Hyperbolic Vortices

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Outline

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1. Abelian Higgs Vortices

- The Abelian Higgs (Ginzburg–Landau) vortex is a two-dimensional static soliton, stabilised by its magnetic flux. Well-known is the Abrikosov vortex lattice in a superconductor.
- Vortices exist on a plane or curved Riemann surface M, with metric
$$ds^2 = \Omega(z, \bar{z})\, dz d\bar{z}\,. \quad (1)$$
$z = x_1 + ix_2$ is a (local) complex coordinate.
- The fields are a complex scalar Higgs field ϕ and a vector potential A_j ($j = 1, 2$) with magnetic field $F = \partial_1 A_2 - \partial_2 A_1$. They don't back-react on the metric.
- Our solutions have N vortices and no antivortices. On a plane, N is the winding number of ϕ at infinity. If M is compact, ϕ and A are a section and connection of a $U(1)$ bundle over M, with first Chern number N.

- The field energy is

$$E = \frac{1}{2}\int_M \left(\frac{1}{\Omega^2}F^2 + \frac{1}{\Omega}|D_j\phi|^2 + \frac{1}{4}(1-|\phi|^2)^2\right)\Omega\, d^2x \qquad (2)$$

where $D_j\phi = \partial_j\phi - iA_j\phi$. The first Chern number is

$$N = \frac{1}{2\pi}\int_M F\, d^2x\,. \qquad (3)$$

- The energy E can be re-expressed as [E.B. Bogomolny]

$$E = \pi N + \frac{1}{2}\int_M \left\{\frac{1}{\Omega^2}\left(F - \frac{\Omega}{2}(1-|\phi|^2)\right)^2 + \frac{1}{\Omega}\left|D_1\phi + iD_2\phi\right|^2\right\}\Omega\, d^2x \qquad (4)$$

where we have dropped a total derivative term.

Taubes Equation

- Minimum energy fields, for given N, satisfy the Bogomolny equations

$$\begin{aligned} D_1\phi + iD_2\phi &= 0\,, && (5) \\ F - \frac{\Omega}{2}(1 - |\phi|^2) &= 0\,. && (6) \end{aligned}$$

- Using eq.(5) to eliminate A_j, eq.(6) becomes the gauge-invariant Taubes equation

$$\nabla^2(\log|\phi|^2) + \Omega(1 - |\phi|^2) = 4\pi \sum_{k=1}^{N} \delta(z - Z_k)\,. \qquad (7)$$

Z_k are the vortex centres, where ϕ is zero.

Area Constraint on N

- The surface area
$$A = \int_M \Omega \, d^2x \tag{8}$$
constrains N. Integrating the second Bogomolny equation gives
$$2\pi N = \frac{1}{2}A - \frac{1}{2}\int_M |\phi|^2 \, \Omega \, d^2x \,, \tag{9}$$
so
$$4\pi N < A \,. \tag{10}$$
If A exceeds $4\pi N$, vortices can be at any N specified locations $Z_1, Z_2, \dots, Z_N$ [Taubes, Bradlow, Garcia-Prada].

2. Hyperbolic Vortices

- The Bogomolny equations are integrable on the hyperbolic plane $\mathbb{H}^2$, with curvature $-\frac{1}{2}$. This was discovered by Witten in connection with $SU(2)$ instantons in $\mathbb{R}^4$ with $SO(3)$ symmetry. Finite-N solutions are rational.
- N.S.M. and N.A. Rink have found trigonometric and elliptic vortex solutions on a hyperbolic trumpet and hyperbolic cylinder $\mathbb{H}^2/\mathbb{Z}$. These correspond to infinite chains of vortices on the cover $\mathbb{H}^2$.
- R. Maldonado and N.S.M have found isolated vortex solutions on compact hyperbolic surfaces. They correspond to an infinite lattice of vortices on a regular tesselation of $\mathbb{H}^2$.

Constructing Solutions

- Write the Bogomolny equations as

$$D_{\bar{z}}\phi = 0\,, \tag{11}$$

$$F_{z\bar{z}} = \frac{i}{4}\Omega(1-\phi\bar{\phi})\,, \tag{12}$$

where $D_{\bar{z}}\phi = \partial_{\bar{z}}\phi - iA_{\bar{z}}\phi$ and $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z$.

- Now let $\phi = \sqrt{H(z,\bar{z})}\,\chi(z)$, where χ is holomorphic and H real. Eq.(11) has solution

$$A_z = \frac{i}{2}\partial_z(\log H)\,, \quad A_{\bar{z}} = -\frac{i}{2}\partial_{\bar{z}}(\log H)\,, \tag{13}$$

and eq.(12) simplifies to the Taubes equation, in the form

$$4\partial_z\partial_{\bar{z}}(\log H) + \Omega(1 - H\chi(z)\overline{\chi(z)}) = 4\pi\sum_{k=1}^{N}\delta(z - Z_k)\,. \tag{14}$$

- The hyperbolic metric $\Omega\, dzd\bar{z}$ has Gaussian curvature $K = -\frac{1}{2}$, so
$$4\partial_z\partial_{\bar{z}}(\log\Omega) = \Omega\,. \tag{15}$$
- To construct a vortex solution on M, take another hyperbolic surface M', with coordinate w, metric $\Omega'(w,\bar{w})\, dwd\bar{w}$, and curvature $-\frac{1}{2}$, so
$$4\partial_w\partial_{\bar{w}}(\log\Omega') = \Omega'\,, \tag{16}$$
and let $f : M \longrightarrow M'$ be a holomorphic map given locally by $w = f(z)$.

- Let
$$\chi(z) = \frac{df}{dz}\,, \tag{17}$$
and let H be the ratio of the metrics at z and $f(z)$,
$$H(z,\bar{z}) = \frac{\Omega'(f(z),\overline{f(z)})}{\Omega(z,\bar{z})}\,. \tag{18}$$
- Then, away from any singularities,
$$\begin{aligned}
4\partial_z\partial_{\bar{z}}(\log H) &= -4\partial_z\partial_{\bar{z}}(\log\Omega) + 4\partial_z\partial_{\bar{z}}(\log\Omega') \\
&= -4\partial_z\partial_{\bar{z}}(\log\Omega) + 4\partial_w\partial_{\bar{w}}(\log\Omega')\frac{df}{dz}\overline{\frac{df}{dz}} \\
&= -\Omega + \Omega'\chi\overline{\chi} \\
&= -\Omega(1 - H\chi\overline{\chi})\,, \qquad (19)
\end{aligned}$$
as required.

Rational Solutions on $\mathbb{H}^2$

- Let's take $f : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$, using the disc model. The metrics are

$$\Omega(z,\bar{z}) = \frac{8}{(1-z\bar{z})^2} \quad \text{and} \quad \Omega'(w,\bar{w}) = \frac{8}{(1-w\bar{w})^2}\,. \tag{20}$$

- Therefore

$$\sqrt{H} = \frac{1-z\bar{z}}{1-f(z)\overline{f(z)}} \tag{21}$$

and the Higgs field is

$$\phi = \frac{1-z\bar{z}}{1-f(z)\overline{f(z)}}\frac{df}{dz}\,. \tag{22}$$

- The vortex centres are the ramification points of f, as ϕ is zero where $\frac{df}{dz}$ is zero.

- For an N-vortex solution, the required map is from $\mathbb{H}^2$ to $\mathbb{H}^2$, mapping boundary to boundary. It is a Blaschke rational function [Witten]

$$f(z) = \prod_{m=1}^{N+1} \frac{z - a_m}{1 - \overline{a_m} z} \tag{23}$$

with $|a_m| < 1\,,\ \forall m$.
- $\frac{df}{dz} = 0$ at N locations inside the disc (and N outside). These are the vortex centres.
- An example is a 1-vortex at the origin. Here $f(z) = z^2$, so

$$\phi = \frac{1 - |z|^2}{1 - |z|^4}\, 2z = \frac{2z}{1 + |z|^2}\,. \tag{24}$$

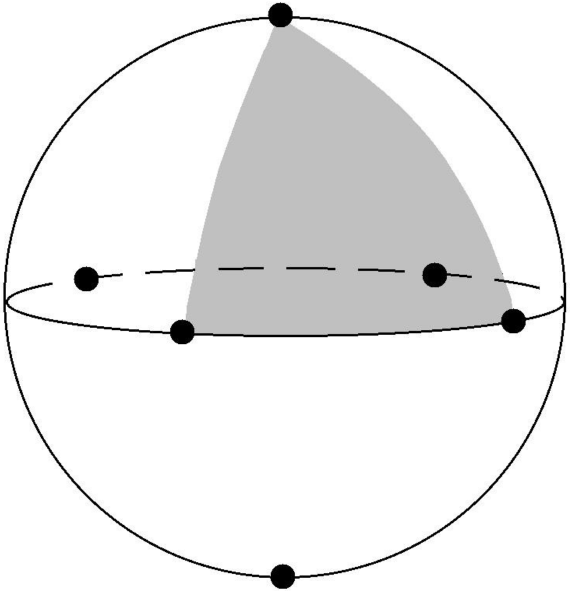
Note that $|\phi| \to 1$ as $|z| \to 1$.

3. 1-Vortex on the Genus-2 Bolza Surface

- Suppose M is compact, of genus $g \geq 2$ and curvature $-\frac{1}{2}$. By Gauss-Bonnet, the area is $A = 8\pi(g-1)$, so the number of vortices is $N < 2(g-1)$.
- If $g = 2$, there can only be 1 vortex. If $g = 3$ there can be 1,2 or 3 vortices, etc.
- The most symmetric genus 2 surface with a hyperbolic metric is the Bolza surface. This is the algebraic curve

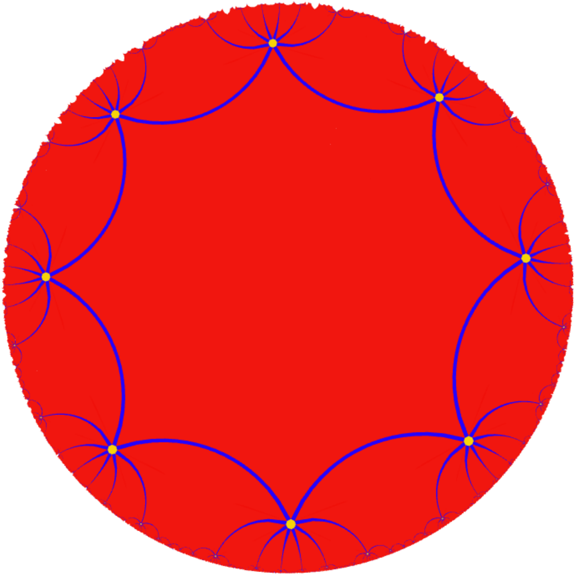
$$y^2 = (x^4 - 1)x \quad (x, y \in \mathbb{C})\,. \tag{25}$$

It double covers the Riemann sphere with six branch points at the vertices of a regular octahedron. Eight equilateral spherical triangles (angles $\frac{\pi}{2}$) are covered by sixteen equilateral hyperbolic triangles (angles $\frac{\pi}{4}$). The symmetry group, excluding reflections, has 48 elements.



Bolza surface double covers the Riemann sphere

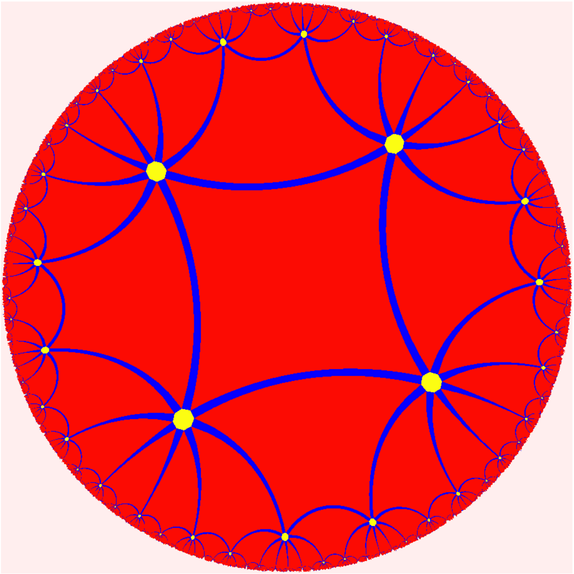
- Cut open, the Bolza surface is a regular octagon in $\mathbb{H}^2$ with vertex angles $\frac{\pi}{4}$ and opposite edges identified. The vertices are all identified to one point.
- The octagon makes just one C_8 subgroup obvious.
- The universal cover of the Bolza surface is $\mathbb{H}^2$, so $\mathbb{H}^2$ is tessellated by Bolza octagons. This is the $\{8, 8\}$ tessellation.



{8,8} tessellation of $\mathbb{H}^2$ by Bolza octagons

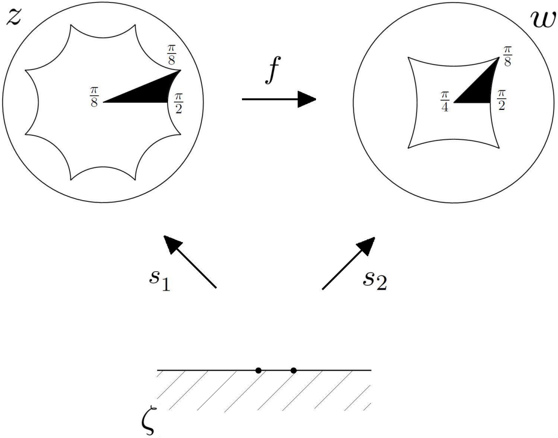
Hyperbolic octagon

- To find a 1-vortex on the Bolza surface, centred at the origin, we need a map f from $\mathbb{H}^2$ to $\mathbb{H}^2$, with $\frac{df}{dz} = 0$ at the origin, compatible with the tessellation into regular octagons.
- f can be found as a 2-1 map from the Bolza octagon of the $\{8,8\}$ tessellation to the square of the $\{4,8\}$ tessellation. (Opposite sides of the square are not identified.)



{4,8} tessellation of $\mathbb{H}^2$ by hyperbolic squares

- f can effectively be constructed as a 1-1 map from a triangle (a sixteenth of the octagon) to another triangle (an eighth of the square).
- A Schwarz triangle map $s(\zeta)$ maps $\mathbb{H}^2$ (in the upper half plane model with coordinate ζ) to a triangle. The triangle-triangle map $f(z)$ is a composition $s_2(s_1^{-1}(z))$.
- f can be analytically continued by reflections across boundaries to a map from $\mathbb{H}^2$ to $\mathbb{H}^2$.
- The vortex has C_8 symmetry, and f has an expansion $f = \alpha z^2 + \beta z^{10} + \dots$.



Map f from hyperbolic octagon to hyperbolic square

- $s(\zeta)$ is known as a ratio of hypergeometric functions [Harmer and Martin]

$$s(\zeta) = \sqrt{\frac{\sin(\pi a')\sin(\pi b')}{\sin(\pi a)\sin(\pi b)}\frac{\Gamma(a')\Gamma(b')\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c')}}\,\zeta^{1-c}\frac{F(a',b';c';\zeta)}{F(a,b;c;\zeta)}\,. \tag{26}$$

The parameters are linear combinations of the triangle angles (divided by π), so are simple rational numbers.

- We can calculate the Higgs field numerically and plot its contours. We can also compute its expansions around the symmetry points of the Bolza surface analytically.
- The Higgs field near the origin (the vortex centre) is

$$\phi = (4\pi)^{-3/2} \sin\left(\frac{\pi}{8}\right) \Gamma^2\left(\frac{1}{8}\right) \Gamma\left(\frac{1}{4}\right) z + \cdots \approx 1.768z\,. \tag{27}$$

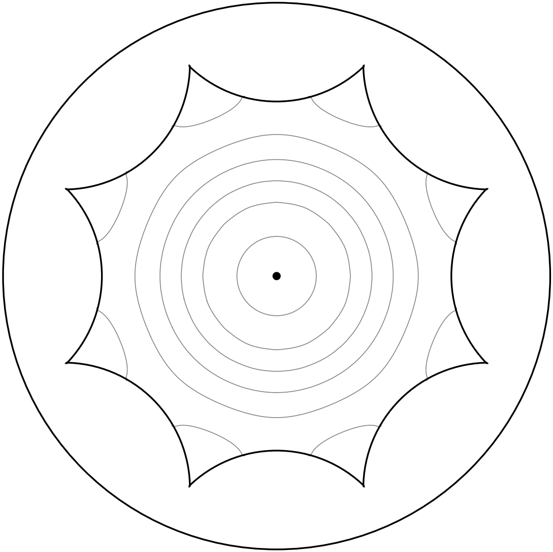
- At an edge mid-point of the octagon (a saddle of $|\phi|$)

$$|\phi| = \frac{\sqrt{2}\,\Gamma\left(\frac{1}{8}\right)\Gamma^2\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{1}{16}\right)\Gamma\left(\frac{3}{16}\right)\Gamma\left(\frac{5}{16}\right)\Gamma\left(\frac{7}{16}\right)} \approx 0.752\,. \tag{28}$$

- At a vertex (maximally far from the vortex)

$$|\phi| = 2^{-1/4} \approx 0.841\,, \tag{29}$$

and this is the maximal value of $|\phi|$.



Contours of $|\phi|^2$ for 1-vortex on Bolza octagon

4. Baptista's Geometric Interpretation of Vortices

- Consider a general surface M with metric $ds^2 = \Omega \, dzd\bar{z}$. Its Gaussian curvature K is given by

$$2K\Omega = -\nabla^2(\log \Omega)\,. \tag{30}$$

- Define a new metric $ds^2 = \Omega' \, dzd\bar{z}$ on M using a vortex solution

$$\Omega' = \Omega|\phi|^2\,. \tag{31}$$

  Ω' has zeros at the vortex centres, so M with the new metric acquires conical singularities with opening angle 4π, and hence deficit angle -2π. The new surface locally double covers the old surface.

- The new Gaussian curvature K' is given by

$$\begin{aligned} 2K'\Omega' &= -\nabla^2(\log\Omega') \\ &= -\nabla^2(\log\Omega + \log|\phi|^2) \\ &= 2K\Omega + \Omega(1-|\phi|^2) - 4\pi\sum\delta(z-Z_k) \\ &= 2K\Omega + \Omega - \Omega' - 4\pi\sum\delta(z-Z_k)\,, \end{aligned} \tag{32}$$

where we used the Taubes equation for $|\phi|$. Therefore

$$(2K'+1)\Omega' = (2K+1)\Omega - 4\pi\sum\delta(z-Z_k)\,. \tag{33}$$

This is Baptista's equation. The vortices define a new metric, preserving $(2K+1)\Omega$ away from the singularities.

- By Gauss–Bonnet, the integrals of both $K\Omega$ and $K'\Omega'$ are $4\pi(1-g)$. Integrating (33) therefore gives

$$A' = A - 4\pi N\,. \tag{34}$$

  A' is less than A but still positive as $4\pi N < A$.
- For vortices, a hyperbolic surface with constant curvature $K = -\frac{1}{2}$ is special. The new metric Ω' is also hyperbolic, with $K' = -\frac{1}{2}$.
- We can verify (34) explicitly for the vortex on the Bolza surface.
- The moduli space of N-vortex solutions on M is equivalent to a moduli space of punctures on M of a special conical type.

5. Conclusions

- Bogomolny vortices are integrable on a hyperbolic surface of curvature $-\frac{1}{2}$. Solutions on the hyperbolic plane are rational.
- On compact hyperbolic surfaces a few explicit solutions are known in the most symmetric cases. The vortex number N is constrained by the genus g.
- Vortices can be interpreted geometrically, as defining hyperbolic metrics with conical singularities of deficit angle -2π on a background smooth surface. The metric on the moduli space of vortices is probably analogous to a Weil-Petersson metric on the moduli space of surfaces with these conical singularities. The details have not been worked out.