\bar{\partial} P = 2\pi i \sum_i k_i \bar{\delta}(z - z_i) dz\,.$$

On the Riemann sphere, and for $n$ null momenta $k_i$,
this is solved by




$$P_0(z) = \sum_{i=1}^{n} \frac{k_i}{z - z_i}\, dz\,.$$

# Scattering Equations and CHY formulae

## Scattering Equations

[Cachazo-He-Yuan, Mason-Skinner]

With $P_0(z) = \sum_{i=1}^{n} \frac{k_i}{z-z_i}\, dz$, we thus obtain the

**Scattering Equations at tree-level:**

$$\text{Res}_{z_i} P_0^2(z) = k_i \cdot P_0(\sigma_i) = \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

Note:

- $(n-3)$ independent equations
- $(n-3)!$ solutions

# Scattering Equations and CHY formulae

## CHY formulae

[Cachazo-He-Yuan]

Representation of the tree-level S-matrix of massless theories:

$$\mathcal{M}_{n,0} = \int \frac{d\sigma^n}{\text{vol}\,SL(2,\mathbb{C})} \prod_i{}' \bar{\delta}\left( \sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} \right) \mathcal{I}_n$$

with $\sigma_i \in \Sigma \cong \mathbb{CP}^1$, $k_i$ null momenta of the scattered particles.

Integration over moduli space $M_{n,0}$ fixed completely by imposing the $(n-3)$ scattering equations [CHY, GM, Witten];

$$\sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

# Scattering Equations and CHY formulae

## CHY formulae

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- gravity: $\mathcal{I}_n =$Pf$'(M)$ Pf$'(\tilde{M})$
- YM: $\mathcal{I}_n = C_n$ Pf$'(M)$

$$M = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}, \qquad \mathrm{Pf}'(M) = \frac{(-1)^{i+j}}{\sigma_{ij}} \mathrm{Pf}(M^{ij}_{ij}),$$

$$A_{ij} = \frac{k_i\cdot k_j}{\sigma_{ij}}, \qquad B_{ij} = \frac{\epsilon_i\cdot\epsilon_j}{\sigma_{ij}}, \qquad C_{ij} = \frac{\epsilon_i\cdot k_j}{\sigma_{ij}},$$

$$A_{ii} = 0, \qquad B_{ii} = 0, \qquad C_{ii} = -\sum_{j\neq i} C_{ij}$$

**Question: Where are these formulae coming from?**

# Ambitwistor Worldsheet models

Upshot:

- correlators of worldsheet models.
- Target space:
  phase space of complex null rays = ambitwistor space.
- ambitwistor strings: new formulae for loop integrands that localise completely on loop-extensions of the scattering equations.

# Ambitwistor Space

Ambitwistor space $\mathbb{A}$ = space of (complex) null rays in $M_{\mathbb{C}}$

- symplectic quotient of cotangent bundle of (supersymmetric) spacetime $(X, P, \Psi) \in T^*M$ by constraints $P^2 = 0$ and $\Psi_r \cdot P = 0$

$$\mathbb{A} := \left\{(X^\mu, P_\mu, \Psi_r^\mu) \in T^*M \,\middle|\, P^2 = 0\right\} \Big/ \{\mathcal{D}_0, \mathcal{D}_r\}$$

  with Hamiltonian vector fields $\mathcal{D}_0 = P \cdot \nabla$, $\mathcal{D}_r = \Psi_r \cdot \nabla + P \cdot \partial_{\Psi_r}$

- $\mathbb{A}$ is a symplectic holomorphic manifold, with symplectic potential

$$\Theta = P \cdot \, \mathrm{d}X + \frac{1}{2} \sum_r \Psi_r \cdot \, \mathrm{d}\Psi_r$$

# The Ambitwistor String

[Mason-Skinner,Berkovits]

- Construct a theory describing maps $\Sigma \to \mathbb{A}$:

$$S = \frac{1}{2\pi}\int P\cdot\bar\partial X + \frac{1}{2}\sum_r \Psi_r\cdot\bar\partial\Psi_r - \frac{e}{2}P^2 - \chi_r P\cdot\Psi_r\,,$$

where $P_\mu \in \Omega^0(\Sigma, K)$. $\Psi^\mu_i \in \Pi\Omega^0(\Sigma, K_\Sigma^{1/2})$.
Geometrically:
  - action from symplectic potential $\Theta$
  - gauge fields $e$ and $\chi_r$ impose the constraints reducing to $\mathbb{A}$.

- Gauge freedom:

$$\delta X^\mu = \alpha P^\mu\,,\quad \delta P_\mu = 0\,,\quad \delta e = \bar\delta\alpha\,.$$

- Quantise using BRST procedure

$$Q = \oint cT + \frac{\tilde c}{2}P^2 + \sum_r \gamma_r P\cdot\Psi_r + \text{ghosts}$$

Anomalies cancel for $d = 10$ as in the usual superstring

# The Ambitwistor String

$S = \frac{1}{2\pi}\int P\cdot\bar\partial X + \frac{1}{2}\sum_r \Psi_r\cdot\bar\partial\Psi_r - \frac{e}{2}P^2 - \chi_r P\cdot\Psi_r$

[Mason-Skinner]

- Simplest vertex operators, describing graviton, B-field and dilaton:

$$V = c\tilde{c}\,\delta^2(\gamma)\,\epsilon_{\mu\nu}\Psi_1^\mu\Psi_2^\nu e^{ik\cdot X}$$

- In the presence of these vertex operators, integrate out $X$.

$$\bar\partial P = 2\pi i\sum_i k_i\bar\delta(z-z_i)dz\,.$$

  Localisation on scattering equations from integrating out moduli of gauge field $e$:

  Scattering equations $\Leftrightarrow$ map to $\mathbb{A}$

- The correlation functions yield the CHY formulae:

$$\mathscr{M} = \int d\mu \prod_i{}' \delta(k_i\cdot P(z_i))\mathcal{I}_L(k_i,\epsilon_i,z_i)\mathcal{I}_R(k_i,\tilde\epsilon_i,z_i)$$

# New Ambitwistor String theories

[Casali-YG-Mason-Monteiro-Roehrig]

| | $S_\Psi$ | $S_{\Psi_1,\Psi_2}$ | $S_{\rho,\Psi}$ | $S_{YM}{}^*$ | $S_{YM_0}{}^*$ |
|---|---|---|---|---|---|
| $S_\Psi$ | Einstein | | | | |
| $S_{\Psi_1,\Psi_2}$ | BI | Galileon | | | |
| $S_{\rho,\Psi}$ | EM | DBI | EMS | | |
| $S_{YM}{}^*$ | EYM | ext. DBI | EYMS | EYMS | |
| $S_{YM_0}{}^*$ | YM | NLSM | YMS | gen.YMS | Scalar |

# Towards Quantum Gravity:
# Loop Integrands from the Riemann Sphere

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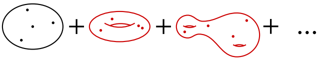
## Worldsheet models

$\mathscr{M} =$ 

- integration over moduli space
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## CHY formulae

$$\mathscr{M} = \sum_{z_i \,|\, S_j(z_i)=0} \frac{\mathcal{I}_L(k_i, \epsilon_i, z_i)\mathcal{I}_R(k_i, \tilde{\epsilon}_i, z_i)}{J}$$

- localized on scattering equations
- algebraic problem

# Upshot

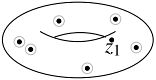
- Problem: Worldsheet formulations of quantum field theories have had a wide ranging impact on the study of amplitudes. However, the mathematical framework becomes very challenging on the higher-genus worldsheets required to describe loop effects.

- Upshot: Derive a framework, applicable in such worldsheet models based on the scattering equations, that transforms formulae on higher-genus surfaces to ones on (nodal) Riemann spheres.

# Scattering Equations on the Torus

Recall: To define the scattering equations, construct a 1-form $P(z, z_i) \in \Omega^0(\Sigma, K_\Sigma)$, such that

$$\bar{\partial} P = 2\pi i \sum_i k_i \bar{\delta}(z - z_i) dz \,.$$




On the torus $\Sigma_q = \mathbb{C}/\{\mathbb{Z} \oplus \mathbb{Z}\tau\}$, this is solved by

$$P = 2\pi i \, \ell dz + \sum_i k_i \left( \frac{\theta_1'(z - z_i)}{\theta_1(z - z_i)} + \frac{\theta_1'(z_i - z_{ref})}{\theta_1(z_i - z_{ref})} + \frac{\theta_1'(z_{ref} - z)}{\theta_1(z_{ref} - z)} \right) dz \,,$$

where $q = e^{2\pi i \tau}$ parametrizes the modulus and $\ell \in \mathbb{R}^d$ the zero-modes of $P$. Using this, we have the

**Scattering Equations on the torus:**

$$\mathrm{Res}_{z_i} P^2(z) := 2k_i \cdot P(z_i) = 0 \,, \qquad P^2(z_0) = 0 \,.$$

# The 1-loop Integrand

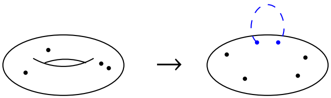
[Adamo-Casali-Skinner, Casali-Tourkine]

The ACS proposal for the 1-loop integrand of type II supergravity takes the form

$$\mathscr{M}_{\text{SG}}^{(1)} = \int d^d\ell \, d\tau \underbrace{\bar{\delta}(P^2(z_0)) \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(z_i))}_{\text{Scattering Equations}} \underbrace{\left( \sum_{\text{spin struct.}} Z^{(1)}(z_i) Z^{(2)}(z_i) \right)}_{\equiv \mathcal{I}_q, \text{ fermion correlator}}$$

- modular invariant
- localises on discrete set of solutions
- conjecture: for $\mathcal{I}_q = 1$, sum over permutations of $n$-gons in $d = 2n + 2$ dimensions

# From the Torus to the Riemann Sphere
## Residue theorem



Residue theorem: elliptic curve $\longrightarrow$ nodal Riemann spere at $q=0$.

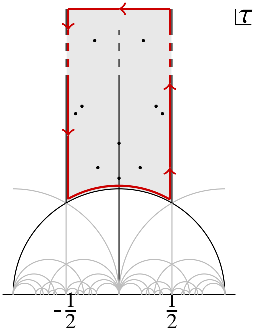
$$\begin{aligned}\mathcal{M}^{(1)}_{SG} &= \frac{1}{2\pi i}\int d^d\ell\, \frac{dq}{q}\,\bar\partial\left(\frac{1}{P^2(z_0)}\right)\prod_{i=2}^{n}\bar\delta(k_i\cdot P(z_i))\,\mathcal{I}_q \\ &= -\frac{1}{2\pi i}\int d^d\ell\,\bar\partial\left(\frac{dq}{q}\right)\frac{1}{P^2(z_0)}\prod_{i=2}^{n}\bar\delta(k_i\cdot P(z_i))\,\mathcal{I}_q \\ &= -\int d^d\ell\,\frac{1}{P^2(z_0)}\prod_{i=2}^{n}\bar\delta(k_i\cdot P(z_i))\,\mathcal{I}_0\Big|_{q=0}\,.\end{aligned}$$

Note that this moves us off ambitwistor space.

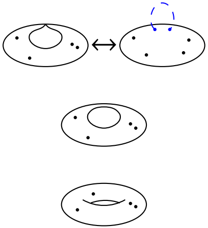
# From the Torus to the Riemann Sphere
## Contour argument

- residue theorem ⇔ contour integral argument in fund. domain
- modular invariance: contributions form sides and unit circle cancel

⇒ localisation on $q = 0$




Contour argument in the fundamental domain



# From the Torus to the Riemann Sphere

Mapping the fundamental domain to the Riemann sphere $\sigma = \mathrm{e}^{2\pi i(z-\tau/2)}$, we obtain

$$P(z) = P(\sigma) = \ell\, \frac{d\sigma}{\sigma} + \sum_{i=1}^{n} \frac{k_i\, d\sigma}{\sigma - \sigma_i}\,.$$

Setting $S = P^2 - \ell^2\, \mathrm{d}\sigma^2/\sigma^2$, the vanishing of the residues of $S$ gives

## the *off-shell scattering equations*

$$0 = \mathrm{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}\,,$$

- manifestly off-shell through loop momentum $\ell$
- closely related to tree-level scattering equations at $n+2$ points
- any $n-1$ equations imply all $n+2$

# From the Torus to the Riemann Sphere

## Off-shell scattering equations

$$0 = \mathrm{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i} + \sum_{j \neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} \,,$$

Using this, we get the following

## One-loop integrand on the nodal Riemann sphere

$$\mathscr{M}^{(1)} = - \int d^d\ell \, \frac{1}{\ell^2} \underbrace{\prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i))}_{\text{off-shell scattering equations}} \frac{d\sigma_i}{\sigma_i^2} \, \mathcal{I}_0 \,,$$

which is our new proposal for the supergravity 1-loop integrand, with $\mathcal{I}_0$ the $q = 0$ limit of the ACS correlator.

# The $n$-gon conjecture

Following the framework derived above, the $n$-gon conjecture becomes

$$\mathscr{M}^{(1)}_{n-\text{gon}} = -\int d^{2n+2}\ell \, \frac{1}{\ell^2} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2} \,,$$

which can be checked to give, with $\mathscr{M}^{(1)} = \int d^d\ell \, \widehat{\mathscr{M}}^{(1)}$,

$$\widehat{\mathscr{M}}^{(1)}_n = \frac{(-1)^n}{\ell^2} \sum_{\sigma \in S_n} \prod_{i=1}^{n-1} \frac{1}{\ell \cdot \sum_{j=1}^{i} k_{\sigma_i} + \frac{1}{2}\left(\sum_{j=1}^{i} k_{\sigma_i}\right)^2} \,.$$

Using partial fraction identities and shifts in the loop momentum, this is indeed equivalent to the sum over permutations of $n$-gons.

$\Rightarrow$ proof of $n$-gon conjecture $(n = 4, 5, 6)$

# The Supergravity 1-loop Integrand

For supergravity, $\mathcal{I}_q \equiv \mathcal{I}(k_i, \epsilon_i, z_i|q) = \mathcal{I}^L_q\, \mathcal{I}^R_q$, where each term is given by a sum over spin structures arising from the fermion correlator. At $q = 0$, this becomes

$$\mathcal{I}^L_0 = 16\,(\mathrm{Pf}(M_2) - \mathrm{Pf}(M_3)) - 2\,\partial_{q^{1/2}}\mathrm{Pf}(M_3)\,.$$

The 1-loop supergravity integrand is thus given by

$$\widehat{\mathscr{M}}^{(1)} = -\int \mathcal{I}^L_0 \mathcal{I}^R_0\, \frac{1}{\ell^2} \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2}\,.$$

At $n = 4$, $\mathcal{I} = t_8 t_8 R^4$ is a constant kinematic tensor and the results coincide with the $n$-gon conjecture. At $n = 5$, the amplitude can be written in terms of pentagon and box integrals.

$\Rightarrow$ proof of ACS 1-loop expression $(n = 4, 5)$

# The super Yang-Mills 1-loop Integrand

Remarkably, this naturally leads to a conjecture for super Yang-Mills scattering amplitudes at 1 loop;

$$\widetilde{\mathscr{M}}^{(1)}(1,\ldots,n) = \int \mathcal{I}_0^L \, PT_n \prod_{i=2}^{n} \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i} \, .$$

Here, the supergravity factor $\mathcal{I}_0^R$ has been replaced by a cyclic sum over Parke-Taylors factor running through the loop,

$$PT_n = \sum_{i=1, i \bmod n}^{n} \frac{\sigma_{0\,\infty}}{\sigma_{0\,i}\sigma_{i\,i+1}\sigma_{i+1\,i+2}\ldots\sigma_{i+n\,\infty}} \, .$$

$\Rightarrow$ This indicates the flexibility of our approach, no 1-loop ambitwistor expression was previously known!

# Proof at one loop:
# Factorisation and Q-cuts

# Proof Part I: Q-cuts

[Baadsgaard, Bjerrum-Bohr, Bourjaily, Caron-Huot, Damgaard, Feng]

Starting from the Feynman diagram expansion,

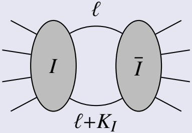
$$\mathcal{M}_{FD}^{(1)} = \frac{N(\ell)}{D_1(\ell)...D_m(\ell)} \,,$$

- shift the loop momentum $\ell \to \ell + \eta$, such that $\eta \cdot \ell = \eta \cdot k_i = 0$, and $\eta^2 = z$. In particular, $D_i(\ell) \to D_i(\ell) + z$.
- use Cauchy Residue theorem.
- shift loop momentum $\ell$ in each term by appropriate sum of external momenta $K_I$.

$$\Downarrow$$

## 'Q-cut' expansion

$$\mathcal{M}_{FD}^{(1)} = \sum_I$$




$$= \sum_I \frac{\mathcal{M}_I^{(0)} \mathcal{M}_{\bar{I}}^{(0)}}{\ell^2(2\ell \cdot K_I + K_I^2)}$$

# Proof Part II: Factorisation



All poles (and residues) of $\mathscr{M}^{(1)}$ are determined from

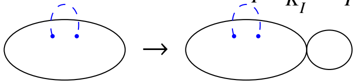
- localising on the boundary of the moduli space

  $\Leftrightarrow$

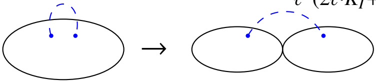
- factorising the nodal Riemann sphere

There are two cases of interest:

- seperating degeneration: $\mathscr{M}^{(1)} \to \mathscr{M}_I^{(1)} \frac{1}{K_I^2} \mathscr{M}_{\bar{I}}^{(0)}$



- 'Q-cut' degeneration: $\mathscr{M}^{(1)} \to \dfrac{\mathscr{M}_I^{(0)}(\ldots,\ell_I,\ell_I+K_I)\mathscr{M}_{\bar{I}}^{(0)}(\ldots,-\ell_I,-\ell_I-K_I)}{\ell^2(2\ell\cdot K_I+K_I^2)}$



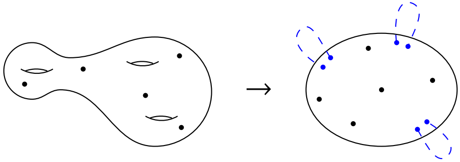
**This concludes the proof:**

$$\mathscr{M}^{(1)} = \mathscr{M}^{(1)}\Big|_{\text{Q-cut representation}}$$

# Extension:
# The All-loop Integrand

# The All-loop Integrand

Starting from the natural extensions of the ACS proposals to Riemann surfaces of genus $g$, we can again use residue theorems to localize on boundary components of the moduli space by contracting $g$ $a$-cycles to obtain a nodal Riemann sphere.



This fixes $g$ moduli, with the remaining $2g-3$ now associated with $2g$ new marked points. The 1-form $P$ is then given by

$$P = \sum_{r=1}^{g} \ell_r \omega_r + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i} \,,$$

where $\omega_r$ is a basis of $g$ global holomorphic 1-forms on the nodal Riemann sphere.

# The All-loop Integrand

Setting $S(\sigma) := P^2 - \sum_{r=1}^{g} \ell_r^2 \omega_r^2$, the multiloop off-shell scattering equations are

$$\mathrm{Res}_{\sigma_i} S = 0\,, \quad i = 1, \dots, n+2g\,.$$

This leads to the following proposal for the

## all-loop integrand;

$$\mathscr{M}_{SG}^{(g)} = \int_{(\mathbb{CP}^1)^{n+2g}} \frac{\mathcal{I}_0^L \mathcal{I}_0^R}{\mathrm{Vol}\, G} \prod_{r=1}^{g} \frac{1}{\ell_r^2} \prod_{i=1}^{n+2g} \bar{\delta}(\mathrm{Res}_{\sigma_i} S(\sigma_i))\,,$$

$$\text{where } \mathcal{I}_0 = \begin{cases} \mathcal{I}_0^L \mathcal{I}_0^R, & \text{gravity} \\ \mathcal{I}_0^L PT_n, & \text{Yang-Mills} \end{cases}.$$

Remarkably, this suggests that $n$-point $g$-loop integrands have a similar complexity to tree amplitudes with $n + 2g$ particles.

# Conclusion

We derived

- a framework to derive formulae for loop integrands on a nodal Riemann sphere using residue theorems, applicable to wide range of theories.
- new formulae for supergravity, super YM and $n$-gon integrands at 1 loop,
- supported on new, off-shell scattering equations that depend on the loop momenta.
- a proposal for the all-loop integrands in supergravity, SYM and biadjoint scalar theories.

This formalism implies that $n$-point $g$-loop scattering amplitudes have the same complexity as $n+2g$-point tree-level amplitudes!

# Outlook

At one loop:

- Can we use this to describe other massless theories at one loop?
- What about a 4d-specific ‘twistorial’ expressions at one loop?
- Can we find a worldsheet description? Importantly, this would allow for a direct derivation of interands (and might extend to higher loops).

Concerning the all-loop integrand conjecture:

- Investigate the known 2-loop integrands for the double box [Adamo-Casali].
- Proof? This could resolve the question concerning the UV behaviour of maximal supergravity!

*Thank you!*