From asymptotics to exact results in Physics and Mathematics

Inês Aniceto

(Jagiellonian University, Krakow)

(Based on work with R. Schiappa, M. Vonk, M. Spaliński)

11^{th} SEMPS, University of Surrey, March 28, 2018

Outline

1. Motivation
2. Setting up Resurgence
3. Applications
   - Summation: Painlevé I and Matrix Models
   - Interpolation: Cusp Anomalous Dimension
   - Prediction: QNM in $\mathcal{N} = 4$ SYM
4. Summary/Future Directions

Perturbation Theory & Asymptotic Series

Perturbation theory: fundamental in computations of

- energies in quantum mechanics
- Solutions on NLODEs
- beta-functions in quantum field theory
- genus expansions of string theory
- large N expansion of non-abelian gauge theories

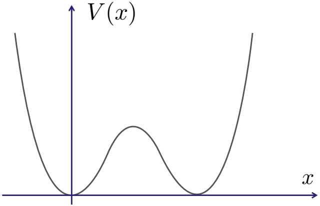
  ...

Goal: to understand analytic properties beyond numerical computations

BUT... most perturbative expansions are asymptotic, i.e. zero radius of convergence!

- Why? due to non–perturbative "semi-classical" effects such as
  - instantons
  - renormalons
  - Other objects not captured by a perturbative analysis

Double Well in Quantum Mechanics



e.g. $V(x) = \frac{x^2}{2}\left(1+\sqrt{g}x\right)^2$

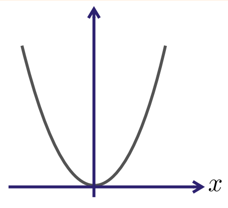
- Hamiltonian

$$H = -\frac{1}{2}\left(\frac{d}{dx}\right)^2 + V(x)$$

- Schrödinger eq

$$H\psi(x,g) = E(g)\psi(x,g)$$

$g = 0 \Rightarrow$ Harmonic oscillator



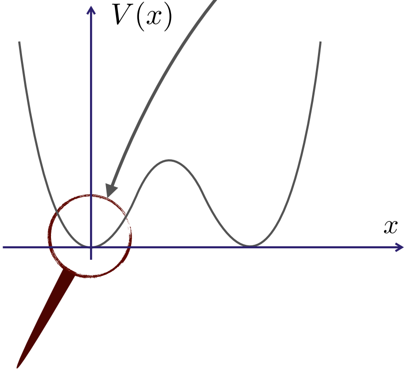
$$V_H(x) = \frac{1}{2}x^2$$

$$E_{\mathrm{g.s.}} = \frac{1}{2}$$

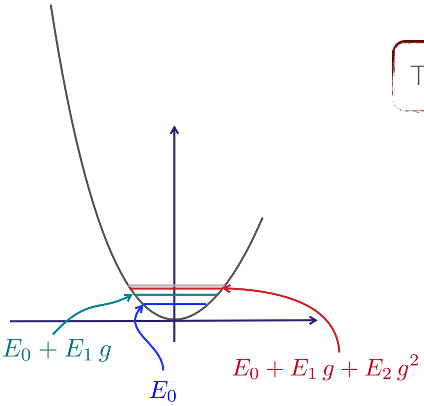
$g > 0$ How can we solve it?

Double Well in Quantum Mechanics

Take coupling g very small



Double Well in Quantum Mechanics



Take coupling g very small

- Ground-state energy:

$$E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n\, g^n$$

where $E_0 = 1/2$

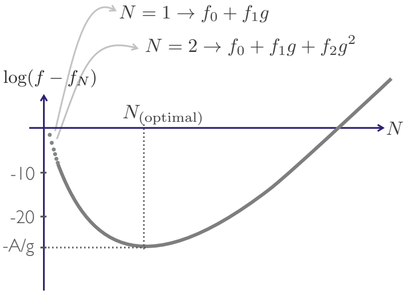
Questions:

- Does the series converge? No! **Asymptotic series**
- Exact results? **Borel transform** & **resummation**

Aside: Asymptotic series

$$f(g) \quad \simeq \quad \sum_{n \geq 0} f_n \, g^n$$

- Divergent! No matter how small g is: $f_n \, g^n \to \infty$
- Truncate at some optimal $n = N$: very good approximation
- Take $g \ll 1$ fixed: define truncation $f_N(g) = \sum_{n=0}^{N} f_n \, g^n$



$$\boxed{N_{(\text{optimal})} \approx A/g}$$

Optimal error:

$$(f - f_N)(g) \sim e^{-A/g}$$

for some value A

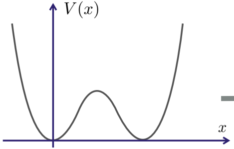
$e^{-A/g}$ Non-perturbative effect: $g \to 0$ invisible in perturbation theory!

Aside: Asymptotic series

$$f(g) \quad \simeq \quad \sum_{n\geq 0} f_n\, g^n$$

- Divergent! No matter how small g is: $f_n\, g^n \to \infty$
- Truncate at some optimal $n = N$: very good approximation
- Take $g \ll 1$ fixed: define truncation $f_N(g) = \sum_{n=0}^{N} f_n\, g^n$

Double-well Potential: $E_n \sim n!\, A^{-n}$ instantons!



$A = 1/6$

If we fix
$g = 1/100$

$N_{\mathrm{op}} = A/g \approx 16$

Optimal error:
$(f - f_N)(g) \sim \mathrm{e}^{-\mathrm{A}/\mathrm{g}}$
$\approx 5.7 \times 10^{-8}$

Analytic properties? Resummation

Perturbative expansion of quantity $F(g)$ in parameter $g \sim 0$

$$F(g) \quad \simeq \quad \sum_{n \geq 0} F_n \, g^{n+1} \,, \qquad \text{Asymptotic series: } F_n \sim n!$$

- How to find $F(g)$?
  - Borel transform $\mathcal{B}[F]$: "remove" the factorial growth
  - Analytically continue $\mathcal{B}[F]$ to full complex plane
  - Define resummation $\mathcal{S}F$ by the inverse Borel transform

Aside: Borel Transform & Resummation

Asymptotic series: $$F(g) \simeq \sum_{n\geq 0} F_n \, g^{n+1}\,, \qquad \text{with} \quad F_n \sim n!$$

- **Borel transform**: $$\mathcal{B}[F](s) = \sum_{n=0}^{\infty} \frac{F_n}{n!}\, s^n$$

  Rule: $\mathcal{B}\left[g^{\alpha+1}\right](s) = s^{\alpha}/\Gamma(\alpha+1)$

  - finite radius of convergence - find function $\mathcal{B}[F](s)$
  - In general $\mathcal{B}[F](s)$ will have singularities

- **Borel resummation** of F is the Laplace transform

$$\mathcal{S}F(g) = \int_0^{\infty} ds\, \mathcal{B}[F](s) \mathrm{e}^{-s/g}$$

Resummation & Ambiguities

$$F(g) \quad \simeq \quad \sum_{n\geq 0} F_n\, g^{n+1}\,, \qquad \text{Asymptotic series: } F_n \sim n!$$

Borel resummation of F along direction θ is the Laplace transform

$$\mathcal{S}_\theta F(g) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} ds\, \mathcal{B}[F](s)\mathrm{e}^{-s/g}$$

- BUT: $\mathcal{S}F$ is just a Laplace transform - needs an integration contour to be properly defined!

- **If we have a singularity in the complex Borel plane:**

> **Nonperturbative ambiguity:** ambiguity in choosing how integration contour will avoid the singularity

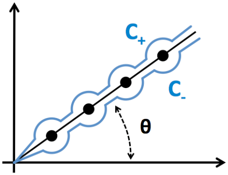
Nonperturbative Ambiguity

Borel resummation of F along direction θ is the Laplace transform

$$\mathcal{S}_\theta F(g) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} ds\, \mathcal{B}[F](s)\mathrm{e}^{-s/g}$$

- Take $\mathcal{B}[F](s)$ with singularities in direction θ:

Nonperturbative ambiguity:



- $\mathcal{B}[F](s) \sim \frac{1}{s-A}$ in direction θ

$$\mathcal{S}_+ F(g) - \mathcal{S}_- F(g) \sim \exp\left(-A/g\right)$$

- around $g \sim 0$ this is non-analytic

- Singularities in the Borel plane occur along Stokes lines

Perturbative series is non-Borel resummable along Stokes lines

Glimpse into Resurgence

- Borel plane singularities:
  - Related to non-perturbative data
  - Govern asymptotic behaviour of original perturbative series

Non-perturbative information **resurges** in the perturbative data!

- Understanding the resurgent properties of our solution:

Obtain a non-ambiguous, global, analytic result
How can we achieve this?

Next

Resurgence

Beyond Perturbation Theory?

Learn from the example of anharmonic potential in QM
[Vainshtein'64, Bender,Wu'73]

- Perturbative series of ground-state energy:

$$E^{(0)}(g) = \sum E_k^{(0)} g^k \,, \quad E_k^{(0)} \sim k!\, A^{-k} \,,\ k \gg 1$$

- Resummation along real axis: singularities and ambiguity!

What happens if we try to include instanton sectors?

- Expanding around each fixed instanton sector

$$n - \text{instanton sector:} \;\; E^{(n)}(g) = \mathrm{e}^{-nA/g} \sum E_k^{(n)} g^k$$

Also asymptotic, with large-order behaviour

$$E_k^{(n)} \sim k!\, (n\, A)^{-k} \,,\ k \gg 1$$

All multi-instanton series suffer from nonperturbative ambiguities!

Problem or Solution?

- Infinite instanton sectors with nonperturbative ambiguities!

  Seems to make the problem with perturbation theory even *worse*!

- **BUT:** for the ground state energy of double-well potential
  [Bogomolny,Zinn-Justin'80-83]
  - ambiguity in 2-instanton sector *precisely* cancels ambiguity in perturbative expansion
  - ambiguity in 3-instanton sector cancels ambiguity in 1-instanton sector
  - ...

  Multi-instantonic ambiguities are the *solution* to our problem!

Beyond Perturbation Theory!

Ground-state energy = sum over all multi-instanton sectors

Ambiguities arising in different sectors conspire to cancel each other
The final result is *real* and *free* from any nonperturbative ambiguities!

How to implement this sum? **Transseries ansatz!**

Transseries: formal power series in two or more variables, each a function of the parameter $z \sim 0$

$$E(g,\sigma) = \sum_{n\geq 0} \sigma^n E^{(n)}(g)\,, \qquad E^{(n)}(g) \simeq \mathrm{e}^{-nA/g} \sum_{k\geq 1} E_k^{(n)} g^k$$

- our case has $\mathrm{e}^{-A/g}$ and g
- σ: instanton counting parameter

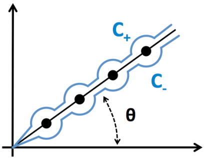
Ambiguities along Stokes lines

$$E(g,\sigma) = \sum_{n\geq 0} \sigma^n E^{(n)}(g)\,, \qquad E^{(n)}(g) \simeq \mathrm{e}^{-nA/g} \sum_{k\geq 1} E_k^{(n)} g^k$$

- If $\mathcal{B}[E^{(n)}]$ has singularities in a direction θ (Stokes line)
  - $E^{(n)}(g)$ has an associated ambiguity: $\left(\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-}\right) E^{(n)} \neq 0$
  - **BUT**: $\mathcal{S}_{\theta\pm} E$ are related:

    $$\mathcal{S}_{\theta^+} E^{(n)} = \mathcal{S}_{\theta^-} \circ \left(E^{(n)} - \mathrm{Disc}_\theta E^{(n)} \right)$$

    - $\mathrm{Disc}_\theta \neq 0$ encodes Stokes transition at θ



- Cancelling ambiguities:
  - Choose $\sigma = \sigma_0$ such that $\left(\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-}\right) E\left(z,\sigma_0\right) = 0$
  - Non-ambiguous result is $\frac{1}{2}\left(\mathcal{S}_{\theta^+} + \mathcal{S}_{\theta^-}\right) E\left(z,\sigma_0\right)$

Calculating Ambiguities and Discontinuities? Via **Resurgence**

Resurgence [Écalle'81]

Cancelation of ambiguities in multi-instanton sectors: larger structure behind perturbation theory!

Resurgence analysis and Transseries

A transseries ($z = \frac{1}{g} \sim \infty$)

$$F(z,\sigma) = \sum_{n\geq 0} \sigma^n F^{(n)} , \qquad F^{(n)}(z) \simeq \mathrm{e}^{-nAz} \sum_{k\geq 0} F_k^{(n)} z^{-k}$$

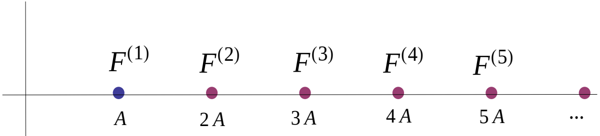
defines a resurgent function if it relates the asymptotics of multi- instanton contributions $F_n^{(\ell)}$ in terms of $F_n^{(\ell')}$ where ℓ' is close to ℓ

How does it work?

Resurgence at play [IA,Schiappa,Vonk'11]

Multi-instanton asymptotic series

$$F(z) = \sum_{n=0}^{\infty} \sigma^n F^{(n)}(z)$$

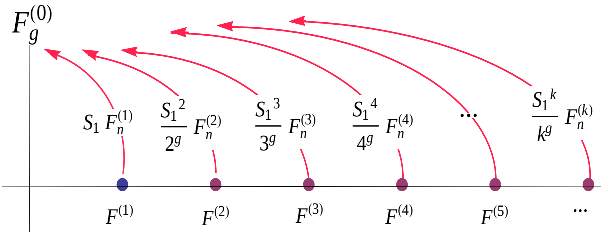


Perturbative series: $$F^{(0)}(z) = \sum_{g=0}^{\infty} F_g^{(0)}\, z^{-g-1}$$

Instanton series: $$F^{(n)}(z) = e^{-nAz} \sum_{g=1}^{\infty} F_g^{(n)}\, z^{-g}$$

Resurgence at play[IA,Schiappa,Vonk'11]

Large-order behaviour - Perturbative series for large g

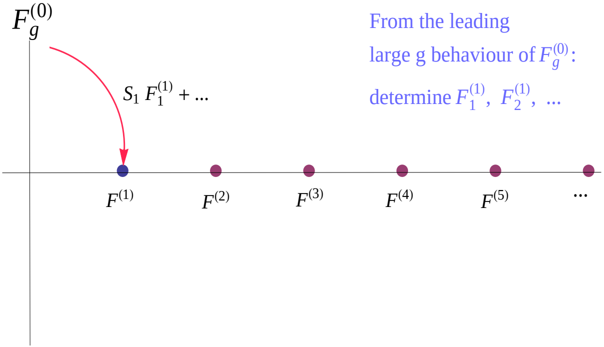


All multi-instanton sectors contribute to the large-order behavior of coefficients F_g^0

$$F_g^{(0)} \sim S_1 \sum_{n>0} a_n(g)\, F_n^{(1)} + 2^{-g} S_1{}^2 \sum_{n>0} b_n(g)\, F_n^{(2)} + \dots$$

Resurgence at play [IA,Schiappa,Vonk'11]

Equivalently: Perturbative series for large g ENCODES all other sectors

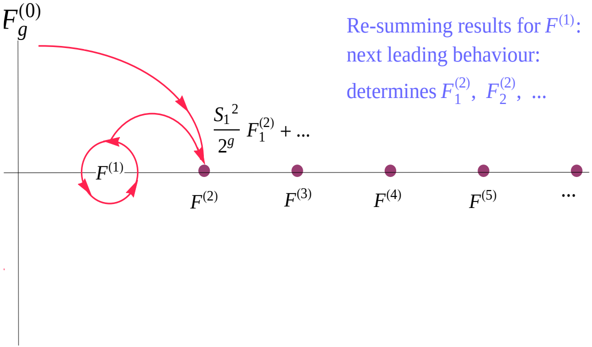


From the leading large g behaviour of $F_g^{(0)}$: determine $F_1^{(1)},\ F_2^{(1)},\ \ldots$

$$F_g^{(0)} \sim S_1\left(F_1^{(1)} + \frac{A}{g-1}\, F_2^{(1)} + \ldots\right) + O(2^{-g})$$

Resurgence at play [IA,Schiappa,Vonk'11]

Equivalently: Perturbative series for large g ENCODES all other sectors



Re-summing results for $F^{(1)}$:
next leading behaviour:
determines $F_1^{(2)}$, $F_2^{(2)}$, ...

$$F_g^{(0)} - S_1 \sum_{n>0} a_n(g)\, F_n^{(1)} \sim 2^{-g}\, S_1{}^2 \left(F_1^{(2)} + \frac{2A}{g-1} F_2^{(2)} + \ldots \right) + O(3^{-g})$$

Resummation and analytic results

Full solution defined by transseries ($z \sim \infty$)

$$F(z,\sigma) \;=\; \sum_{n\geq 0} \sigma^n \mathrm{e}^{-nAz} \Phi^{(n)}(z)\,, \qquad \Phi^{(n)}(z) \;\simeq\; z^{\beta_n} \sum_{k\geq 0} F_k^{(n)} z^{-k}$$

How to evaluate it? Depends on the value of $z \in \mathbb{C}$

- If $\mathrm{Re}(A\,z) > 0$, non-perturbative sectors exponentially suppressed:

  Borel summation

  $$\mathcal{S}_\theta F(z,\sigma) \;=\; \mathcal{S}\Phi^{(0)}(z) + \sigma \mathrm{e}^{-Az}\, \mathcal{S}\Phi^{(1)}(z) + \mathcal{O}\left(\mathrm{e}^{-2Az}\right)$$

  - we can obtain results for large **AND** small couling ($z \ll 1$)

- If $\mathrm{Re}(A\,z) = 0$, all sectors of the same order:

  Analytic transseries summation

  $$\mathcal{S}F(z,\sigma) = \sum_{n\geq 0} \sigma^n \mathrm{e}^{-nAz} z^{\beta_n}\, F_0^{(n)} + \frac{1}{z} \sum_{n\geq 0} \sigma^n \mathrm{e}^{-nAz} z^{\beta_n}\, F_1^{(n)} + \mathcal{O}\left(z^{-2}\right)$$

  - we can obtain anaytic information, e.g. zeros of the solution

Next

Applications!

Resurgence in Quantum Theories

- Many recent applications of resurgence
  - Ordinary integrals and non-linear differential equations
  - Quantum Mechanics: Exact WKB, ambiguity cancelations
  - QFTs: fractional instantons, UV renormalons, OPEs
  - Matrix models: generalised instanton sectors
  - String theory: holomorphic anomaly equation

Next:

1. Analytic summation [Garoufalidis,Its,Kapev,Mariño,IA,Schiappa,Vaz,Vonk,'10 -'18]
   - Global solutions of NLODEs: Painlevé I
   - Asymptotics of matrix models at large N
2. Ambiguity cancelation and interpolation [IA,'15]
   - Cusp anomalous dimension at large coupling
3. Prediction of nonperturbative phenomena [IA,Spaliński'15, on-going]
   - Quasi-normal modes in $\mathcal{N} = 4$ SYM

Painlevé I and Matrix Models

Painlevé I, 2d Gravity and Matrix models

- Matrix models:
  - NP description of string theory in simpler backgrounds: non-critical strings and Dijkgraaf-Vafa type topological strings[Dijkgraaf,Vafa '02]
  - Simper models for studying NP structure behind large N 't Hooft expansions
  - Can help us understand large-N duality
- 2d quantum gravity is obtained by taking a double scaling limit: large N and small coupling g_s[Douglas,Shenker '90][Brézin,Kazakov '90][Gross,Migdal '90]
- Free energy of 2d gravity related to the Painlevé I NLODE

$$u^2 - \frac{1}{6}u'' = z$$

  - $u(z) = -F''(z)$ where $z^{-5/4} \sim g_s$.
- Study Painlevé I: simpler model, already showing major features from string theory
  - Asymptotic series with $(2g)!$ growth $\Rightarrow$ g_s^2 expansion

General solution for Painlevé I

Use a **2-parameter transseries**: [Garoufalidis,Its,Kapaev,Mariño '10] [IA,Schiappa,Vonk '11]

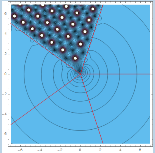
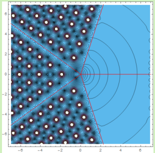
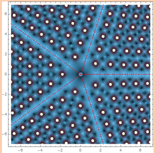
$$u(x;\sigma_1,\sigma_2) = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m \, \mathrm{e}^{-\frac{(n-m)A}{x}} \, \Phi^{(n|m)}(x)$$

- Two instanton actions $A = \pm 8\sqrt{3}/5$: evidence of resonance, many sectors with same exponential grading
- $x = z^{-5/4} \sim g_s$ is open string coupling; σ_i are boundary data
- Asymptotic series: $\Phi^{(n|n)}(\mathrm{x})$ have a topological genus expansion (g_s^2), $\Phi^{(n|m)}$, $n \neq m$ have expansions in g_s: evidence of resonance

- Sectorial solutions in Painlevé I: specified by boundary data σ_i
  - Different σ_i determine different solutions and asymptotics
- Stokes phenomena: "glue" different sectors to build global solutions

Painlevé I solutions

$$u(x,\boldsymbol{\sigma}) = \sum_{\mathbf{n}\in\mathbb{N}_0^2} \boldsymbol{\sigma}^{\mathbf{n}} \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}/x}\Phi_{\mathbf{n}}(x), \quad \mathbf{A} \equiv (A,-A),\ \boldsymbol{\sigma}^{\mathbf{n}} \equiv \sigma_1^n\sigma_2^m$$

0 parameter: Tritronquée	1 parameter: Tronquée	2 parameter
4 empty "quintants" 	2 empty "quintants" 	General 

- Can we "sum" the transseries into a function? Take $\sigma_2 = 0$
  - If $\mathrm{e}^{-A/x}$ is exp. suppressed: Borel-Padé summation
  - If $\mathrm{e}^{-A/x} \sim 1$: analytic transseries summation $\Rightarrow$ analytical data
    - Mathematical interpretation: anti-Stokes line
    - Physical interpretation: phase transition

Painlevé I Partition function $\mathcal{Z}$

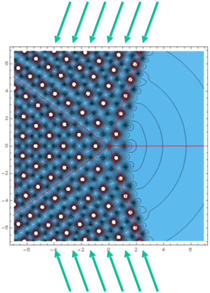
- Define Partition function: $\mathcal{Z}(x,\boldsymbol{\sigma}) = \mathrm{e}^F$ with $F'' \equiv u$
- Analytic transseries summation: Allows us to go inside the "filled sectors"

$$\mathcal{Z}(x,\boldsymbol{\sigma}) = \sum_{n=0}^{+\infty} \left(\sigma_1 \mathrm{e}^{-A/x}\right)^n x^{\beta_n} F_0^{(n)} + x \sum_{n=0}^{+\infty} \left(\sigma_1 \mathrm{e}^{-A/x}\right)^n x^{\beta_n} F_1^{(n)} + \cdots$$

- Sectors with poles of u, zeros of $\mathcal{Z}$.
- Find locations of **all** zeros of the partition function from the transseries [Costin et al, '95-13; IA,Schiappa,Vonk, on-going]

$$\mathcal{Z}_0\left(\zeta, q\right) = \sum_{n=0}^{+\infty} G_2(n+1)\zeta^n q^{n^2},\ \zeta \sim \sigma_1 \mathrm{e}^{-A/x};\ q \sim x^{\frac{1}{2}}$$

- Only works for the adjoining sectors: to get to fifth sector: Stokes phenomena



Summation and analytic results

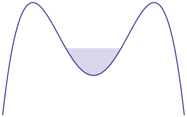
Large N quartic matrix model

Quartic matrix model

Quartic model partition function ($N \times N$ matrix M)

$$\mathcal{Z}(N, g_s) \propto \int dM \, \exp\left(-\frac{1}{g_s}\mathrm{Tr}\, V(M)\right), \quad V(z) = \frac{1}{2}z^2 - \frac{1}{24}\lambda z^4$$

Local solutions in "Stokes regions": saddle point analysis around **1-cut solution**



Free energy has perturbative genus expansion at large N

$$F \equiv \log Z \simeq \sum_{g \geq 0} F_g(t)\, g_s^{2g-2}, \; t = g_s N$$

- Obey a NP **finite difference eq**: string equation

$$\mathcal{R}(t)\left(1 - \frac{\lambda}{6}(\mathcal{R}(t-g_s) + \mathcal{R}(t) + \mathcal{R}(t+g_s))\right) = t\,, \quad \mathcal{R}(n\, g_s) = r_n$$

where $r_n = \frac{\mathcal{Z}_{n+1}\mathcal{Z}_{n-1}}{\mathcal{Z}_n^2}$ and $\mathcal{R}(t)$ is directly related to the free energies

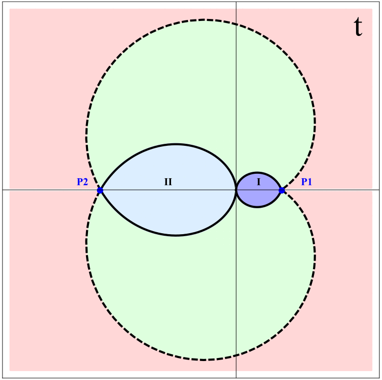
Quartic matrix model

$\mathcal{R}(t)$ has **resurgent properties**:

$$\mathcal{R}(t,\sigma_1,\sigma_2) = \sum_{n,m\geq 0} \sigma_1^n \sigma_2^m \mathrm{e}^{-N\,(n-m)\,\frac{A(t)}{t}}\, t^{\beta_{nm}} R_{(n|m)}(t)$$

- $R_{(n|m)}(t)$ asymptotic expansions
- Instanton action $A(t)$ and coefficients $R_g^{(n|m)}(t)$ are functions.
- Large-N phase diagram (first studied in [Bertola '07,Bertola,Tovbis '11]): study the leading contributions to the exponentials:
  - Stokes lines $\mathbb{Im}\,(A\,(t)\,/t) = 0$: instanton contributions maximally suppressed
  - Anti-Stokes lines $\mathbb{Re}\,(A\,(t)\,/t) = 0$: all contributions of same order
- Recover analytic data from the transseries:
  - Finite N results via Borel-Padé summation [Couso-Santamaría,Schiappa,Vaz '15]
  - Lee-Yang zeros via analytic transseries summation [IA,Schiappa,Vonk, on-going]

Phase Diagram



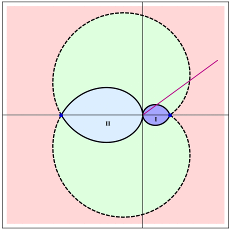
- light blue: Stokes regions, standard 't Hooft large N expansion
  - I: 1-cut solution is dominant
  - II: 2-cut sym solution dominant
- green: anti-Stokes region,dominated by 3-cuts solution, modular properties; no genus expansion [Bonnet,David,Eynard '00]
- light red: trivalent tree-like configuration dominant

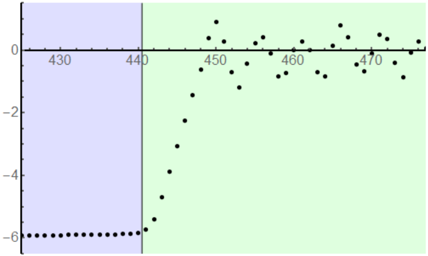
- Re line in I and II: Stokes lines, exponentially suppressed saddles are maximally suppressed
- P1 (P2): DS point described by Painlevé I (II) equation

Evidence of different phases?
What local solutions are associated with each phase?
How to obtain analytic data? Global Solutions?

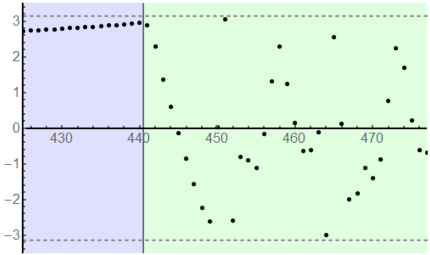
The anti-Stokes phase: numerical evidence

- Numerically calculate the recursion coefficients r_n with the boundary condition of the 1-cut configuration
- Take $N = 1000$ $\arg t = \frac{\pi}{12}$ fixed, change $|t|$ from the 1-cut phase into anti-Stokes
- r: normalization factor (classical solution $g_s = 0$)





(r_n−r): log of absolute value

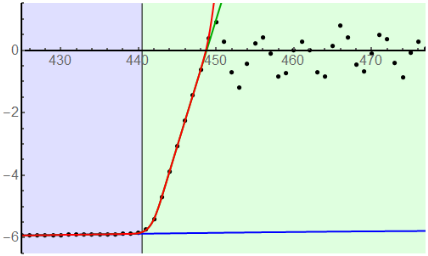


(r_n−r): phase

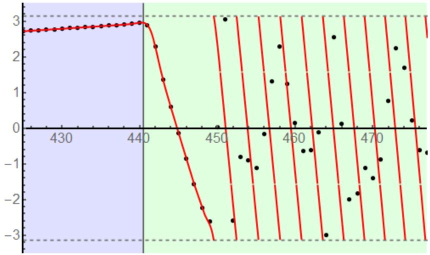
Evidence of **different phases**: they lead to different asymptotics of the $\mathcal{R}(t)$ in different regions

The anti-Stokes phase: numerical evidence

- Perform optimal truncation to the one-parameter sectors of $\mathcal{R}(t, \sigma_1, 0)$:
  - perturbative $R_{(0,0)}(t)$ plus n-instantons $R_{(n,0)}(t)$, for $n = 1, 2, 3$
- Compare to the numerical results for the r_n



$(r_n - r)$ vs $(R_{3-\text{inst}} - r)$: log of absolute value

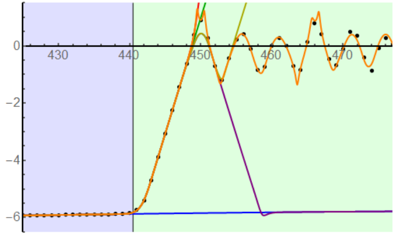


$(r_n - r)$ vs $(R_{3-\text{inst}} - r)$: phase

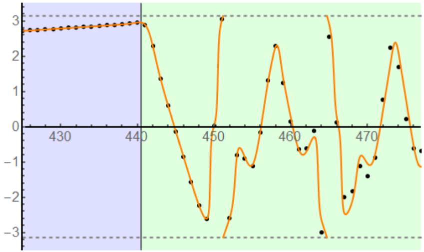
Adding the first three instanton correction to the $\mathcal{R}(t)$, we cannot reach far into the anti-Stokes region: all instanton contributions are of the same order and need to be included.
Can we do better? Perform **analytic transseries summation**

Analytic transseries summation

- Perform analytic transseries summation for the one-parameter partition function $\mathcal{Z}(t) = \mathrm{e}^{\mathcal{F}}$
- Sum the leading terms in g_s for $\mathcal{Z}(t)$
- Determine the $\mathcal{R}(t)$ from these results



$(r_n - r)$ vs $(R_{\mathrm{quad.ATS},1} - r)$: log of absolute value



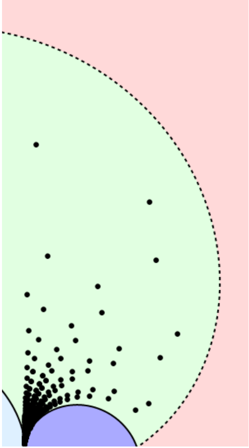
$(r_n - r)$ vs $(R_{\mathrm{quad.ATS},1} - r)$: phase

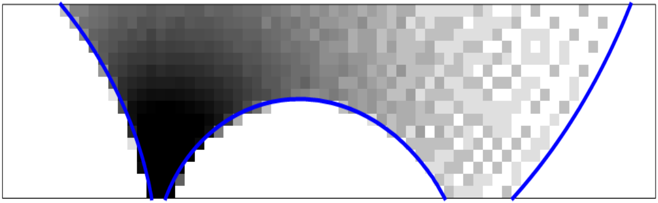
Leading g_s analytic transseries summation for $\mathcal{Z}(t)$ follows the numerical results far into the anti-Stokes region!

Zeroes of the partition function

Use the analytic transseries summation to predict Lee-Yang zeros?

- **Left:** prediction of zeros of $\mathcal{Z}(t)$ obtained from analytic transseries summation with $N = 10$ eigenvalues
- **Down:** numerical calculation of zeros from direct calculation of the matrix integral ($N = 100$). The grayscale is proportional to number of zeros





Leading g_s quadratic transseries summation for $\mathcal{Z}(t)$ predicts analytic results deep into the anti-Stokes region!

Ambiguity cancelation and Interpolation

Cusp Anomalous Dimension

Cusp Anomalous Dimension

- Appears in $\mathcal{N} = 4$ SYM and strings in $AdS_5 \times S^5$
- Scaling behaviour of the anomalous dimension of a Wilson loop with a light-like cusp in the integration contour

$$\langle W \rangle \sim \mathrm{e}^{-\Gamma_{\mathrm{cusp}} \log \frac{\Lambda_{\mathrm{UV}}}{m_{\mathrm{IR}}}}$$

- Scaling dimension of a twist-2 operator $\mathrm{tr}(X^I D_{\mu_1} \cdots D_{\mu_S} X^I)$, at large spin S;
- Dispersion relation of long folded spinning strings in AdS:

$$\Delta - S = f(g) \log S$$

$f(g)$: universal scaling function

Cusp Anomalous Dimension

- From integrability it obeys the BES integral equations [Beisert,Eden,Staudacher,07]

$$\frac{\gamma(2gt)}{2gt} = K\left(2gt,0\right) - 2g\int_0^\infty \frac{dt'}{e^{t'}-1} K(2gt,2gt')\gamma(2gt')$$

  - $K\left(t,t'\right)$ is so-called BES Kernel [Eden,Staudacher,06]

- Cusp anomalous dimension given by

$$\Gamma_{\rm cusp}\left(g\right) = 8\lim_{t\to 0}\frac{\gamma(2gt)}{2gt}.$$

- Weak coupling result $g \ll 1$ known
- **Resurgent analysis:** for $g \gg 1$ expansion is asymptotic! [Basso,Korchemsky,Kotanski,07]

Transseries and ambiguities [IA,15]

- Up to 2-instantons: 1-parameter transseries ansatz $(x = 8\pi g \gg 1)$

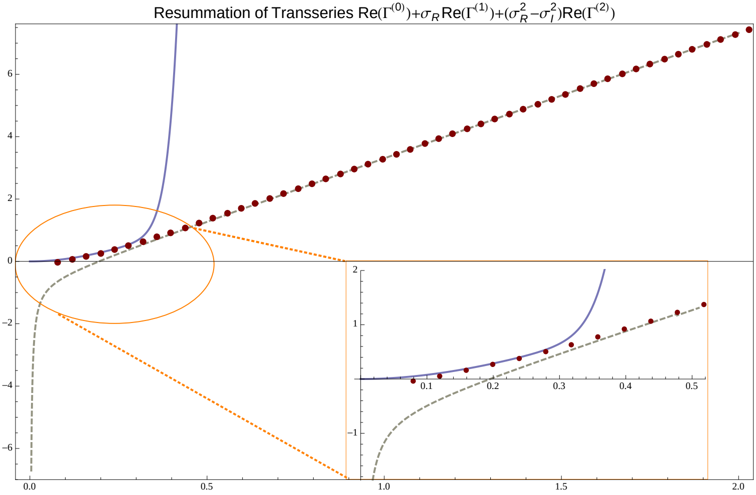
$$\frac{\Gamma_{\rm cusp}(g,\sigma)}{2g} - 1 = \sum_{m=0}^{+\infty} \sigma^m \mathrm{e}^{-mA\frac{x}{2}} \Gamma^{(m)}(x)\,; \quad \Gamma^{(m)}(x) \simeq x^{-m/2} \sum_{k=0}^{+\infty} \Gamma_k^{(m)} \left(\frac{x}{2}\right)^{-k}$$

- $\Gamma^{(m)}(x)$ are asymptotic series. Resurgent transseries? Yes!
  - sectors $\Gamma^{(0)}$, $\Gamma^{(1)}$ and $\Gamma^{(2)}$ related via large order relations
- g real and positive: resummation of each sector $\mathcal{S}_{\theta=0}\Gamma^{(m)}(x)$
  - **But**: $\theta = 0$ direction has singularities - it is a Stokes line!
  - We have an imaginary ambiguity: $(\mathcal{S}_{0^+} - \mathcal{S}_{0^-})\,\Gamma^{(m)}(x) \neq 0$
- Use resurgence to cancel ambiguity: fix $\sigma_0 = \sigma_R + \mathrm{i}\,\sigma_I$
  - $\Gamma_{\rm cusp}(g,\sigma_0)$ no longer has imaginary part!
- Can we resum the transseries and obtain results for g finite?

Yes: via the Borel-Padé resummation

Resummation and Results at Weak Coupling [IA,15]

- Resum the results up to 2nd nonperturbative order:



  - dashed – truncated sum of the perturbative expansion $\Gamma^{(0)}$
  - Blue – known small coupling expansion (7 loops) of $\Gamma(g)$

Prediction of NP phenomena

Hydrodynamics

Hydrodynamic gradient expansion

- Evolution equations for energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0$$

- In hydrodynamic theories the E-M tensor is given by

$$T^{\mu\nu} = \mathcal{E}\, u^\mu u^\nu + \mathcal{P}(\mathcal{E})(\eta^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu},$$

  - $\mathcal{E}$ is energy density
  - $\Pi^{\mu\nu}$ is the shear stress tensor
  - $\mathcal{P}(\mathcal{E}) = \mathcal{E}/3$ is pressure in $d = 4$ conformal theories
  - u is flow velocity - timelike eigenvector of the E-M tensor

- **Hydrodynamic gradient expansion:** approximate $\Pi^{\mu\nu}$ by series of corrections to ideal fluid behaviour

Fluid-gravity correspondence

- Hydrodynamic gradient expansion can be determined via the microscopic theory associated to the fluid
- For relativistic hydrodynamics with boost invariant flow: microscopic theory is large N $\mathcal{N} = 4$ SYM at strong coupling
- Objective: determine energy density from gauge-gravity duality, by solving Einstein's equations with appropriate metric ansatz
- Non-hydrodynamic d.o.f. are exponentially decaying sectors of a transseries-type ansatz for the metric components, quasi-normal modes (QNM)
- Determine perturbative part to very high order (240 terms) [Heller,Janik,Witaszczyk,'13]
- Determine non-perturbative sectors to high order [IA,Jankowski,Meiring,Spaliński,Witaszczyk,on-going]

Non-hydrodynamic modes and gradient expansion

Borel transform for the perturbative part of gradient expansion:

[Heller,Janik,Witaszczyk,'13] [IA,Jankowski,Meiring,Spaliński,Witaszczyk,on-going]

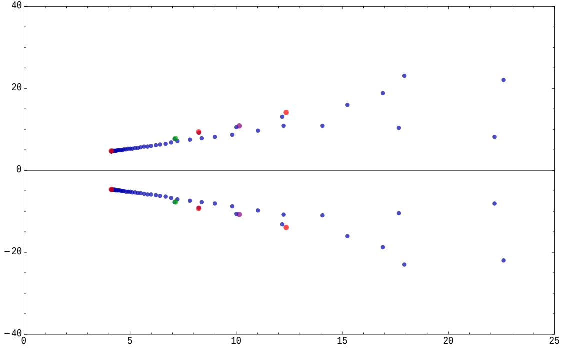
QNM:

ω_1; $2\omega_1$; $3\omega_1$

ω_2;

ω_3;

$\overline{\omega}_i$



$$
\begin{aligned}
\omega_1 &= 3/2\,(2.746676 + 3.119452\mathrm{i}); \\
\omega_2 &= 3/2\,(4.763570 + 5.169521\mathrm{i}); \\
\omega_3 &= 3/2\,(6.769565 + 7.187931\mathrm{i});
\end{aligned}
$$

Resurgence?

Transseries and NP predictions

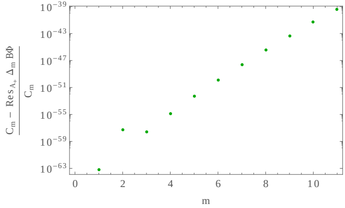
- Multi-parameter transseries ansatz for the energy density

$$\epsilon(\tau,\boldsymbol{\sigma}) = \sum_{\mathbf{n}} \boldsymbol{\sigma}^{\mathbf{n}} \mathrm{e}^{-\mathbf{n}\cdot\mathbf{A}(\omega_i)\tau^{2/3}} \Phi_{\mathbf{n}}(\tau)$$

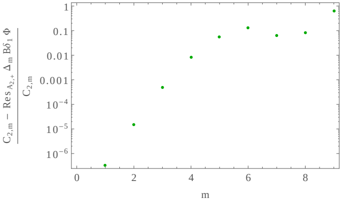
- Analyse the large order behaviour of the hydrodynamic series

$$\Phi_{\mathbf{0}}(\tau) \simeq \tau^{-4/3} \sum_{k=0}^{+\infty} \epsilon_k^{(0)} \tau^{-2k/3}$$

Convergence of $\epsilon_k^{(0)}$ to first coefficients of ω_1 sector



Convergence of resummed $\epsilon_k^{(0)}$ to first coefficients of ω_2 sector



Summary

Introduction to resurgence and applications to physical problems

- Resurgence analysis:
  - Transseries solutions
  - Predictions and large-order relations
  - Ambiguity cancelations
  - Summation and analytic results
- Applications:
  - Painlevé I NLODE and Large N dynamics of matrix models
  - Strong coupling of cusp anomalous dimension
  - Strongly coupled fluid in $\mathcal{N} = 4$ SYM and gravitational QNM

Current work

- Analysis of phase diagram of quartic matrix model
  - Stokes transitions;
  - modular properties of the transseries
- Stokes transitions in Painlevé I
- Algebra structure of multi-parameter resurgent transseries, interplay between
  - coupling g_{YM}, $g_s \to 0$;
  - rank of gauge group $N \to \infty$;
  - 't Hooft coupling $\lambda = g_{YM}^2 N$ fixed: large, small
- Applications of resurgence in string theory observables:
  - Bremsstrahlung function;
  - Lüscher corrections and the thermodynamic Bethe ansatz

Thank you!