# Generalised geometry and consistent truncations

Louise Anderson

[based on ArXiv 1711:04711]

# Outline

- Supergravity and Consistent truncations
  ( Motivation for physicists ?)
- Generalised geometry
  ( Motivation for mathematicians?)
- Supergravity and generalised geometry
- Adjoint orbits and consistent truncations

# Consistent truncations

( Motivation for physicists ?)

# Something quick about string theory

**(Super) String theory** D=10,11

# Something quick about string theory

(Super) String theory D=10,11

**IIA**

**IIB**

**$E_8$ x $E_8$ Heterotic**

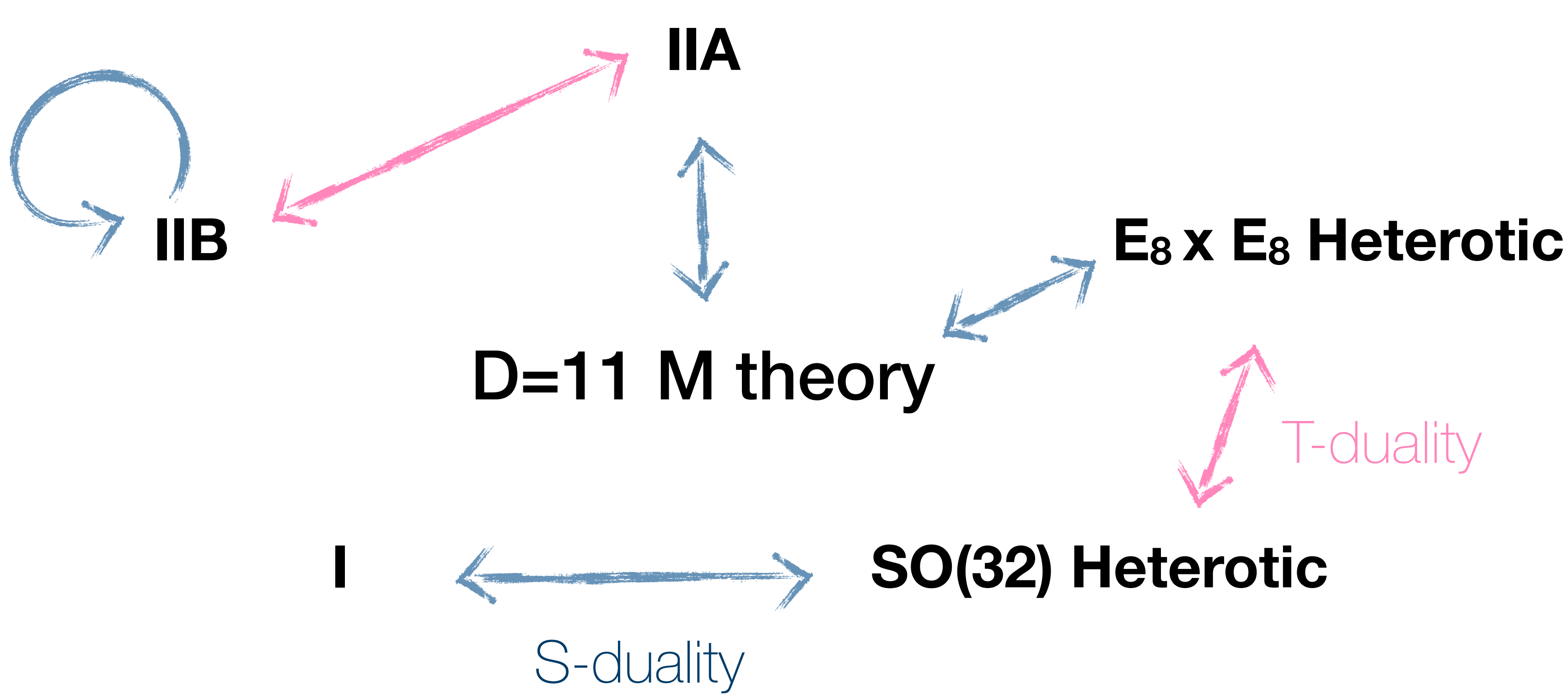
5 different ones

**I**

**SO(32) Heterotic**

# Something quick about string theory

(Super) String theory D=10,11




[Witten -95]

# Something quick about string theory

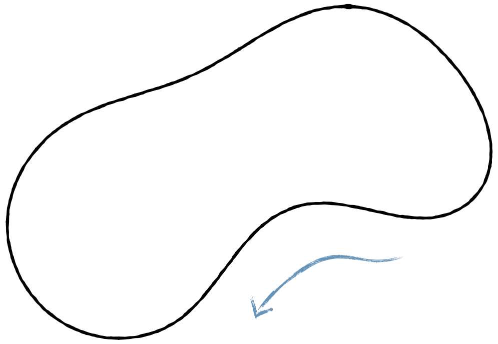
**(Super) String theory** D=10,11

**IIB** $\longleftrightarrow$ **IIA**

massless DOF

R-R

NS-NS

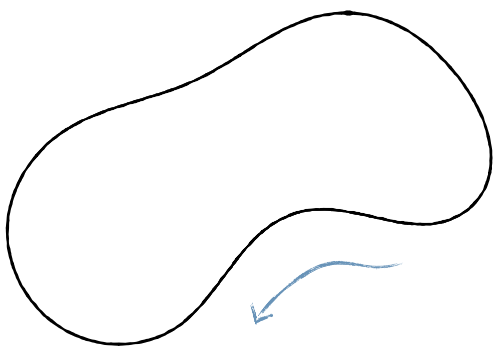


# Supergravity

The low-energy limit D=10,11

| | IIB | IIA |
|---|---|---|
| massless DOF | | |
| R-R | $F_{1,3,5}$ +fermions | $F_{0,2,4}$ +fermions |
| NS-NS | $(g_{\mu\nu},\ \phi,\ B_{\mu\nu})$ | $(g_{\mu\nu},\ \phi,\ B_{\mu\nu})$ |
| | chiral | non-chiral |

128 Fermionic + 128 Bosonic



Fourier modes

(keep only the lowest modes)

# Supergravity

## The NS-NS-sector and symmetries

$$(g_{\mu\nu},\, \phi,\, B_{\mu\nu})$$

Action: $$S = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[e^{-2\phi}\left(\mathcal{R} + 4\,(\partial\phi)^2 - \frac{1}{12}H^2\right)\right]$$

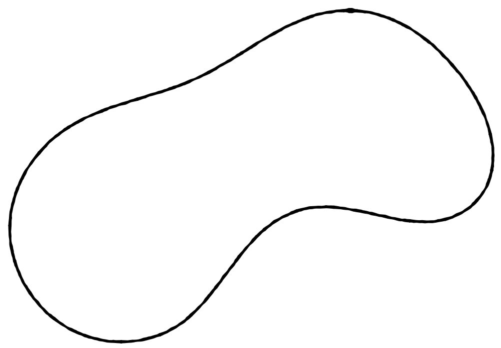
Only 3-form field strength appears!

$\Longrightarrow$ Gauge symmetry: $B \to B + d\Lambda$

$$\Omega^2(\mathcal{M})_{cl} \rtimes \mathrm{Diff}(\mathcal{M})$$

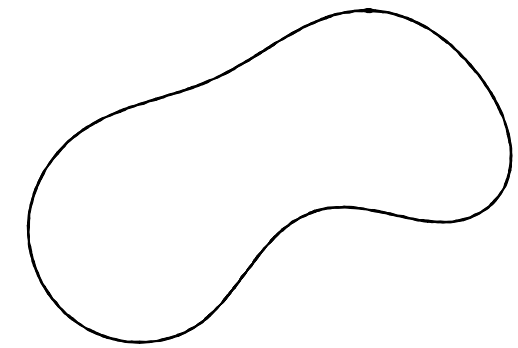
# Dimensional reduction

A simple example: KK-reduction



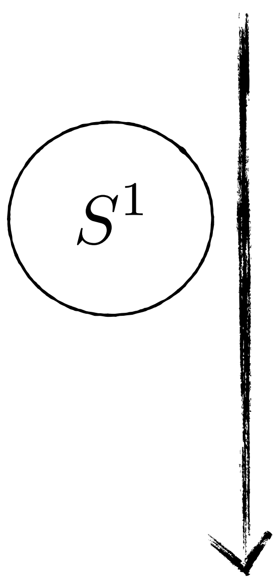
What about
lower dimensions?

# Dimensional reduction



## A simple example: KK-reduction

D



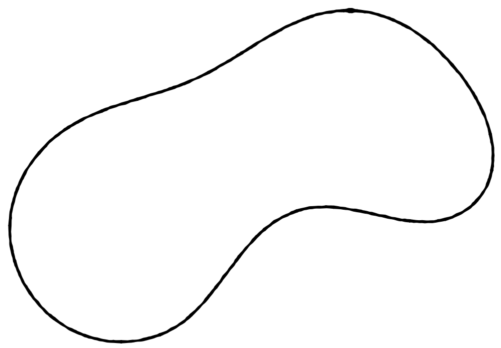

d=D-1

Assume one spatial direction is compact.

Decompose into Fourier modes in that direction.

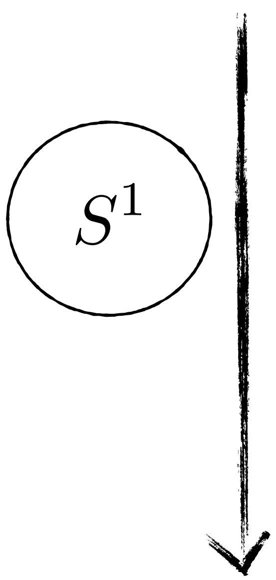
$\Longrightarrow$ infinite tower of massive particles

# Dimensional reduction



## A simple example: KK-reduction

D



$S^1$

d=D-1

Assume one spatial direction is compact.

Decompose into Fourier modes in that direction.

$\Longrightarrow$ infinite tower of massive particles

$R_{S^1} \to 0$ $\Longrightarrow$ truncates to massless modes

# Dimensional reduction

In general:

D=10,11 | Solutions of EOMs


d=D-k | Solutions of EOMs

# Dimensional reduction

In general:

D=10,11

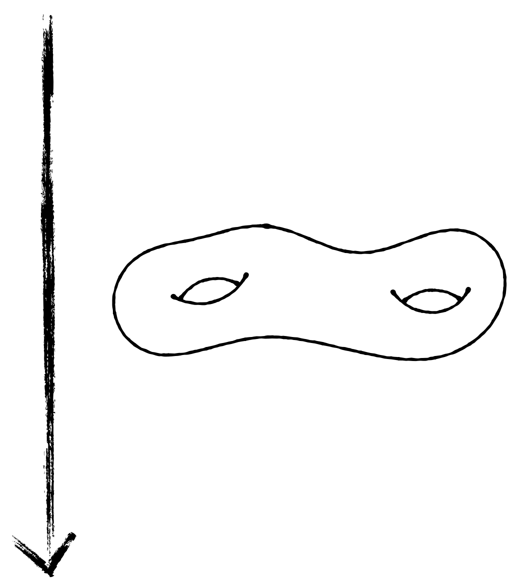
Solutions of EOMs?


d=D-k

Solutions of EOMs

# Consistent Truncations

D=10,11



d=D-k

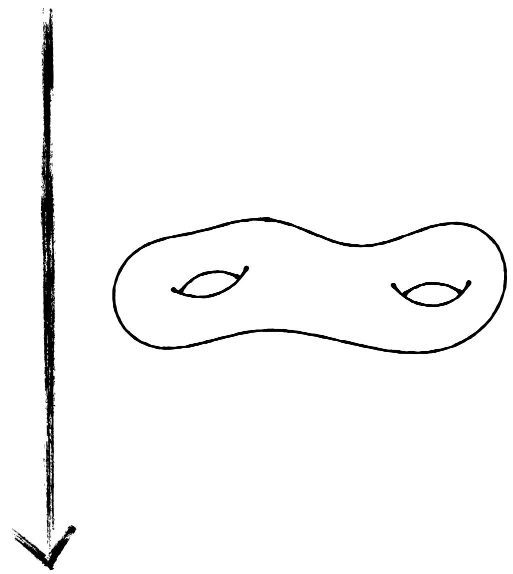
Solutions of EOMs!

**Consistent truncation!**

Solutions of EOMs

# Consistent Truncations

D=10,11



d=D-k

Solutions of EOMs!

**Consistent truncation!**

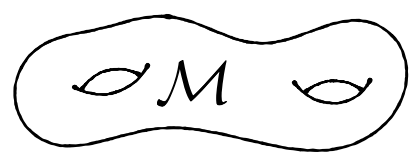
Solutions of EOMs

# Consistent Truncations

Which properties must $\mathcal{M}$ have for this to be possible?

**Complicated problem!**

D=10,11




d=D-k

Solutions of EOMs!

**Consistent truncation!**

Solutions of EOMs

# Consistent Truncations

Which properties must $\mathcal{M}$ have for this to be possible?

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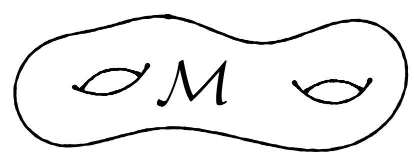


d=D-k

Solutions of EOMs!

**Consistent truncation!**

Solutions of EOMs

# Scherk-Schwartz Reductions

Which properties must $\mathcal{M}$ have for this to be possible?

D=10,11

local group manifold,

ex $\mathcal{M} = S^1,\ S^3$

$\mathcal{M}$

d=D-k

# Scherk-Schwartz Reductions

Which properties must $\mathcal{M}$ have for this to be possible?

D=10,11

$\Big\downarrow$

d=D-k

local group manifold,
ex $\mathcal{M} = S^1,\ S^3$

$\mathcal{M}$

$\mathcal{M}$ group manifold

$\Longrightarrow$ parallelisable

…and, the frames satisfy a condition inherited from the Lie algebra.

# Known exceptional cases:

Which properties must $\mathcal{M}$ have for this to be possible?

D=10,11

$\downarrow$

d=D-k

local group manifold,

ex $\mathcal{M} = S^1,\ S^3$

$\mathcal{M}$

…but it also works for $\mathcal{M} = S^d,\ H^d, \ldots$

Why?

# Generalised Geometry

( Motivation for mathematicians ?)

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London

2018-03-28

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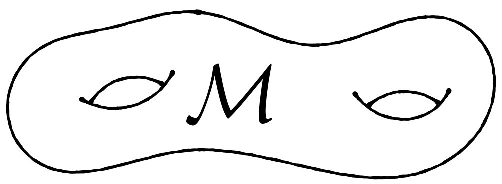
# Intro to Generalised Geometry




What we're used to: $T\mathcal{M}$

vector $v$

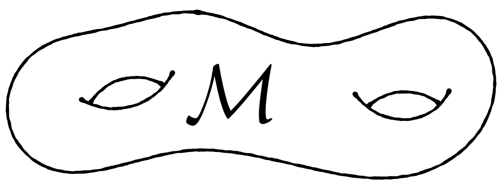
Generalised geometry: $T\mathcal{M} \oplus T^*\mathcal{M}$

generalised vector

$$V = (v,\, \lambda)$$

[Courant -90] [Hitchin et al. -03,-04,-10]

# Intro to Generalised Geometry

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vector $\quad v$

Riemannian metric $g$

Generalised geometry:

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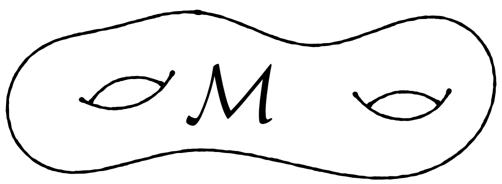
generalised vector

$$V = (v,\ \lambda)$$
$$W = (w,\ \mu)$$

natural metric $\quad \eta$

generalised metric $\quad \mathcal{G}$

# The linear algebra of $T\mathcal{M} \oplus T^*\mathcal{M}$

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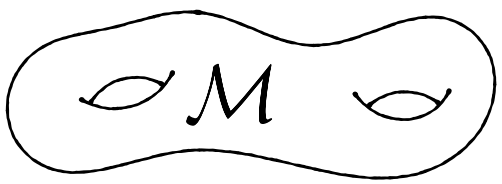
natural metric $\eta$

$$2\eta\,(V, W) = i_v\mu + i_w\lambda$$

generalised metric $\mathcal{G}$

# The linear algebra of $T\mathcal{M} \oplus T^*\mathcal{M}$

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natural metric $\eta$

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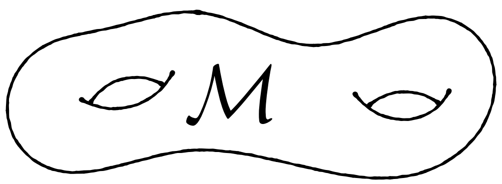
as a matrix:

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \simeq \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

indefinite, preserved by O(d,d)-transformations

# The linear algebra of $T\mathcal{M} \oplus T^*\mathcal{M}$

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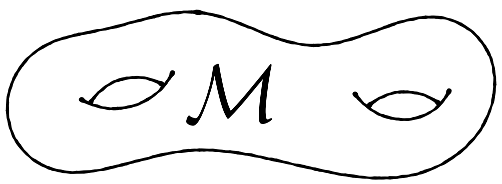
# O(d,d) Generalised Geometry

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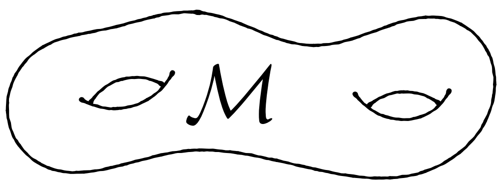


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natural metric $\quad \eta$

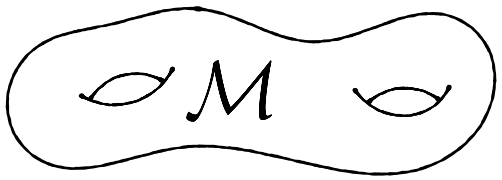
Canonical orientation

$$\Longrightarrow$$

structure group:

O(d,d) $\longrightarrow$ SO(d,d)

# O(d,d) Generalised Geometry




Generalised geometry:

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

$$V = (v,\ \lambda)$$
$$W = (w,\ \mu)$$

structure group: SO(d,d)

generated by:

$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$$

GL(d)-action

$$e^{-B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

$B$-transform

$B \in \Lambda^2 T^* M$

$$e^{-\beta} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

$\beta$-transform

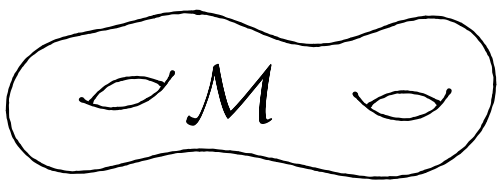
$\beta \in \Lambda^2 TM$

2 extra 'symmetries'

$$e^B V = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = v + \lambda + i_v B$$

$$e^\beta V = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = v + \lambda + i_\lambda \beta$$

# Courant Algebroid




What we're used to:

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vector $v$

Riemannian metric $g$

natural metric $\eta$

Lie bracket

Courant bracket

Generalised geometry:

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

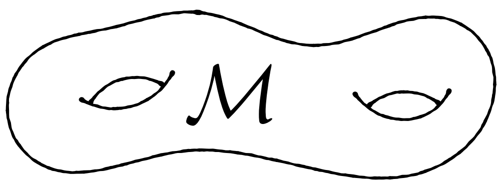
$$V = (v,\ \lambda)$$
$$W = (w,\ \mu)$$

$$[[v+\lambda, w+\mu]] = [v,w] + \mathcal{L}_v\mu - \mathcal{L}_w\lambda - \frac{1}{2}d\left(i_v\mu - i_w\lambda\right)$$

note: $\pi\left([[V,W]]\right) = [[\pi(V), \pi(W)]]$

$$\pi : T\mathcal{M} \oplus T^*\mathcal{M} \to T\mathcal{M}$$

# Courant Algebroid




What we're used to:

$$T\mathcal{M}$$

vector $v$

structure group: SO(d,d)

$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \qquad e^{-B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \qquad e^{-\beta} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

GL(d)-action

$B$-transform
$B \in \Lambda^2 T^* M$

$\beta$-transform
$\beta \in \Lambda^2 TM$

Generalised geometry:

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

$$V = (v,\, \lambda)$$
$$W = (w,\, \mu)$$

Courant bracket

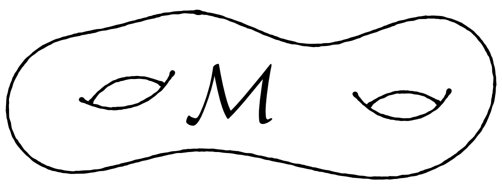
$$[[v + \lambda, w + \mu]] = [v, w] + \mathcal{L}_v \mu - \mathcal{L}_w \lambda - \frac{1}{2} d\left(i_v \mu - i_w \lambda\right)$$

This is invariant under diffeomorphisms…

…and under $B$-transforms with *closed* $B$

(not under $\beta$-transforms)

# Courant Algebroid

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$B$-transform

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$\beta$-transform

$\beta \in \Lambda^2 TM$

Generalised geometry:

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

$$V = (v,\, \lambda)$$
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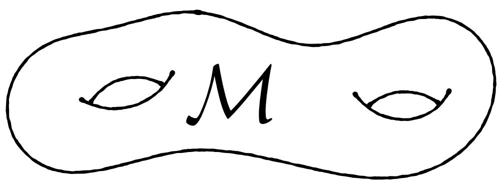
Courant bracket

$$[[v+\lambda, w+\mu]] = [v,w] + \mathcal{L}_v \mu - \mathcal{L}_w \lambda - \frac{1}{2} d\left(i_v \mu - i_w \lambda\right)$$

There is an overall action of

$$\Omega^2(\mathcal{M})_{cl} \rtimes \mathrm{Diff}(\mathcal{M})$$

# Courant Algebroid




What we're used to:

$T\mathcal{M}$

vector $\quad v$

Riemannian metric $g$

natural metric $\eta$

Lie bracket

Courant bracket

Generalised geometry:

$T\mathcal{M} \oplus T^*\mathcal{M}$

generalised vector

$V = (v,\, \lambda)$

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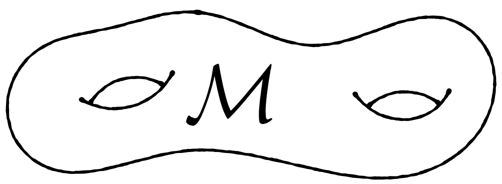
$$[[v+\lambda, w+\mu]] = [v,w] + \mathcal{L}_v\mu - \mathcal{L}_w\lambda - \frac{1}{2}d\left(i_v\mu - i_w\lambda\right)$$

…but this does not satisfy the Jacobi identity

# Courant Algebroid




$$T\mathcal{M} \oplus T^*\mathcal{M}$$

natural metric $\eta$

Courant bracket

$\pi : T\mathcal{M} \oplus T^*\mathcal{M} \to T\mathcal{M}$

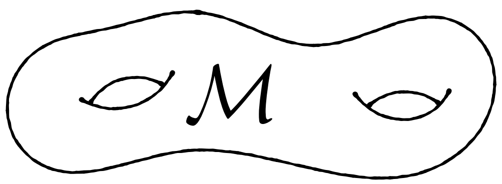
$\Longrightarrow$

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

is a Courant algebroid

[Roytenberg -99]

# Courant Algebroid




What we're used to:

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vector $\quad v$

Riemannian metric $g$

Lie bracket

Generalised geometry:

natural metric $\eta$

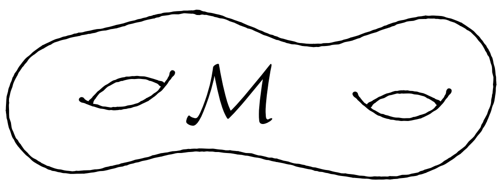
Courant bracket

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

$$V = (v,\, \lambda)$$
$$W = (w,\, \mu)$$

# A generalised Lie derivative




What we're used to:

$$T\mathcal{M}$$

vector $v$

Riemannian metric $g$ — natural metric $\eta$

Lie bracket — Courant bracket

Lie derivative — Dorfmann derivative

Generalised geometry:

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

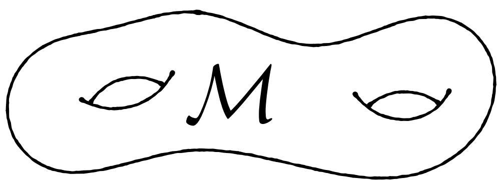
generalised vector

$$V = (v,\ \lambda)$$
$$W = (w,\ \mu)$$

$$L_V W := [v, w] + \mathcal{L}_v \mu - i_w (d\lambda)$$

Satisfies Leibniz rule

# A generalised Lie derivative




What we're used to: Riemannian metric $g$ Lie bracket Lie derivative

$T\mathcal{M}$

vector $v$

Generalised geometry: natural metric $\eta$ Courant bracket Dorfmann derivative

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

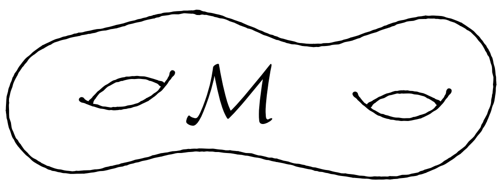
$$V = (v,\ \lambda)$$
$$W = (w,\ \mu)$$

$$L_V W := [v, w] + \mathcal{L}_v \mu - i_w\left(d\lambda\right)$$

$$L_{e^B V} e^B W = e^B \left(L_v W\right) - i_v i_w \left(dB\right)$$

$B$-transforms is a symmetry of $\boldsymbol{L}$ iff $B$ is *closed*

# A generalised Lie derivative




| | | | |
|---|---|---|---|
| What we're used to: | Riemannian metric $g$ | Lie bracket | Lie derivative |
| $T\mathcal{M}$ | | | |
| vector $v$ | | | |
| Generalised geometry: | natural metric $\eta$ | Courant bracket | Dorfmann derivative |

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

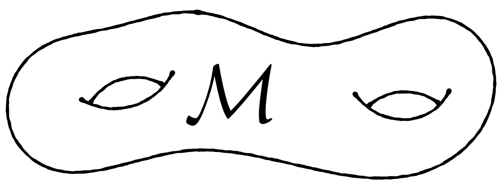
$$V = (v,\, \lambda)$$
$$W = (w,\, \mu)$$

$$L_V W := [v, w] + \mathcal{L}_v \mu - i_w\,(d\lambda)$$

This can be thought of as generating the symmetries of the Courant algebroid.

$$\Omega^2(\mathcal{M})_{cl} \rtimes \mathrm{Diff}(\mathcal{M})$$

# A generalised Lie derivative




What we're used to:

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vector $v$

Riemannian metric $g$

Lie bracket

Lie derivative

natural metric $\eta$

Courant bracket

Dorfmann derivative

Generalised geometry:

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generalised vector

$$V = (v,\ \lambda)$$
$$W = (w,\ \mu)$$

$$L_V W := [v, w] + \mathcal{L}_v \mu - i_w\,(d\lambda)$$

The antisymmetrisation of the Dorfmann derivative is the Courant bracket.

# Patching structure

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

natural metric $\eta$

Courant bracket

Dorfmann derivative

Consider patches of $\mathcal{M}$ with a courant algeborid structure.

Stitch them together using $B$-transforms.

Symmetries:

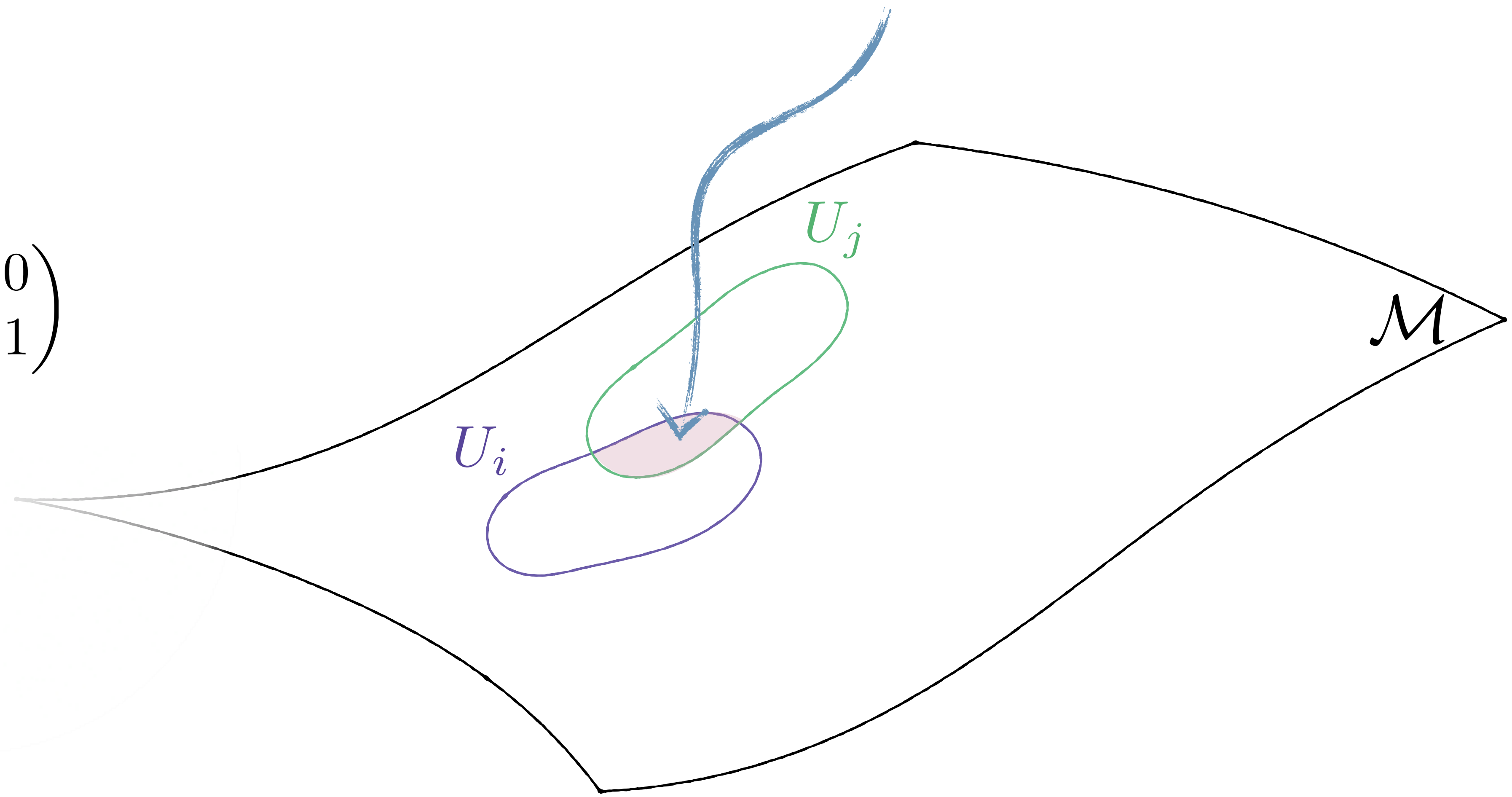
$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \qquad e^{-B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

GL(d)-action

$B$-transform

$B \in \Lambda^2 T^* M$

$$\Omega^2(\mathcal{M})_{cl} \rtimes \mathrm{Diff}(\mathcal{M})$$



# Patching structure

$$T\mathcal{M} \oplus T^*\mathcal{M}$$

natural metric $\eta$
Courant bracket
Dorfmann derivative

Stitch them together using $B$-transforms.
Cocycle condition: $B_{(ij)} + B_{(jk)} + B_{(ki)} = 0$

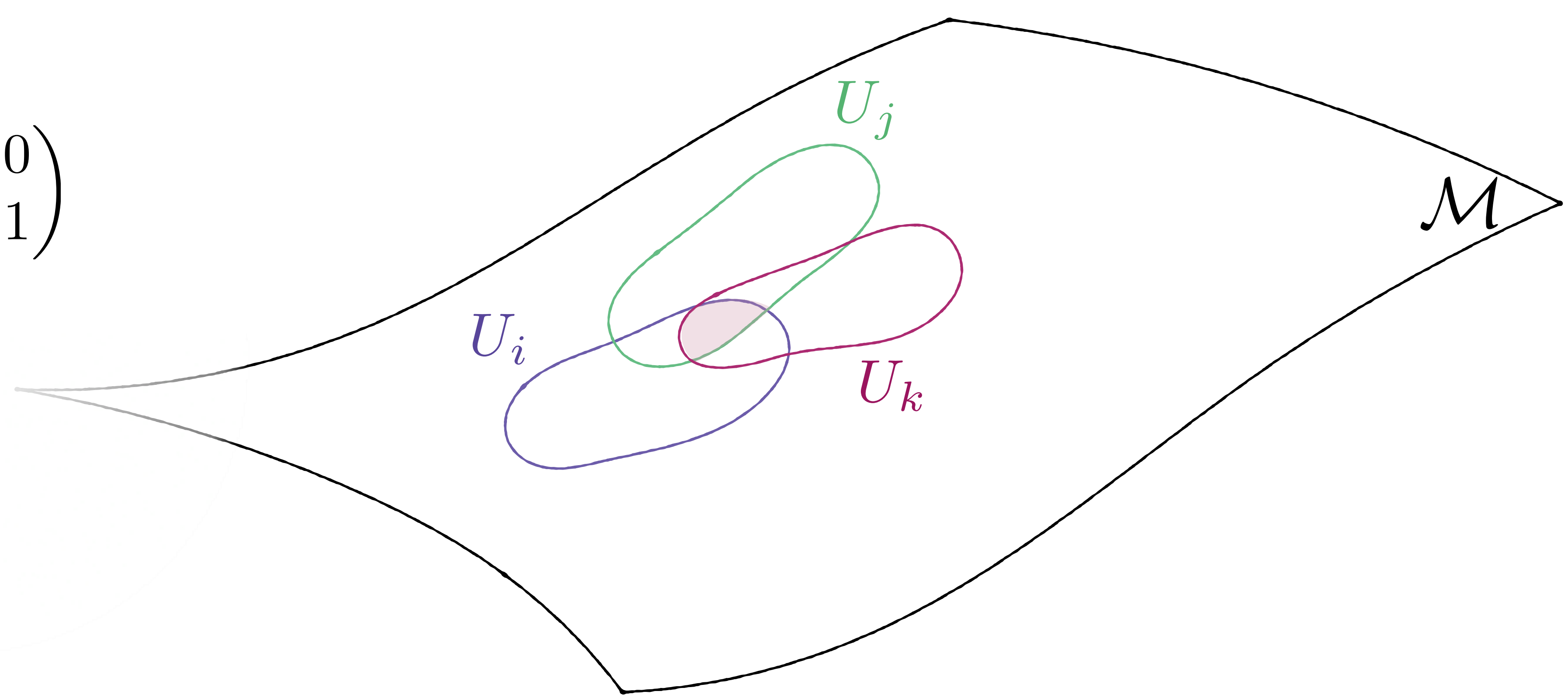
In this way, we create a new bundle,
$$E \sim T\mathcal{M} \oplus T^*\mathcal{M}$$

Symmetries:

$$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \qquad e^{-B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$

GL(d)-action

$B$-transform
$B \in \Lambda^2 T^* M$




$$\Omega^2(\mathcal{M})_{cl} \rtimes \mathrm{Diff}(\mathcal{M})$$

# The generalised tangent bundle

natural metric $\eta$

Courant bracket

Dorfmann derivative

In this way, we create a new bundle,

$$E \sim T\mathcal{M} \oplus T^*\mathcal{M}$$

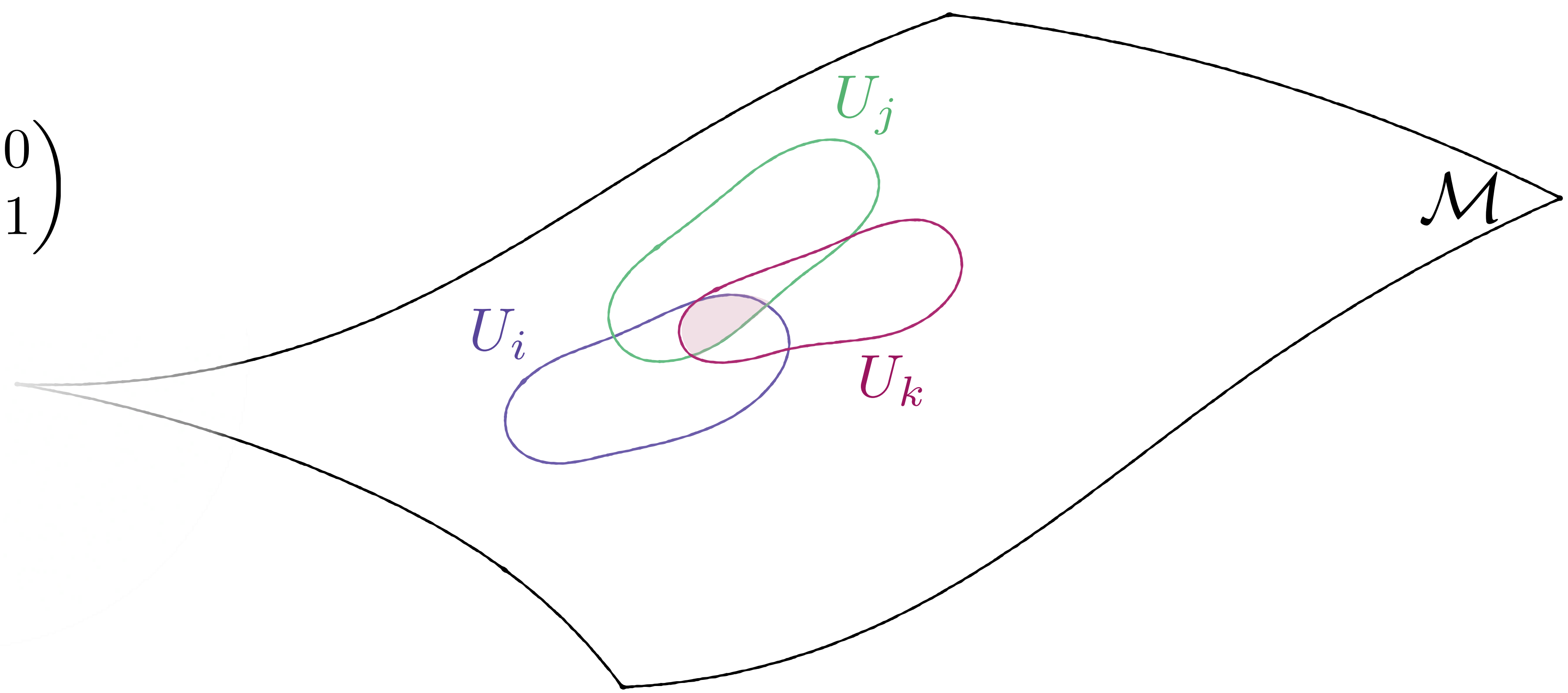
$$0 \longrightarrow T^*\mathcal{M} \xrightarrow{\iota} \boxed{E} \xrightarrow{\pi} T\mathcal{M} \longrightarrow 0$$

Symmetries:

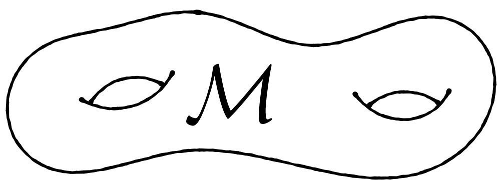
$\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$ GL(d)-action

$e^{-B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$ $B$-transform $B \in \Lambda^2 T^* M$

$$\Omega^2(\mathcal{M})_{cl} \rtimes \mathrm{Diff}(\mathcal{M})$$



# The generalised tangent bundle




What we're used to:

$$T\mathcal{M}$$

vector $\quad v$

| What we're used to | Generalised geometry |
| --- | --- |
| Riemannian metric $g$ | natural metric $\eta$ |
| Lie bracket | Courant bracket |
| Lie derivative | Dorfmann derivative |

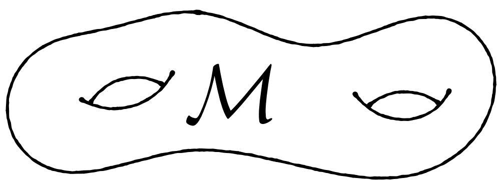
Generalised geometry:

$$E \sim T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

$$V = (v,\, \lambda)$$
$$W = (w,\, \mu)$$

# The generalised metric

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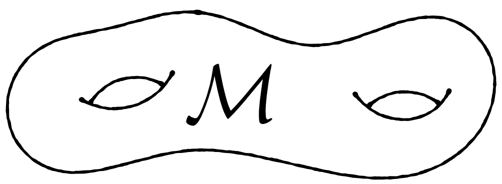


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$T\mathcal{M}$

vector $v$

Riemannian metric $g$

Generalised geometry:

$E \sim T\mathcal{M} \oplus T^*\mathcal{M}$

generalised vector

$V = (v,\, \lambda)$

$W = (w,\, \mu)$

natural metric $\eta$

maximally indefinite

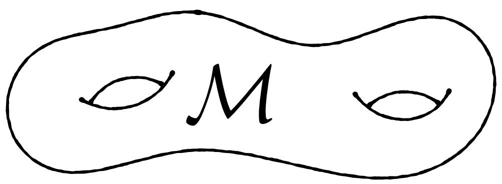
generalised metric $\mathcal{G}$

$$\eta\Big(\,(v, gv)\,,(v, gv)\,\Big) = \quad g(v,v)$$

$$\eta\Big(\,(v, -gv)\,,(v, -gv)\,\Big) = -\,g(v,v)$$

$$\Longrightarrow \quad O(d,d) \to O(d) \times O(d)$$

# The generalised metric




What we're used to:

$$T\mathcal{M}$$

vector $\quad v$

Riemannian metric $g$ $\qquad g : T\mathcal{M} \to T^*\mathcal{M}$

$$\eta\Big( (v, gv)\,, (v, gv) \Big) = \quad g(v,v)$$

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generalised vector

$$V = (v,\, \lambda)$$

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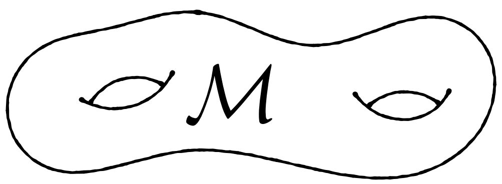
natural metric $\eta$

generalised metric $\quad \mathcal{G}$

Positive definite?

"$\mathcal{G}\,(.,.) = \eta\,(.,.)\,\big|_{C+} - \eta\,(.,.)\,\big|_{C-}$"

# The generalised metric




What we're used to:

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Riemannian metric $\; g$

$$g : T\mathcal{M} \to T^*\mathcal{M}$$

Generalised geometry:

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generalised vector

$$V = (v,\, \lambda)$$

$$W = (w,\, \mu)$$

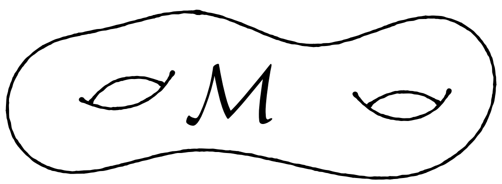
natural metric $\; \eta$

generalised metric $\; \mathcal{G}$

Positive definite?

$$2\mathcal{G} = \left(e^{-B}\right)^T \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} e^{-B}$$

# Generalised Curvature




What we're used to:

$$T\mathcal{M}$$

vector $v$

Riemannian metric $g$

Ricci tensor/scalar $\mathcal{R}_{ab}/\mathcal{R}$

Generalised geometry:

$$E \sim T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised vector

$$V = (v,\, \lambda)$$

$$W = (w,\, \mu)$$

generalised metric $\mathcal{G}$

Gen. Ricci tensor $R_{ab}$ / $R$

$$R_{ab} = \mathcal{R}_{ab} - \frac{1}{4} H_{abc} H_b{}^{cd} + \frac{1}{2} \nabla^c H_{abc}$$

$$R = \mathcal{R} - \frac{1}{12} H^2 \qquad H = dB$$

# Supergravity and Generalised Geometry

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Generalised geometry:

$$E \sim T\mathcal{M} \oplus T^*\mathcal{M}$$

generalised metric $\mathcal{G}$

Symmetries:

$\Omega^2(\mathcal{M})_{cl} \rtimes \mathrm{Diff}(\mathcal{M})$

$$R = \mathcal{R} - \frac{1}{12} H^2$$

$$H = dB$$

Recall: NS-NS sector of type II supergravity:

$$(g_{\mu\nu}, \cancel{\phi,} B_{\mu\nu})$$

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} \left[ e^{-2\phi} \left( \mathcal{R} + 4 (\partial\phi)^2 - \frac{1}{12} H^2 \right) \right]$$

Bosonic Symmetries:

$\Omega^2(\mathcal{M})_{cl} \rtimes \mathrm{Diff}(\mathcal{M})$

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[Coimbra, Strickland-Constable, Waldram -11]

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Consider the slight generalisation $\tilde{E} = \det\left(T^*\mathcal{M}\right) \otimes E$

$\Longrightarrow$ $O(d,d) \times \mathbb{R}^+$ structure

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modifies Vol(***G***) and ***R*** to account for the dilation

# Supergravity and Generalised Geometry

Supergravity $\xrightarrow[\text{geometry}]{\text{Generalised}}$ Gen. Einstein-Hilbert

[Coimbra, Strickland-Constable, Waldram -11,-12]

This seems to be the right language to use for studying supergravity…

So what about consistent truncations?

# Generalised parallelisable spaces

Recall: Consistent truncations are known to exist on
local group manifolds + exceptional examples.

D=10,11

…but it also works
for $\mathcal{M} = S^d,\ H^d, \ldots$

local group
manifold,
ex $\mathcal{M} = S^1,\ S^3$

$\mathcal{M}$

Why?

d=D-k

# Reduction on a local group manifold

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$\mathcal{M}$

d=D-k

Left-invariant vector fields gives a globally defined frame satisfying $[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c$

$g$ Cartan-Killing metric

All dependance on internal coordinates encoded in the left-invariant vielbeins

[Scherk, Schwarz -79]

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$\Longrightarrow$ dependence on these factor out of action and EOM's

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(constants)

dependence on these factor out of action and EOM's

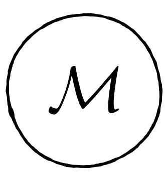
$\Longrightarrow$ Consistent truncation

[Scherk, Schwarz -79]

# The generalised analog

The important properties:

D=10,11




d=D-k

existence of a global frame $\{\hat{e}_a\}$

Lie algebra relation:

$$[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c$$

(constants)

existence of a global generalised frame $\{\hat{E}_A\}$

Leibniz algebra relation:

$$L_{\hat{E}_A} \hat{E}_B = F_{AB}{}^C \hat{E}_C$$

(constants)

D=10,11


d=D-k

[Lee, Strickland-Constable, Walrdam -14]

# Generalised parallelisable spaces

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D=10,11

local group manifold,
ex $\mathcal{M} = S^1,\ S^3$

$\mathcal{M}$

d=D-k

…but it also works for $\mathcal{M} = S^d,\ H^d, \ldots$

Why?

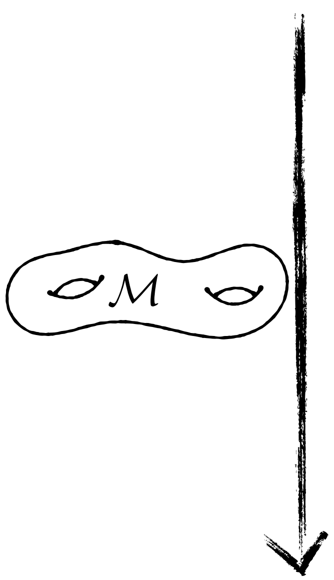
**Turns out they are all *generalised* (Liebniz) parallelisable**

[Lee, Strickland-Constable, Walrdam -14] [Hohm, Samtleben -15]
[Baguet, Pope, Samtleben -16] [Inverso, Samtleben, Trigiante -16]

# Generalised parallelisable spaces

Towards a classification of consistent truncations? [Inverso -17]

D=10,11

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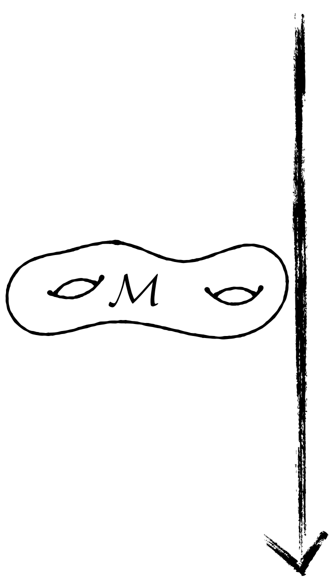
# New examples via adjoint orbits

Here: new examples of gen. parallelisable spaces

D=10,11




d=D-k

Clue: An Inönü-Wigner contraction

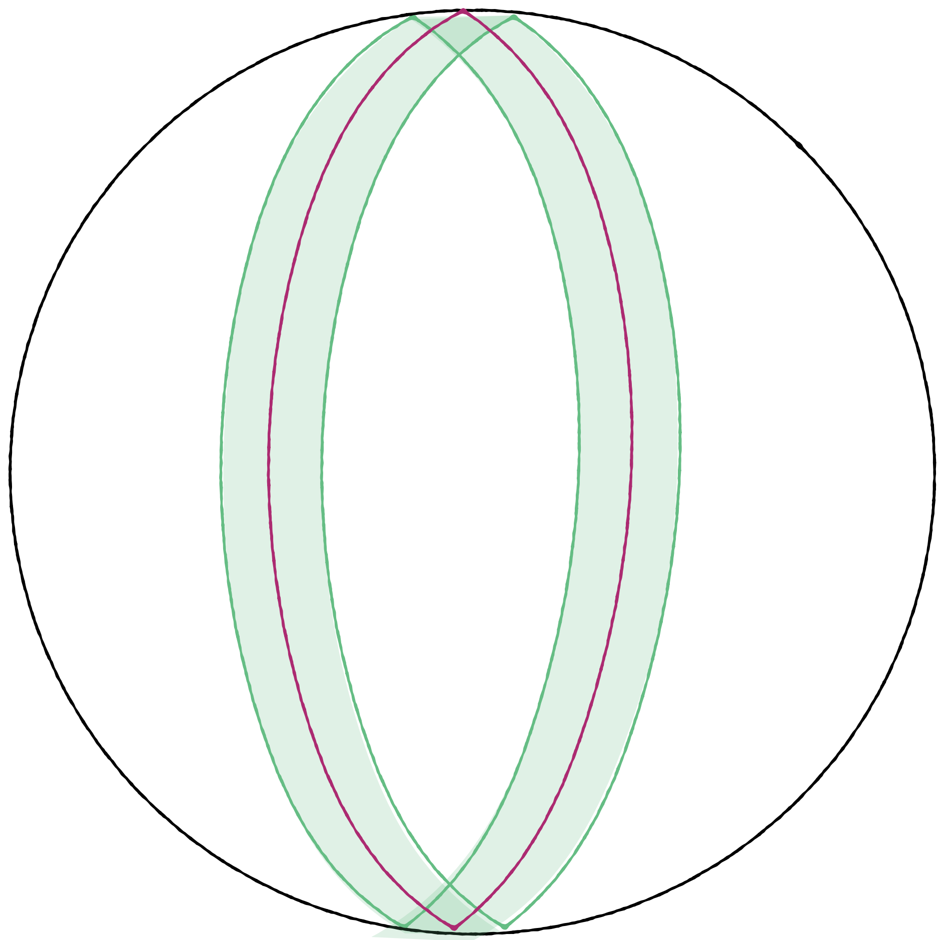
$$S^3 \longrightarrow S^2 \times \mathbb{R}$$

$$\frac{SO(4)}{SO(3)} \to \frac{SO(3) \ltimes \mathbb{R}^3}{SO(2) \ltimes \mathbb{R}^2}$$

“Equatorial belt approximation”

[deFelice -14] [Cavallari -16][Anderson -17]

# New examples via adjoint orbits



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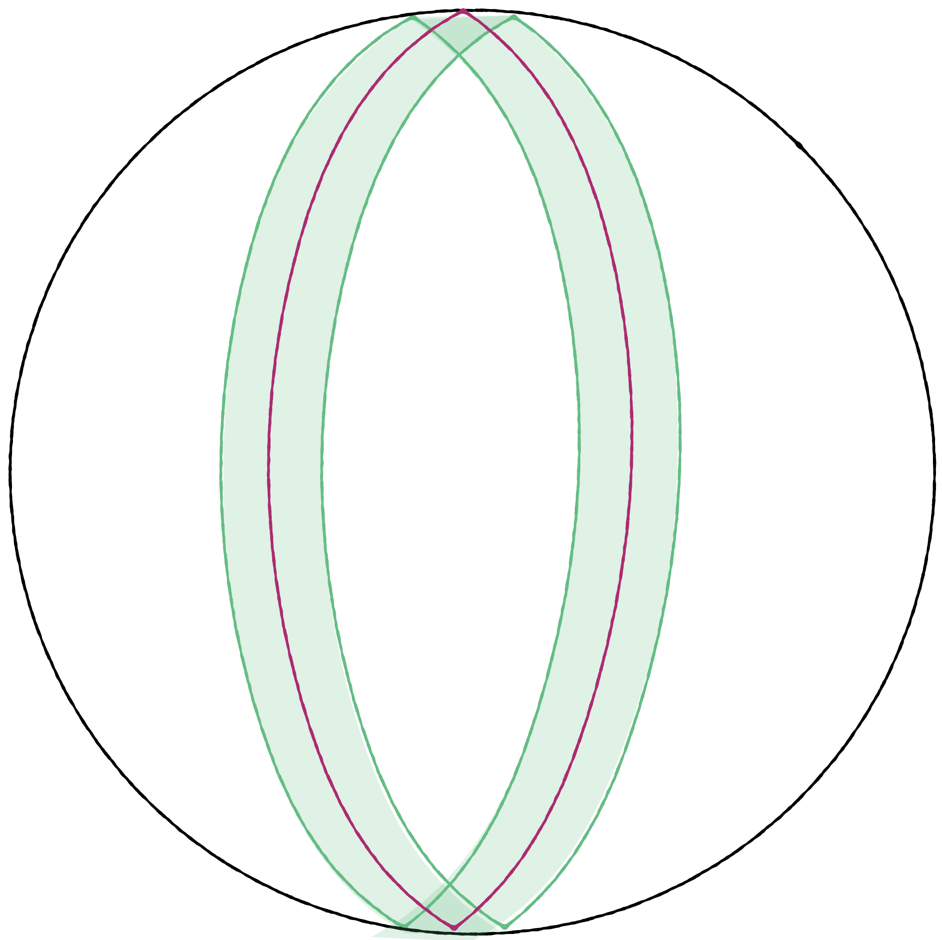
“Equatorial belt approximation”

$$S^3 \hookrightarrow \mathbb{R}^4 \qquad y_4 \to \epsilon y_4$$

$$\sum_a y_a y_a = \sum_{i \neq 4} y_i y_i + \epsilon^2 y_4 y_4$$

# New examples via adjoint orbits



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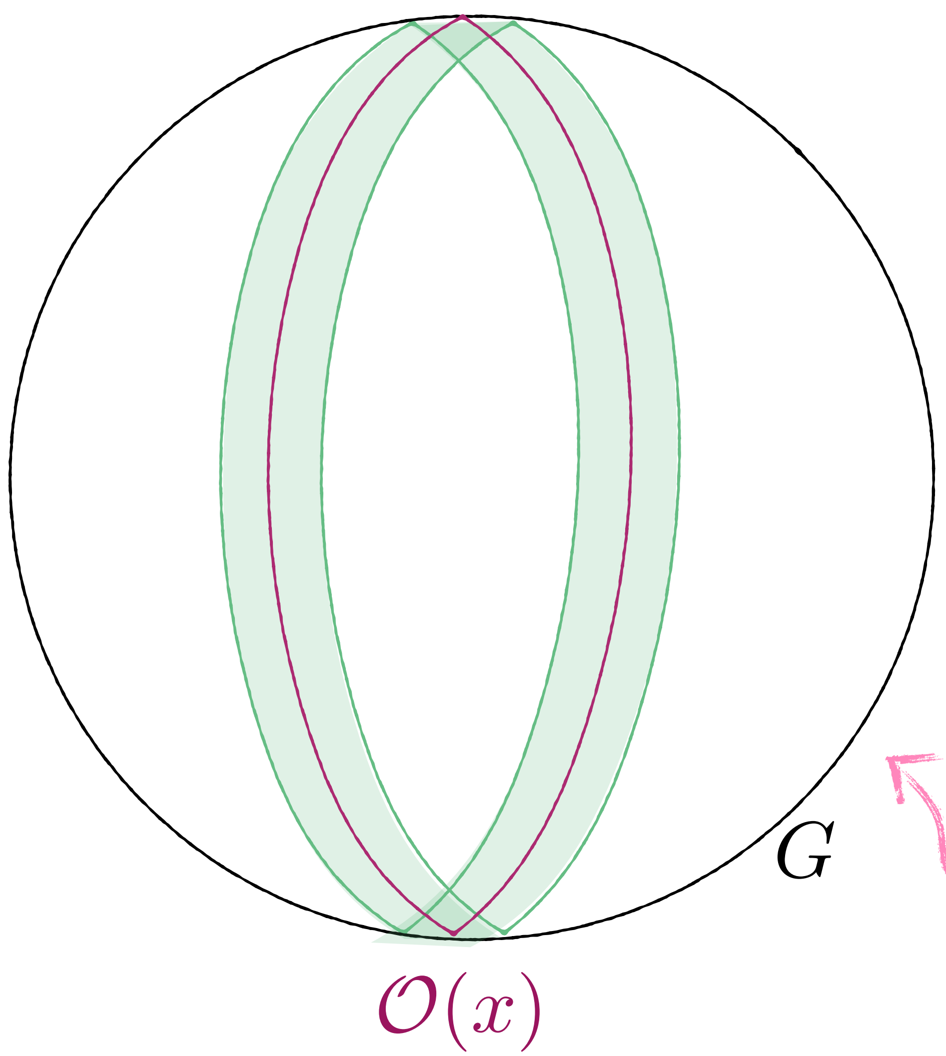
# The orbit picture

Which spaces can we get this way?




Given: $G$ $H \subset G$

If $H = \mathrm{Stab}(x)$

$$\Longrightarrow \quad \mathcal{O}(x) \simeq G/H$$

Close to $\mathcal{O}(x)$, the space looks like $\mathcal{O}(x) \times$ normal directions

Known: gen. parallelisation on $G$

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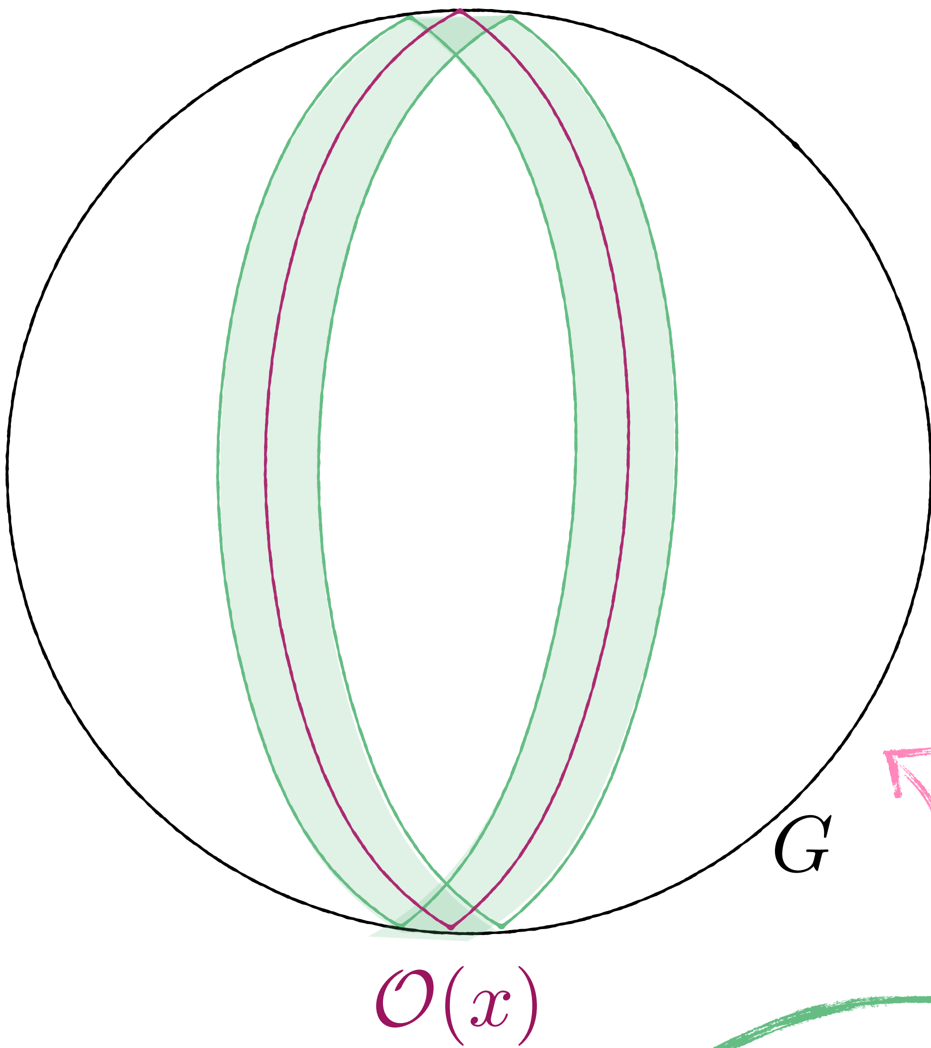


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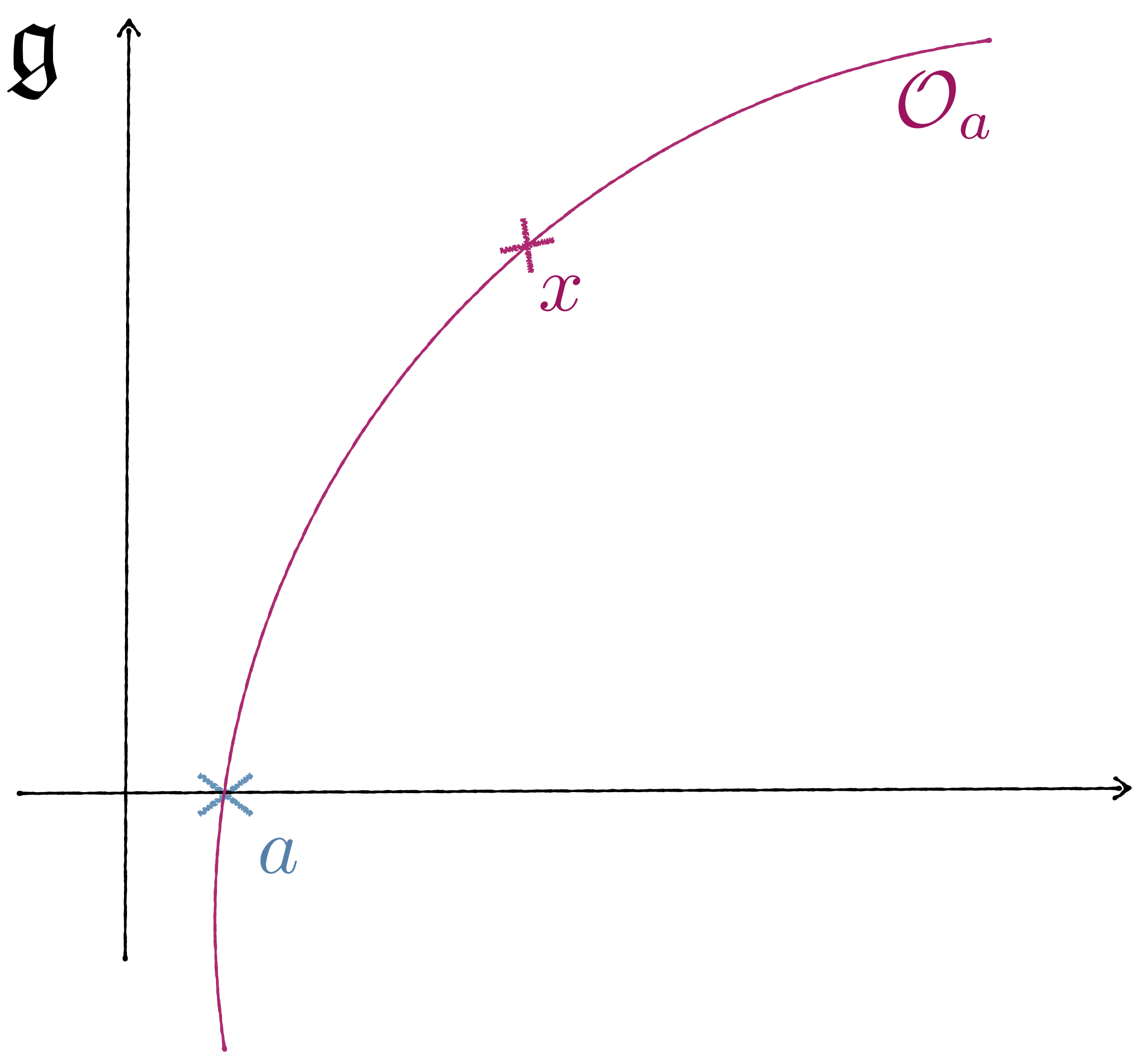
…what about here?

# Adjoint orbits in $\mathfrak{g}$




Consider orbits in $\mathrm{Lie}(G) = \mathfrak{g}$ under the adjoint action of $G$

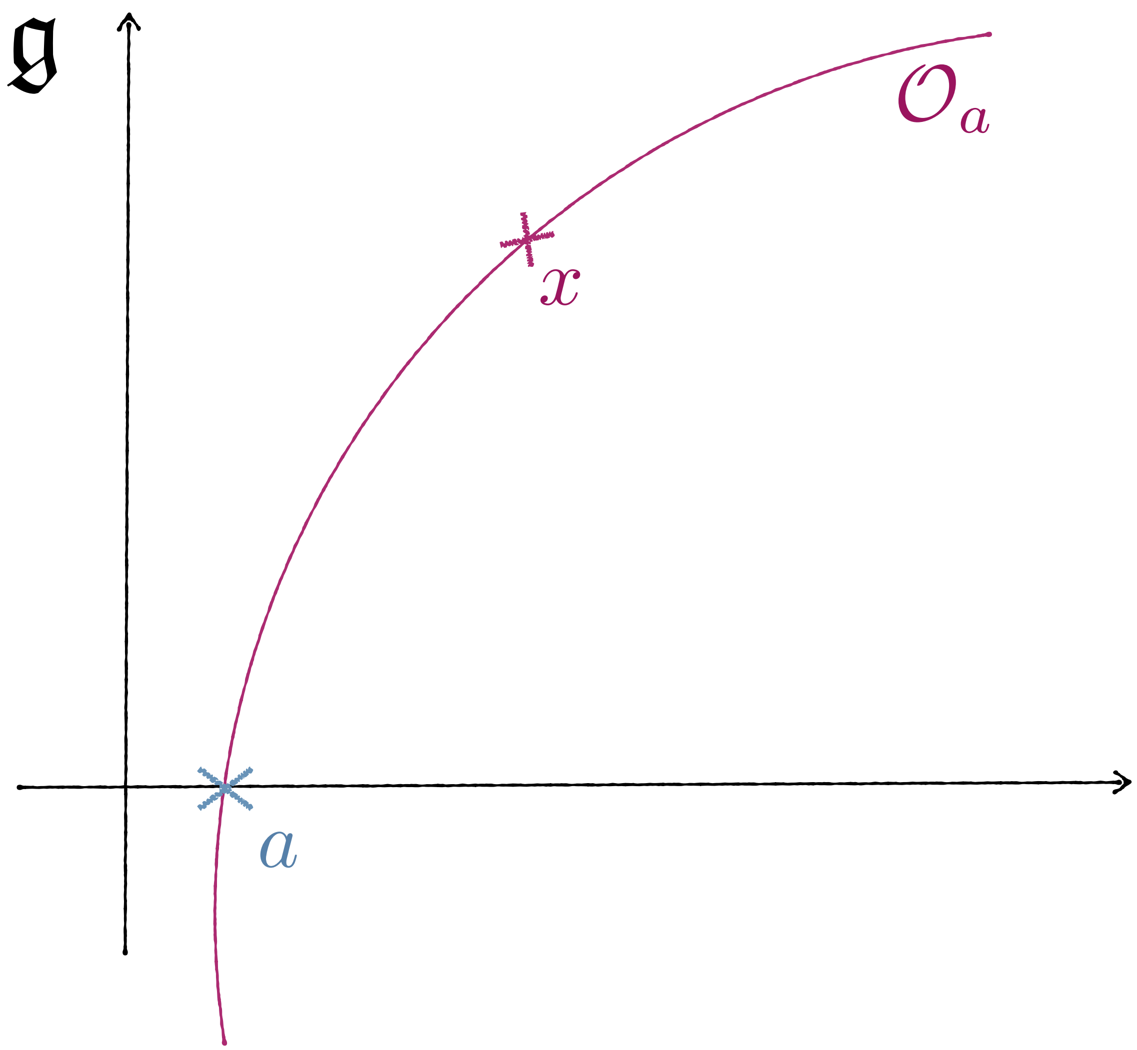
Every such orbit can be parametrised by $a \in$ (closure of) fund. Weyl chamber

$a \in$ interior of fund. Weyl chamber

$\implies$ $\mathcal{O}_a$ is regular.

Else, it is degenerate.

# Adjoint orbits in $\mathfrak{g}$



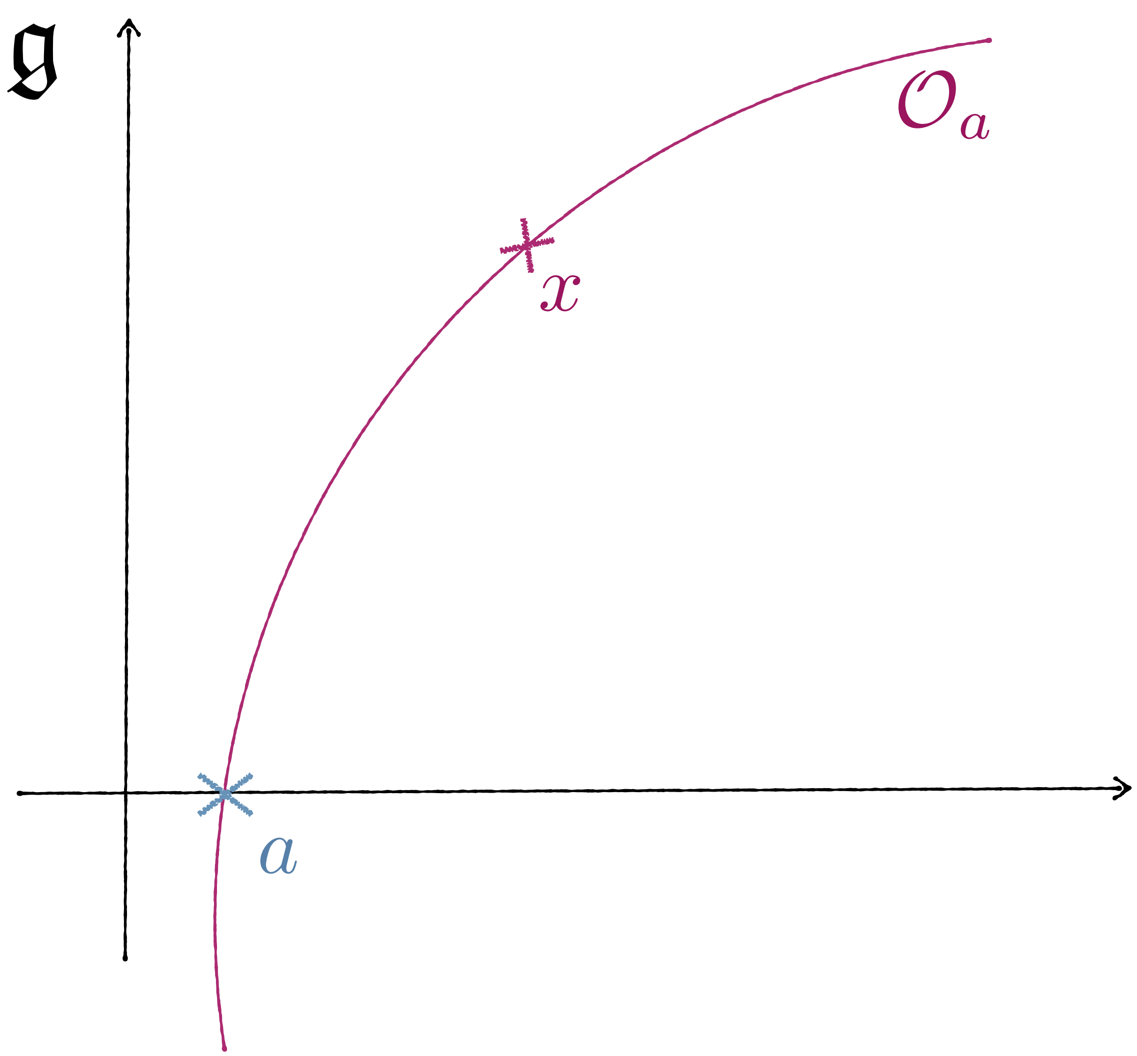

Consider orbits in $\mathrm{Lie}(G) = \mathfrak{g}$
under the adjoint action of $G$

$$H = \mathrm{Stab}(a) \Rightarrow \mathcal{O}_a \simeq G/H$$

As usual, we have:

$$T_e H = \mathrm{ann}(a) = \{\xi \in \mathfrak{g},\ [\xi, a] = 0\} \equiv \mathfrak{h}$$

# Adjoint orbits in $\mathfrak{g}$




Consider orbits in $\mathrm{Lie}(G) = \mathfrak{g}$ under the adjoint action of $G$

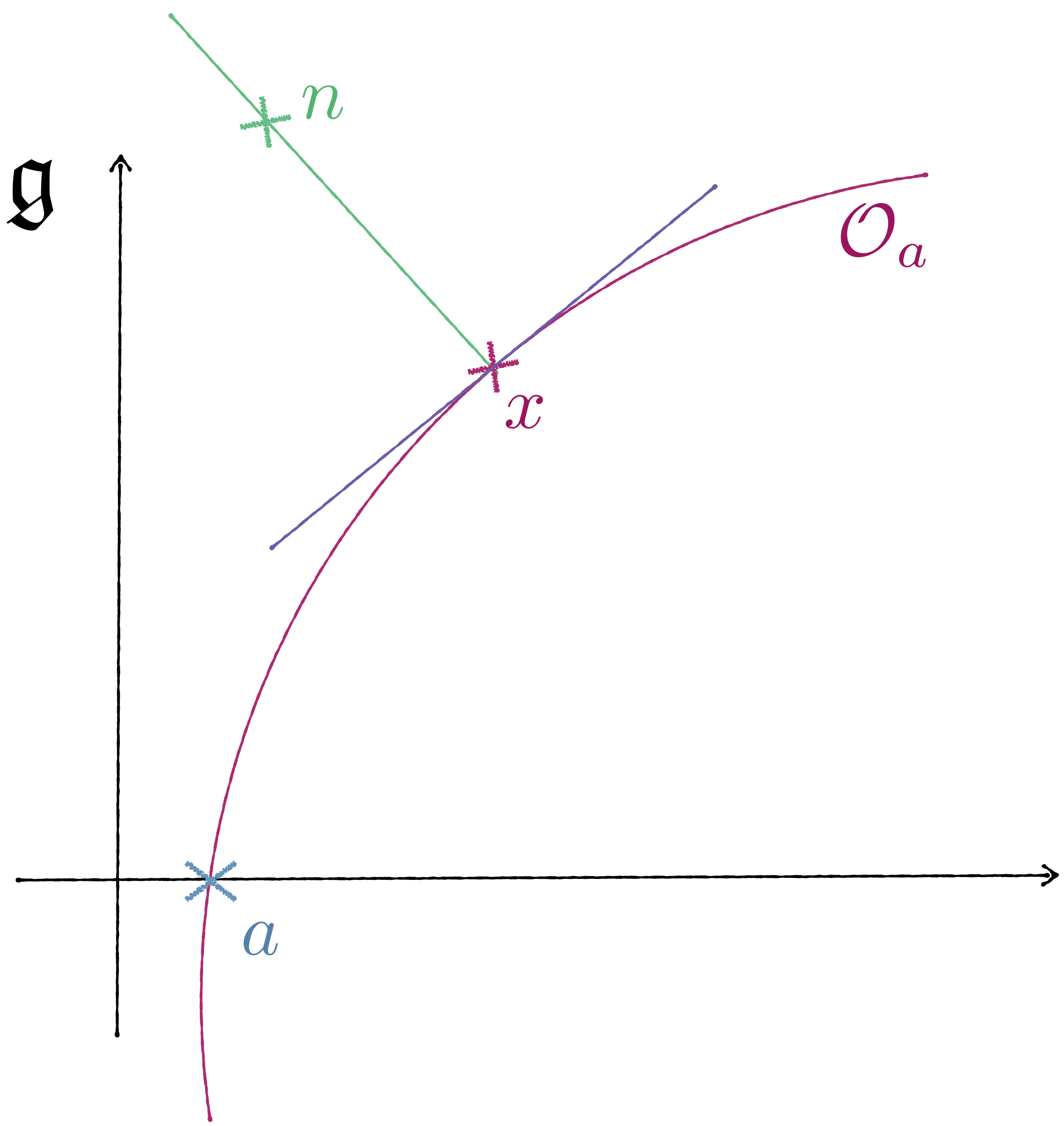
- $\mathrm{Ad}_G$-invariant inner product on $\mathfrak{g}$
- Natural embedding $\mathcal{O}_a \hookrightarrow \mathfrak{g}$

$\Longrightarrow$

There is an orthogonal splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{v}$

$\Longrightarrow$ $T_{\pi(e)}\mathcal{O}(a) \sim \mathfrak{v}$

# Adjoint orbits in $\mathfrak{g}$




Consider now the normal bundle

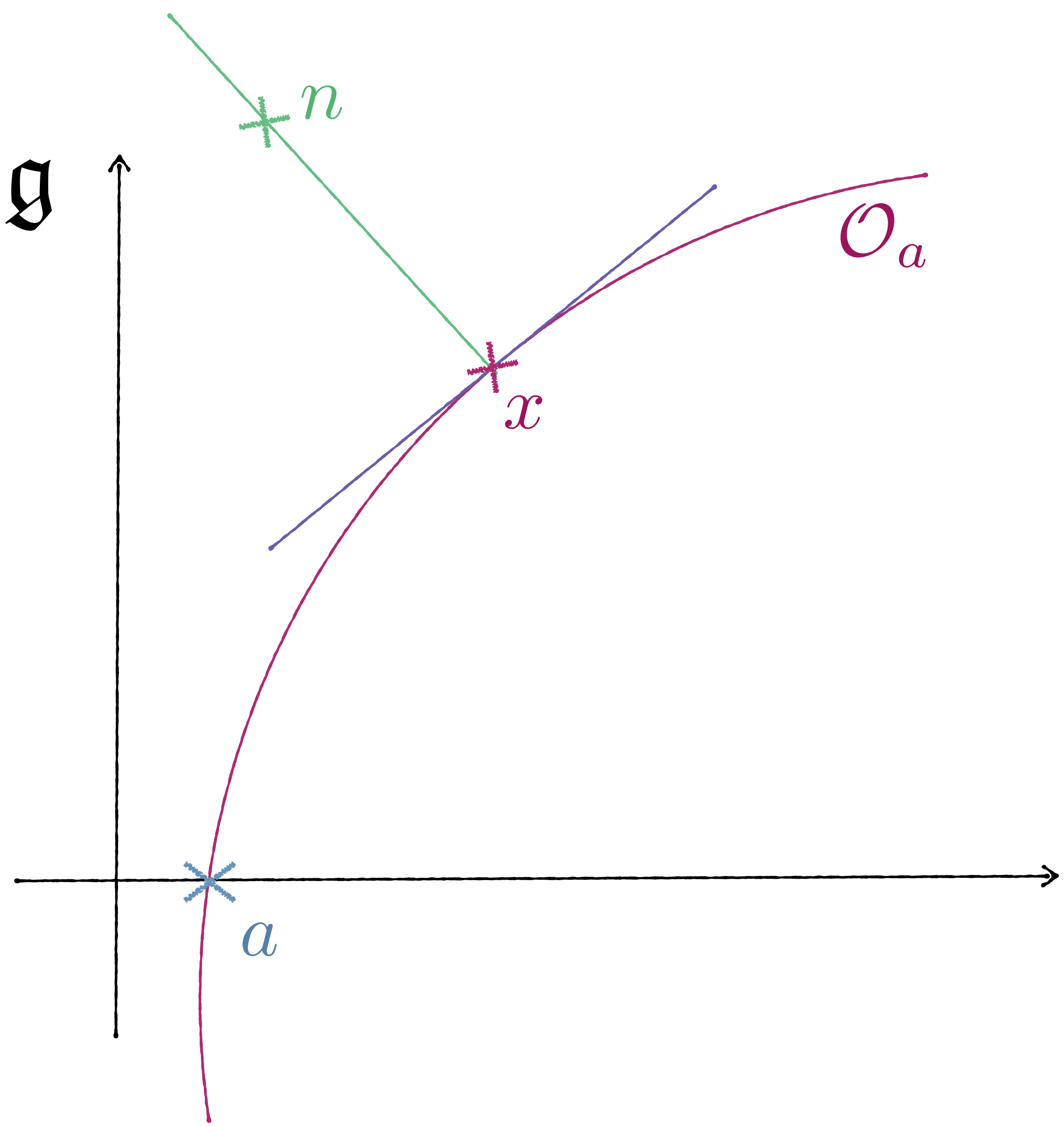
$$N\mathcal{O}_a \hookrightarrow \mathfrak{g} \otimes \mathfrak{g}$$

Parametrise this by pairs $(x, n) \in \mathfrak{g} \otimes \mathfrak{g}$

There is an orthogonal splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{v}$

$$\Longrightarrow T_{\pi(e)}\mathcal{O}(a) \sim \mathfrak{v}$$

# Adjoint orbits in $\mathfrak{g}$

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at any point $x$, we have

$$T_x\mathcal{O} \simeq \mathfrak{v} \qquad N_x\mathcal{O} \simeq \mathfrak{h}$$

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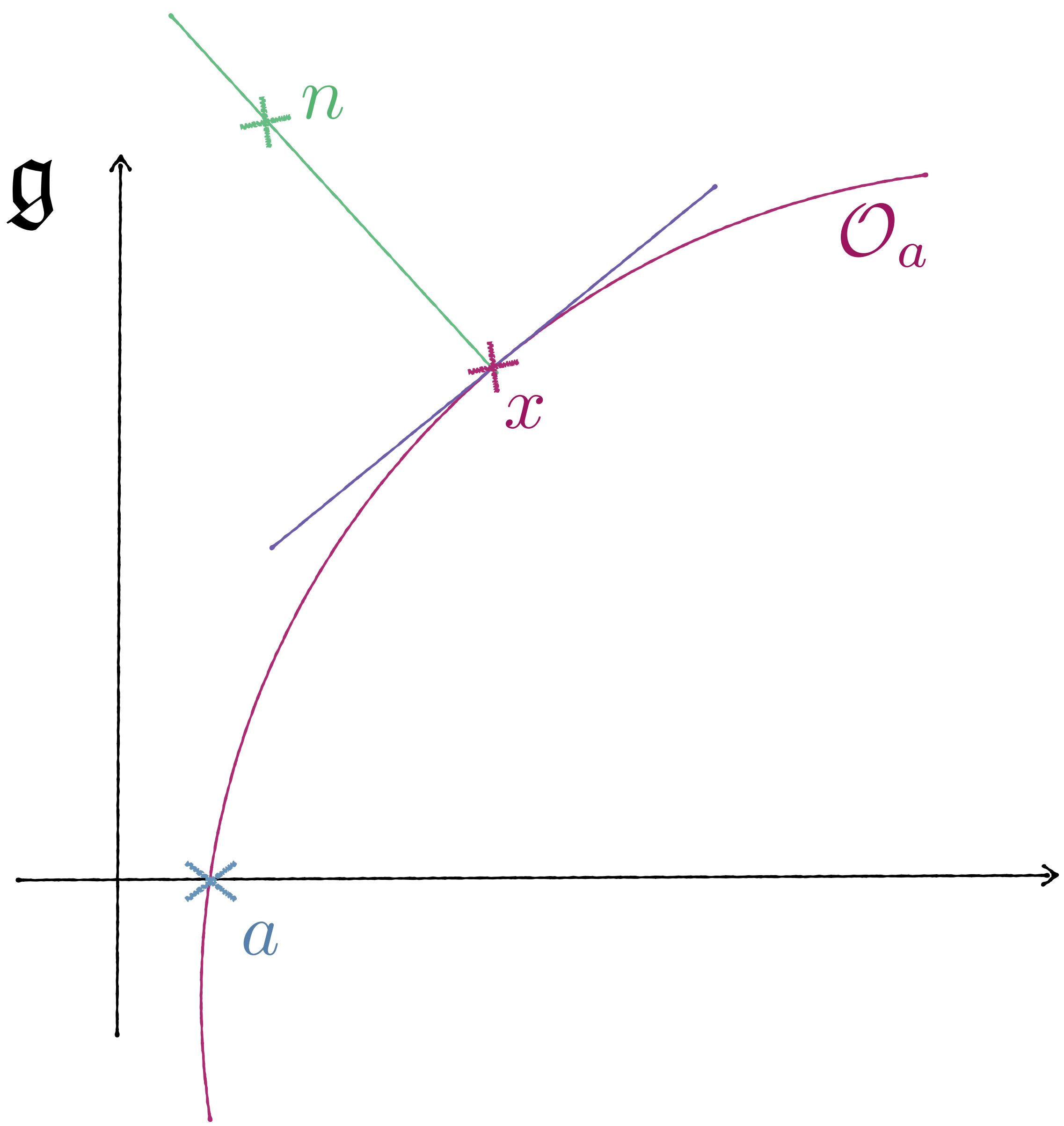
# Adjoint orbits in $\mathfrak{g}$

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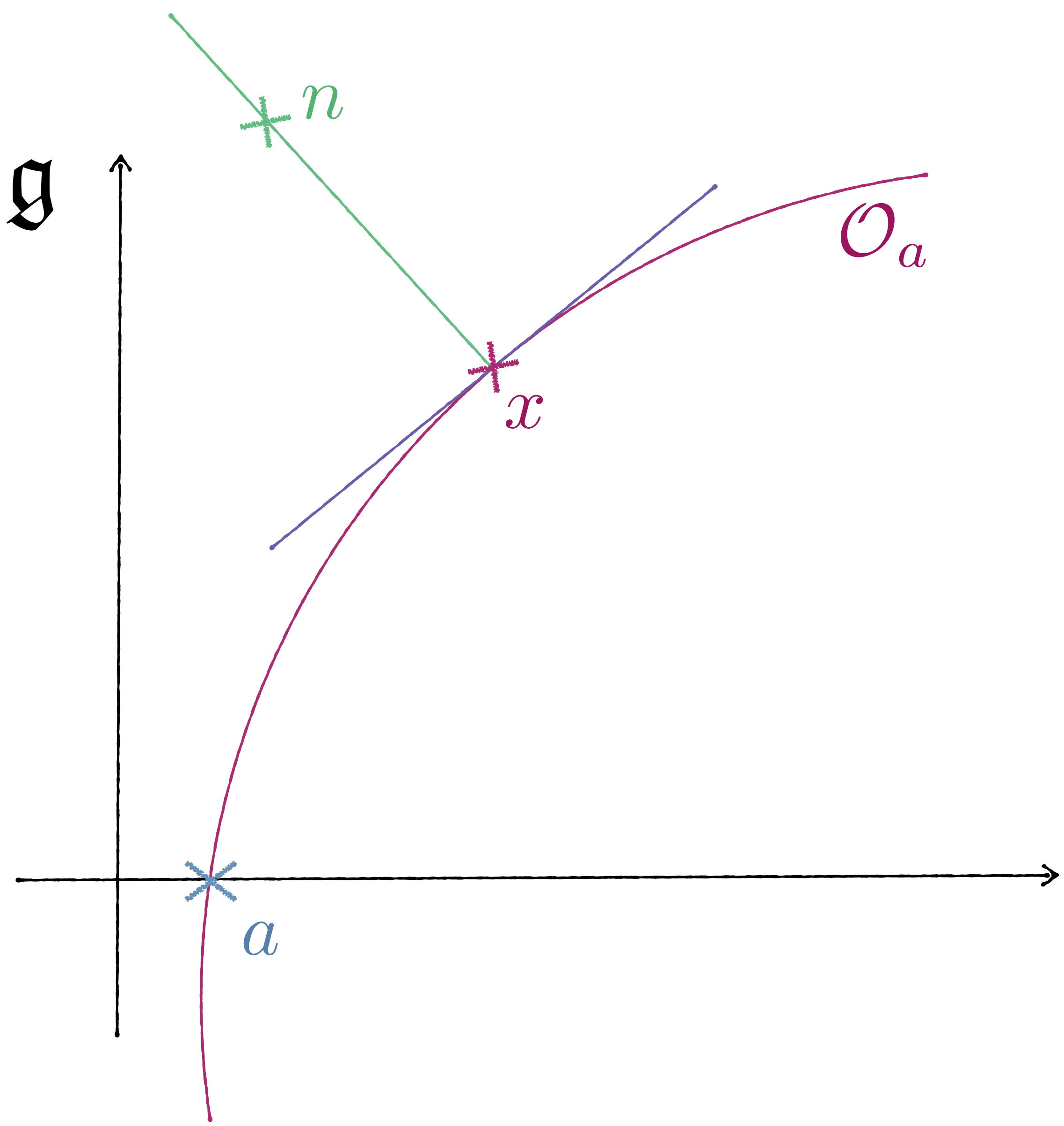
at any point $x$, we have
$$T_x\mathcal{O} \simeq \mathfrak{v} \qquad N_x\mathcal{O} \simeq \mathfrak{h}$$

Want: 2d vectors satisfying $\mathrm{Lie}\,(G \ltimes \mathfrak{g})$
(In analogy with the $S^2 \times \mathbb{R}$-case)

Is there an action of $G \ltimes \mathfrak{g}$ on $N_x\mathcal{O}$ ?

# Adjoint orbits in $\mathfrak{g}$




An action of $G \ltimes \mathfrak{g}$ on $N_x\mathcal{O}$

The product:

$$(g',\, \eta')(g,\, \eta) = (gg',\, Ad_{g'}\eta + \eta')$$

The Lie bracket:

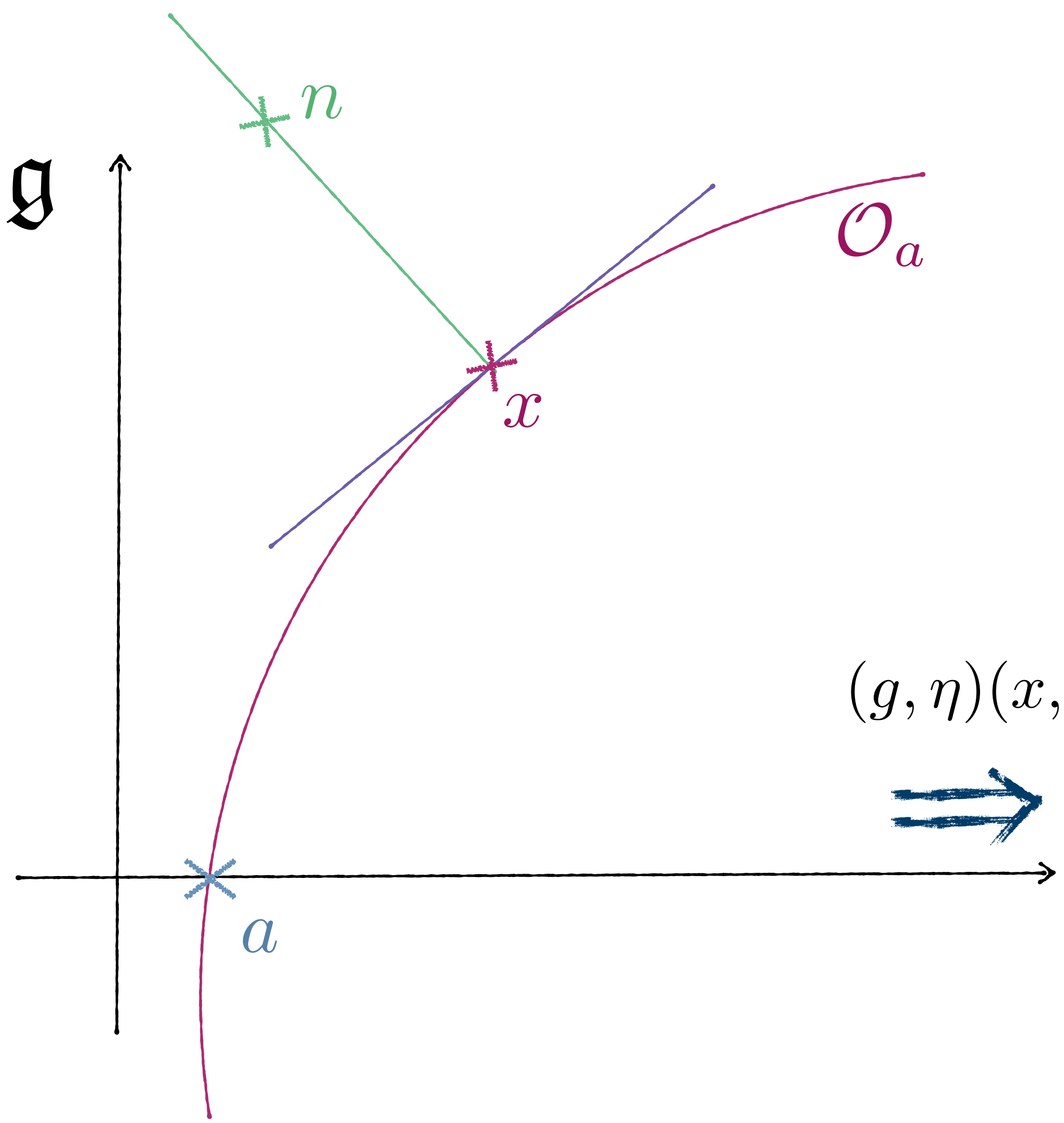
$$[(\gamma_a, \tilde{\gamma}_a), (\gamma_b, \tilde{\gamma}_b)] = ([\gamma_a, \gamma_b],\, [\gamma_a, \tilde{\gamma}_b] - [\gamma_b, \tilde{\gamma}_a])$$

orthogonal proj.

Define the action:

$$(g, \eta)(x, n) = \big(Ad_g x,\, Ad_g n + \Pi_{\mathrm{ann}(Ad_g x)}\eta\big)$$

# Adjoint orbits in $\mathfrak{g}$




An action of $G \ltimes \mathfrak{g}$ on $N_x\mathcal{O}$

Define the action:

$$(g,\eta)(x,n) = \left(Ad_g x,\ Ad_g n + \Pi_{\mathrm{ann}(Ad_g x)}\eta\right)$$

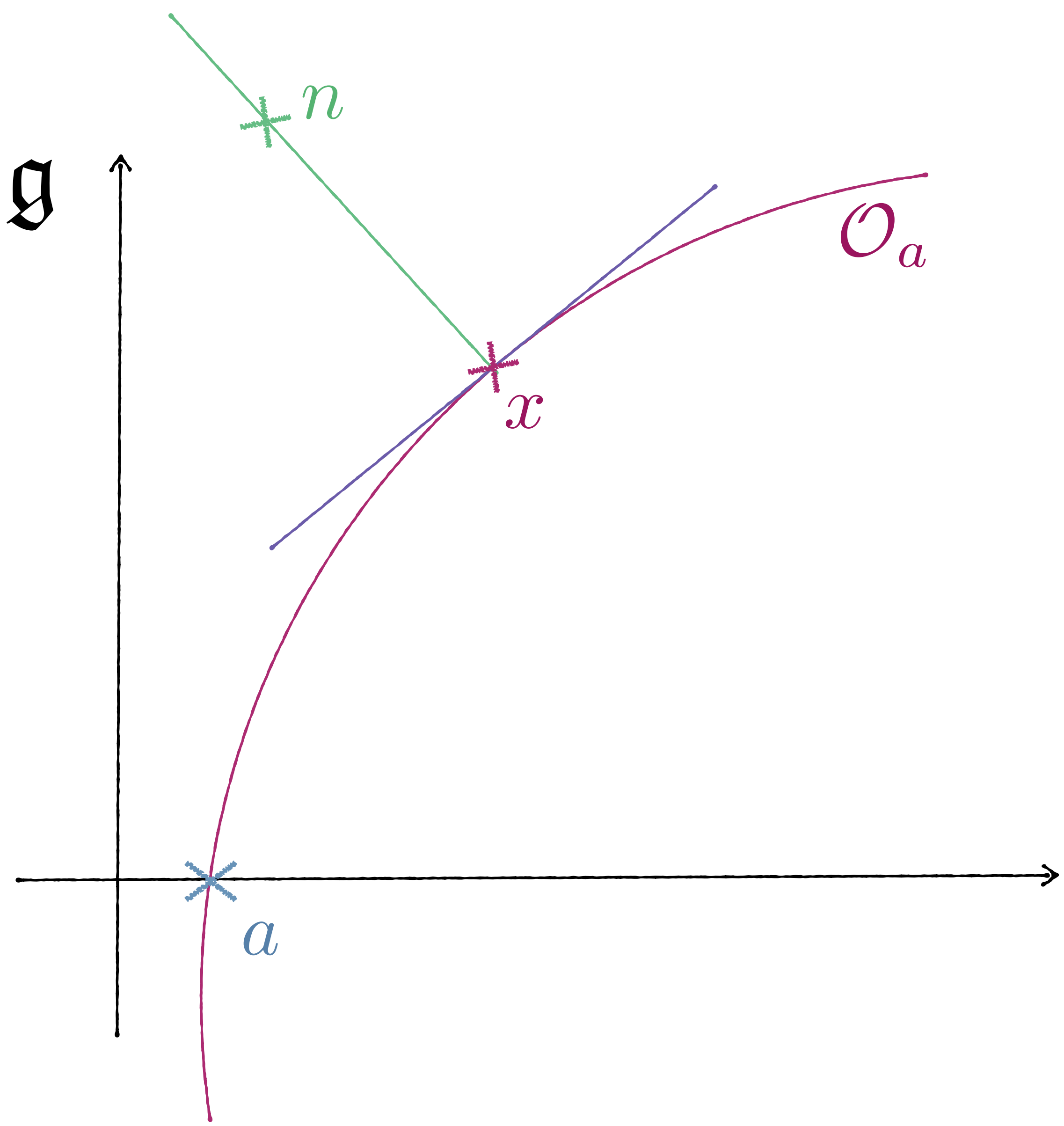
Which are the vectors generated by this action?

take $g = e^{t\gamma}$ , $\eta = t\eta$

$$(g,\eta)(x,n) = (x + t[\gamma,x] + \dots,\ n + t[\gamma,n] + t\Pi_{ann(x)}\eta + \dots)$$

$$\Longrightarrow \quad \frac{d}{dt}\left((g,\eta)(x,n)\right)\Big|_{t=0} = \left([\gamma,x],\ [\gamma,n] + \Pi_{ann(x)}\eta\right)$$

# Adjoint orbits in $\mathfrak{g}$




An action of $G \ltimes \mathfrak{g}$ on $N_x\mathcal{O}$

Define the action:

$$(g,\eta)(x,n) = \left(Ad_g x,\ Ad_g n + \Pi_{\mathrm{ann}(Ad_g x)}\eta\right)$$

Which are the vectors generated by this action?

take $g = e^{t\gamma}$ , $\eta = t\eta$

$$\frac{d}{dt}\left((g,\eta)(x,n)\right)\Big|_{t=0} = \left([\gamma,x],\ [\gamma,n] + \Pi_{ann(x)}\eta\right)$$

Let $\{\sigma_a, \tilde{\sigma}_{\tilde{a}}\}$ be a basis for $\mathrm{Lie}\,(G \ltimes \mathfrak{g})$.

This gives us 2dim$(G)$ vectors via:

$$\hat{e}_{(a,\tilde{a})} = \left([\sigma_a, x]\,, [\sigma_a, n] + \Pi_{\mathrm{ann}(x)}\tilde{\sigma}_{\tilde{a}}\right)$$

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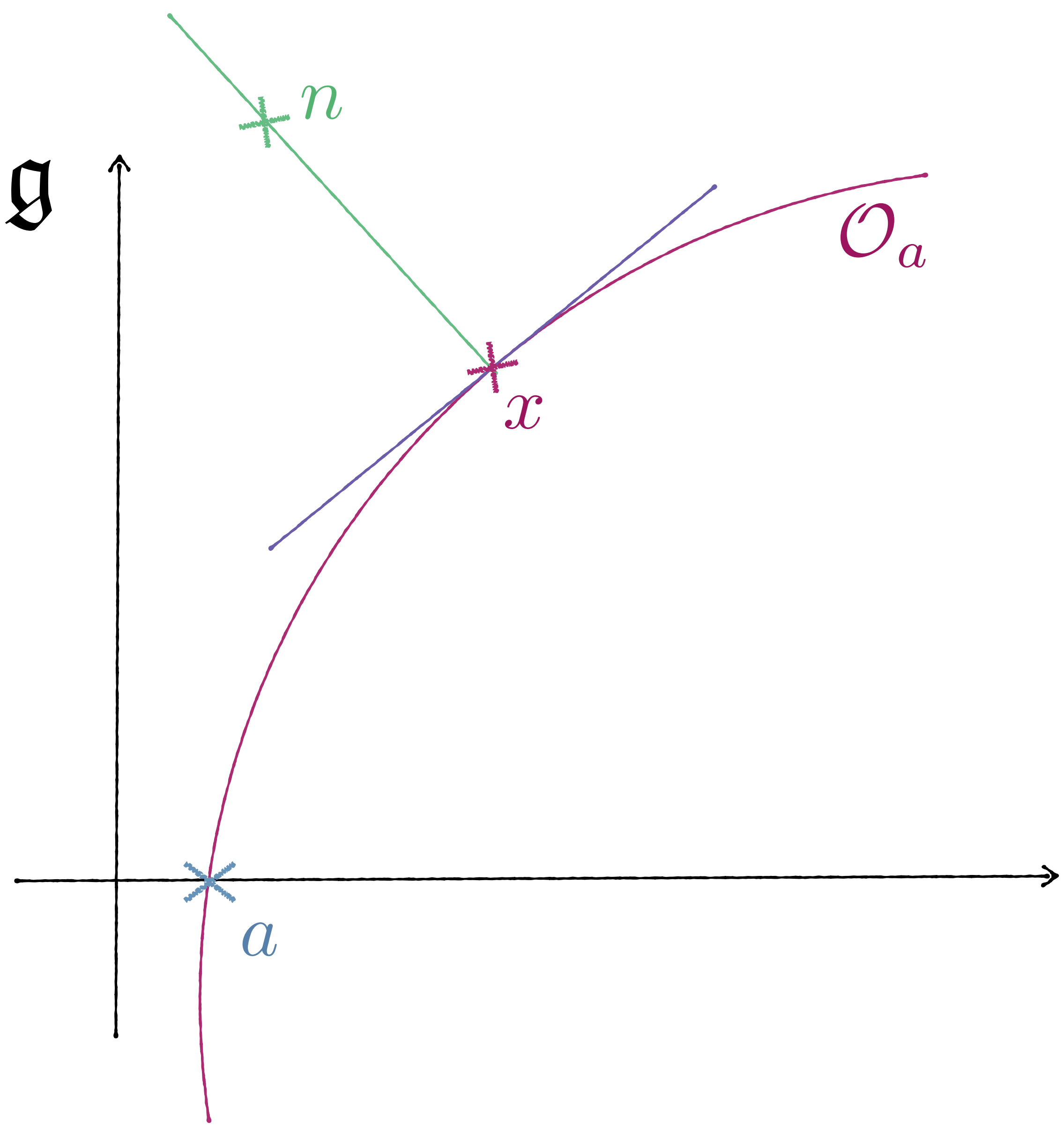


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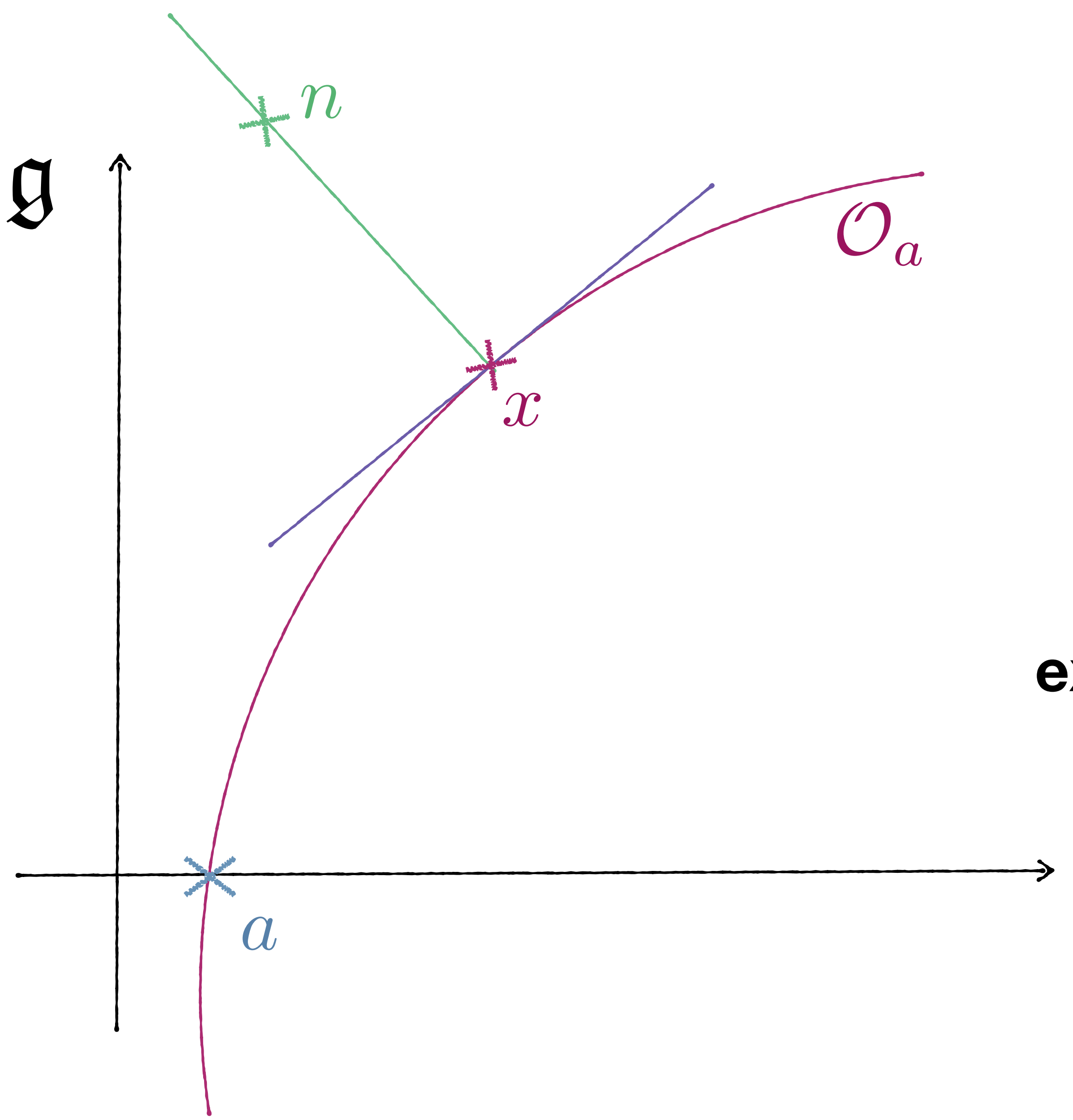
Or more conveniently (after relabelling):

$$\hat{e}^v_a = \hat{e}_{(a,0)} = ([\sigma_a, x], [\sigma_a, n])$$
$$\hat{e}^h_a = \hat{e}_{(0,\tilde{a})} = \left(0, \Pi_{\mathrm{ann}(x)}\sigma_a\right)$$

# Adjoint orbits in $\mathfrak{g}$

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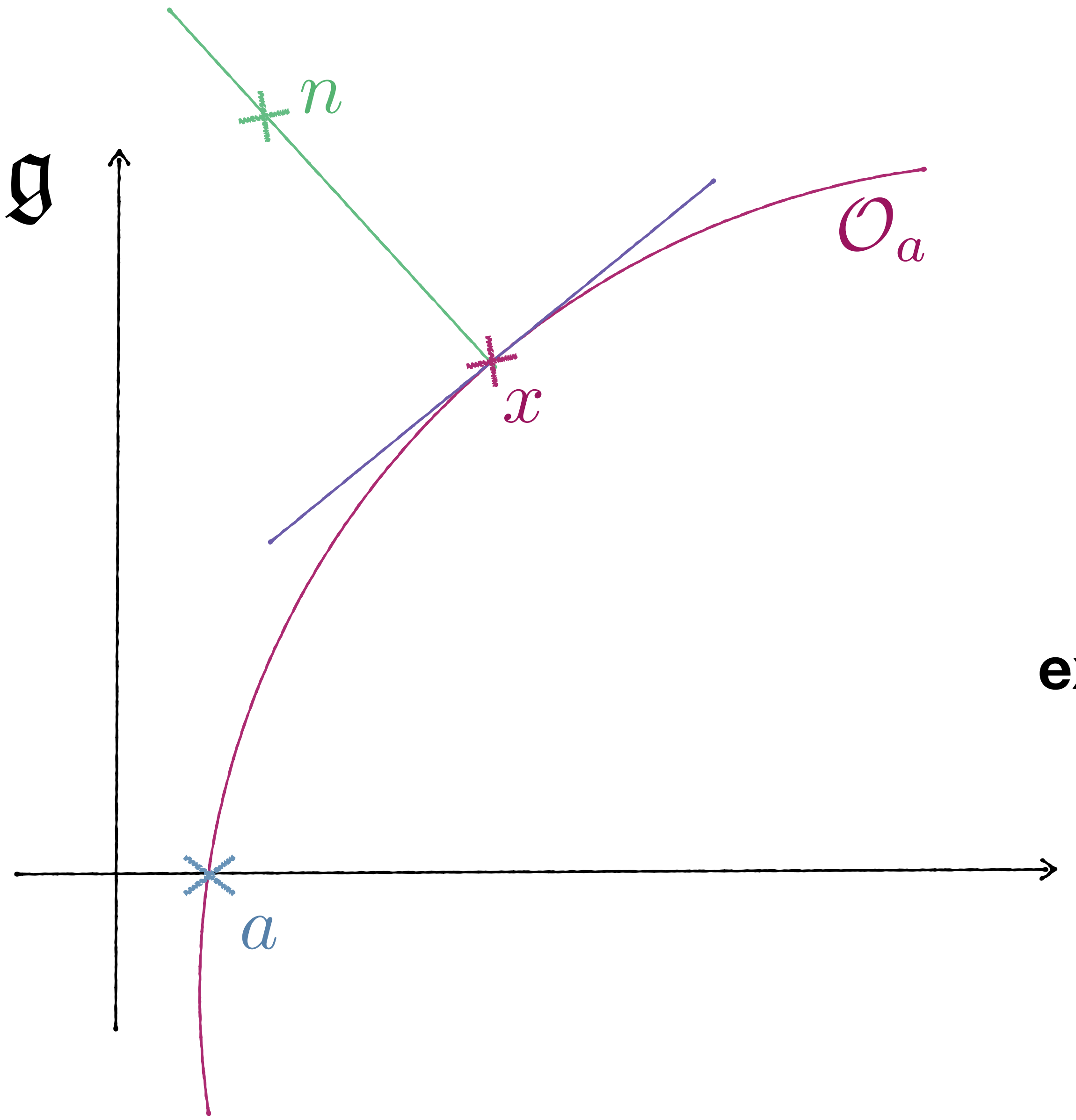
**ex:** $H$ abelian

# Adjoint orbits in $\mathfrak{g}$




An action of $G \ltimes \mathfrak{g}$ on $N_x\mathcal{O}$

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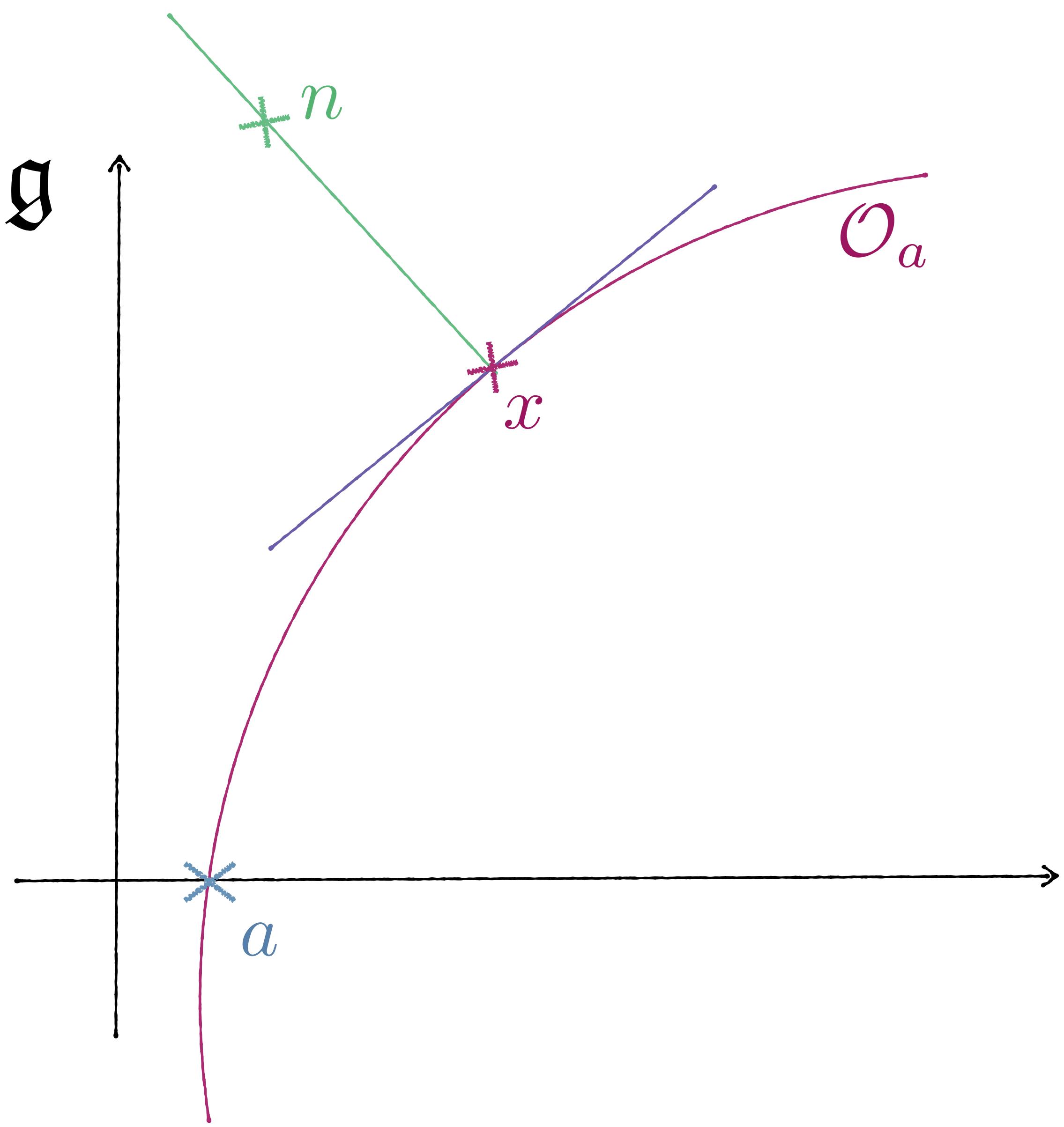
**ex:** $H$ abelian $\Longrightarrow$ $\{\hat{e}^v_a\}$, $\{\hat{e}^h_a\}$

orthogonal in the sense of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$

# Adjoint orbits in $\mathfrak{g}$

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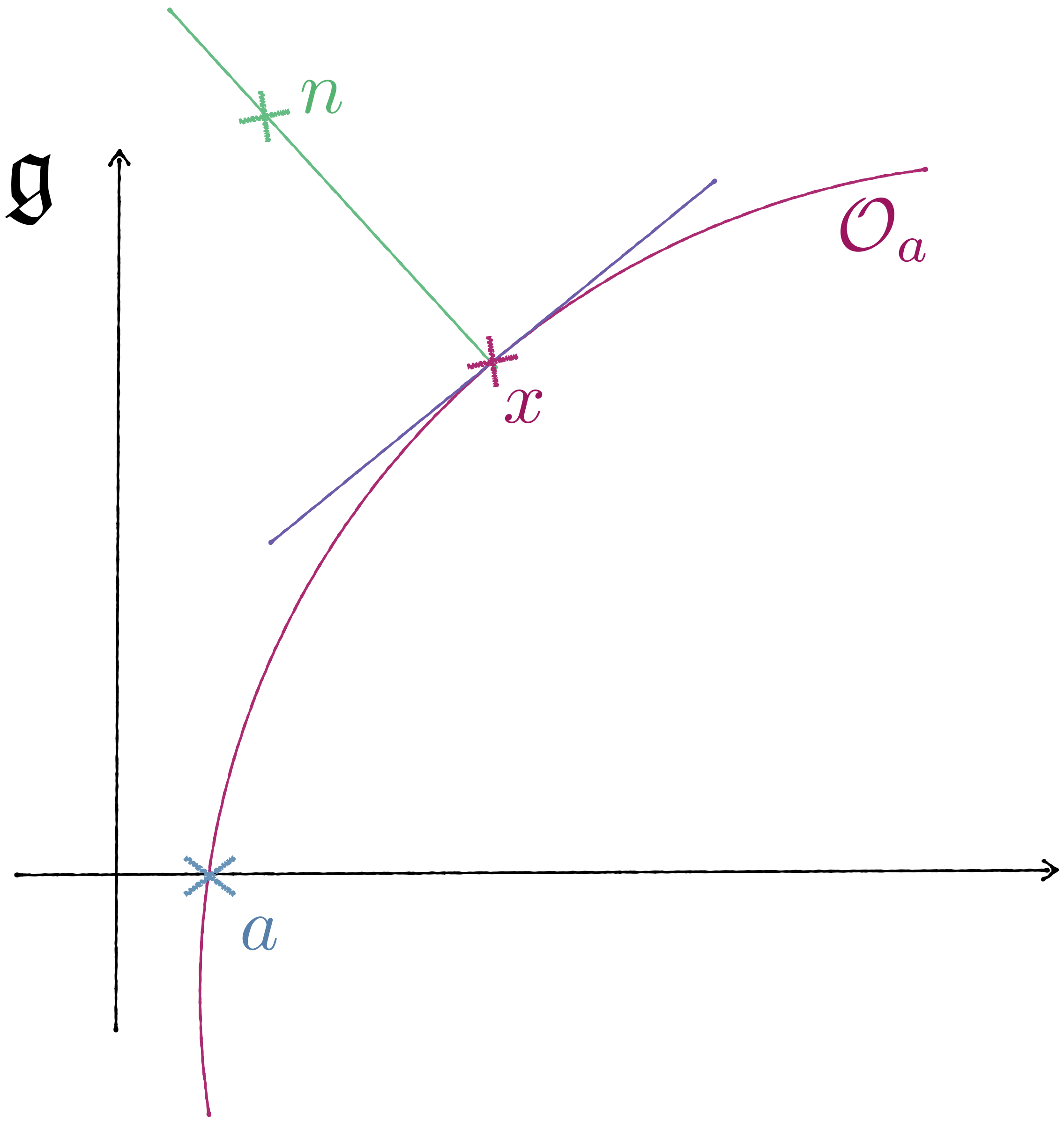
Commutation relations:

$$[\hat{e}_a^v, \hat{e}_b^v] = f_{ab}{}^c \hat{e}_c^v \qquad [\hat{e}_a^v, \hat{e}_b^h] = f_{ab}{}^c \hat{e}_c^h$$

Structure constants of $G$

$$[\hat{e}_a^h, \hat{e}_b^h] = 0$$

# Adjoint orbits in $\mathfrak{g}$

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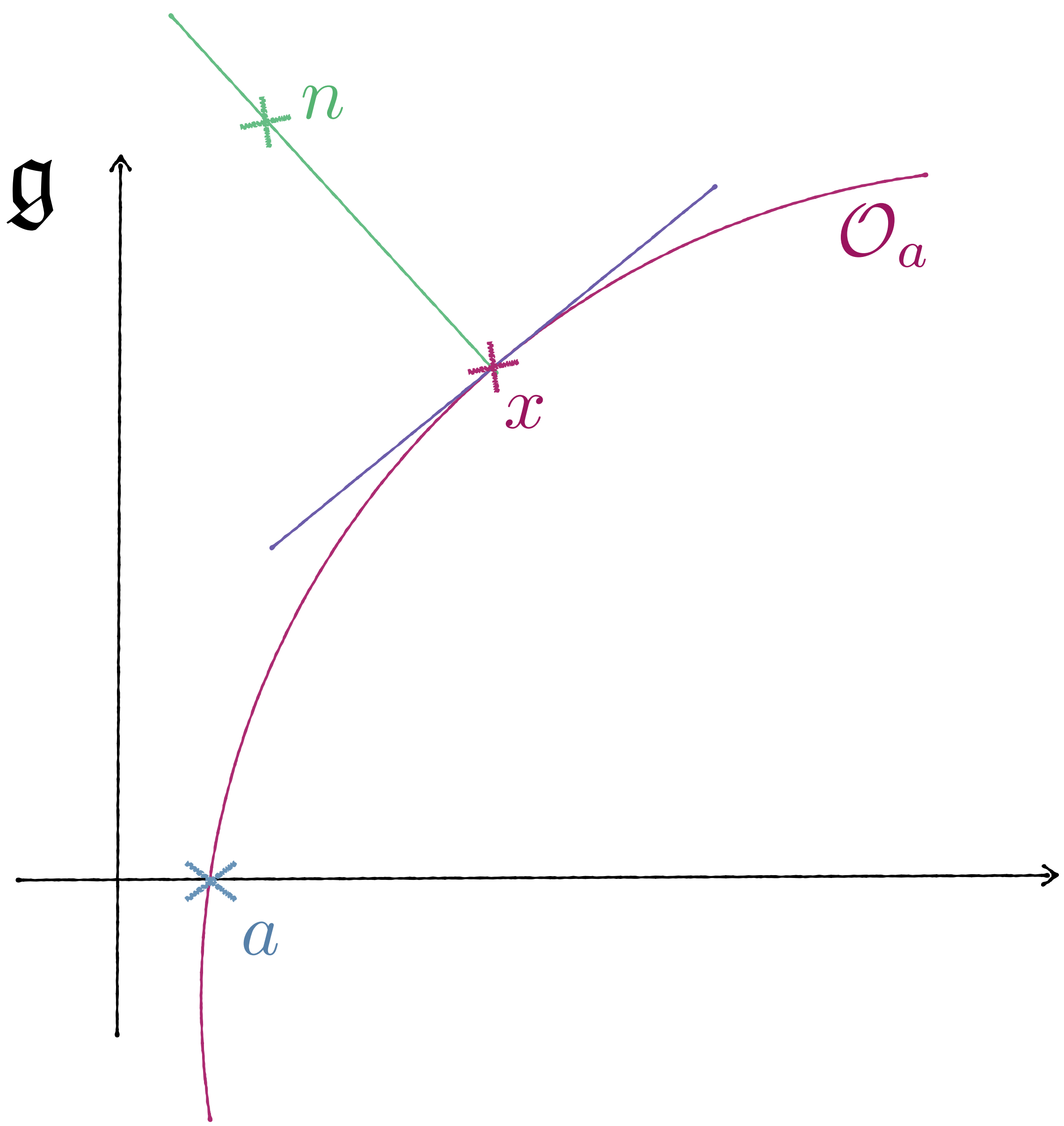
So we have 2dim$(G)$ vectors that satisfies the desired commutation relations…

…problem is they're not globally defined!

# Adjoint orbits in $\mathfrak{g}$

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So we have 2dim$(G)$ vectors that satisfies the desired commutation relations…

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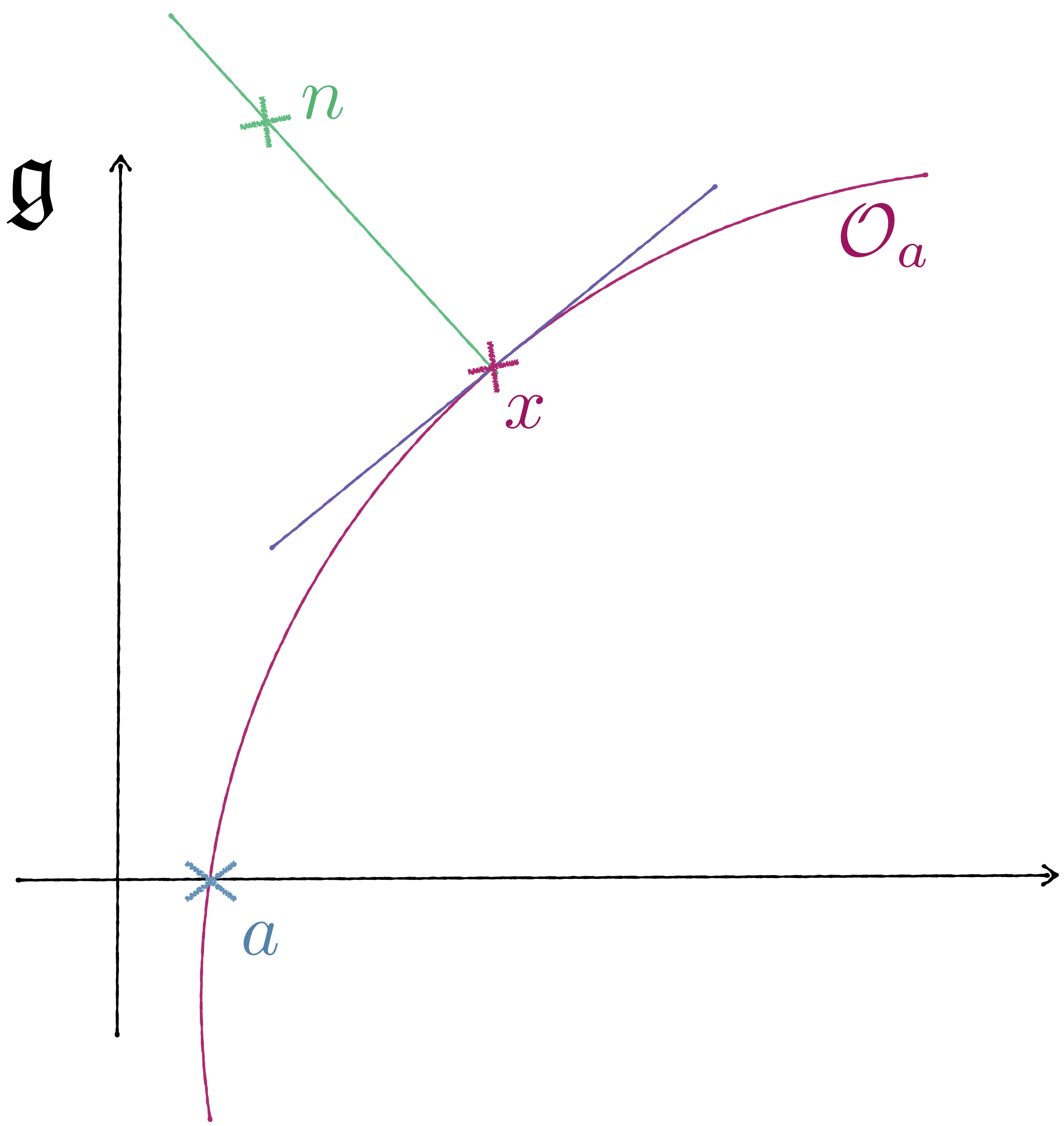
Take linear combinations of them:

$$\hat{e}^{\pm}_a = \hat{e}^v_a \pm \hat{e}^h_a$$

# A generalised frame on $N\mathcal{O}$




So we now have 2dim$(G)$ globally defined vectors $\hat{e}_a^{\pm} = \hat{e}_a^{v} \pm \hat{e}_a^{h}$

**Need to find:**

- Suitable metric
- 2-form B

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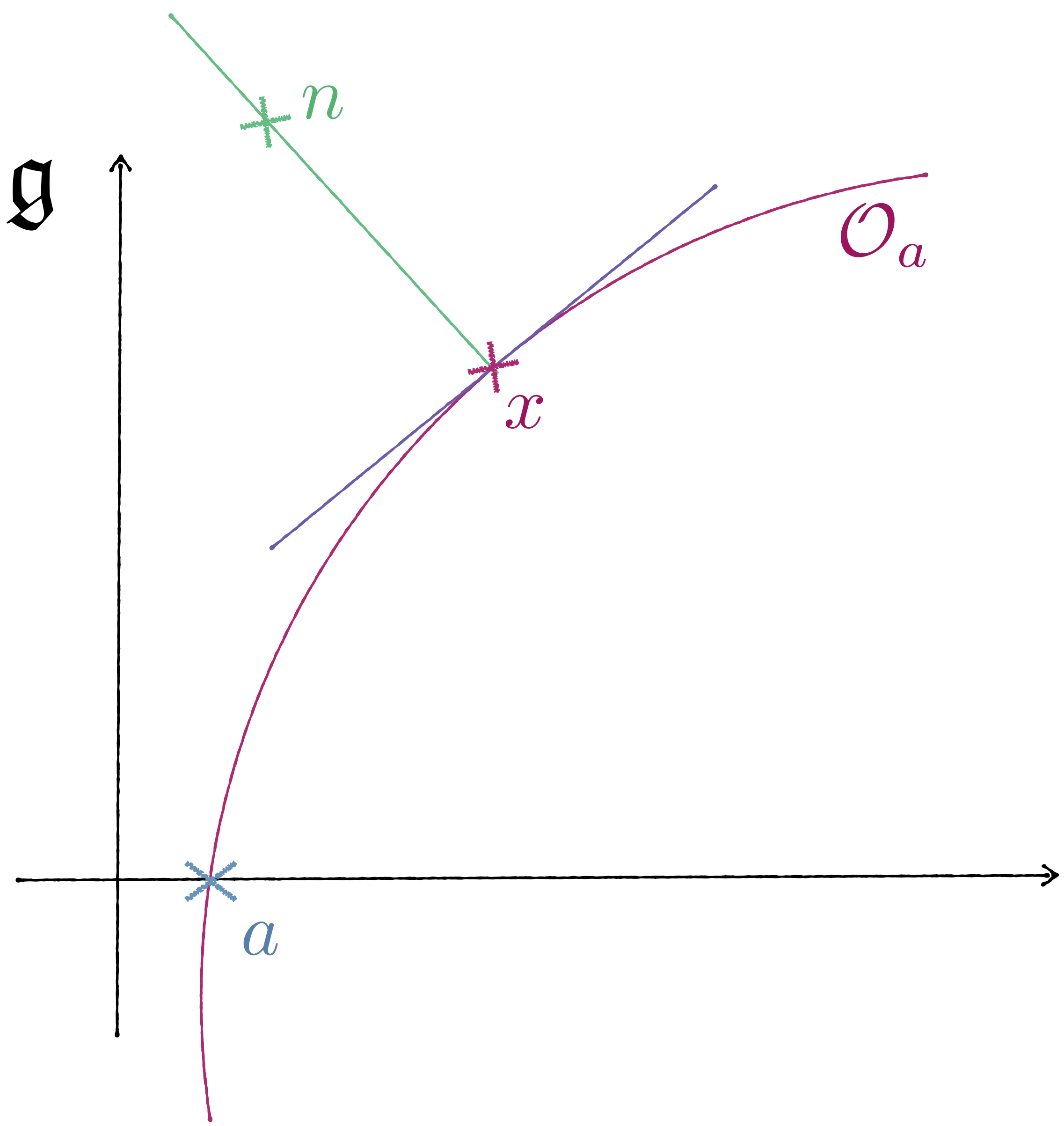


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the tricky part!

- 2-form B

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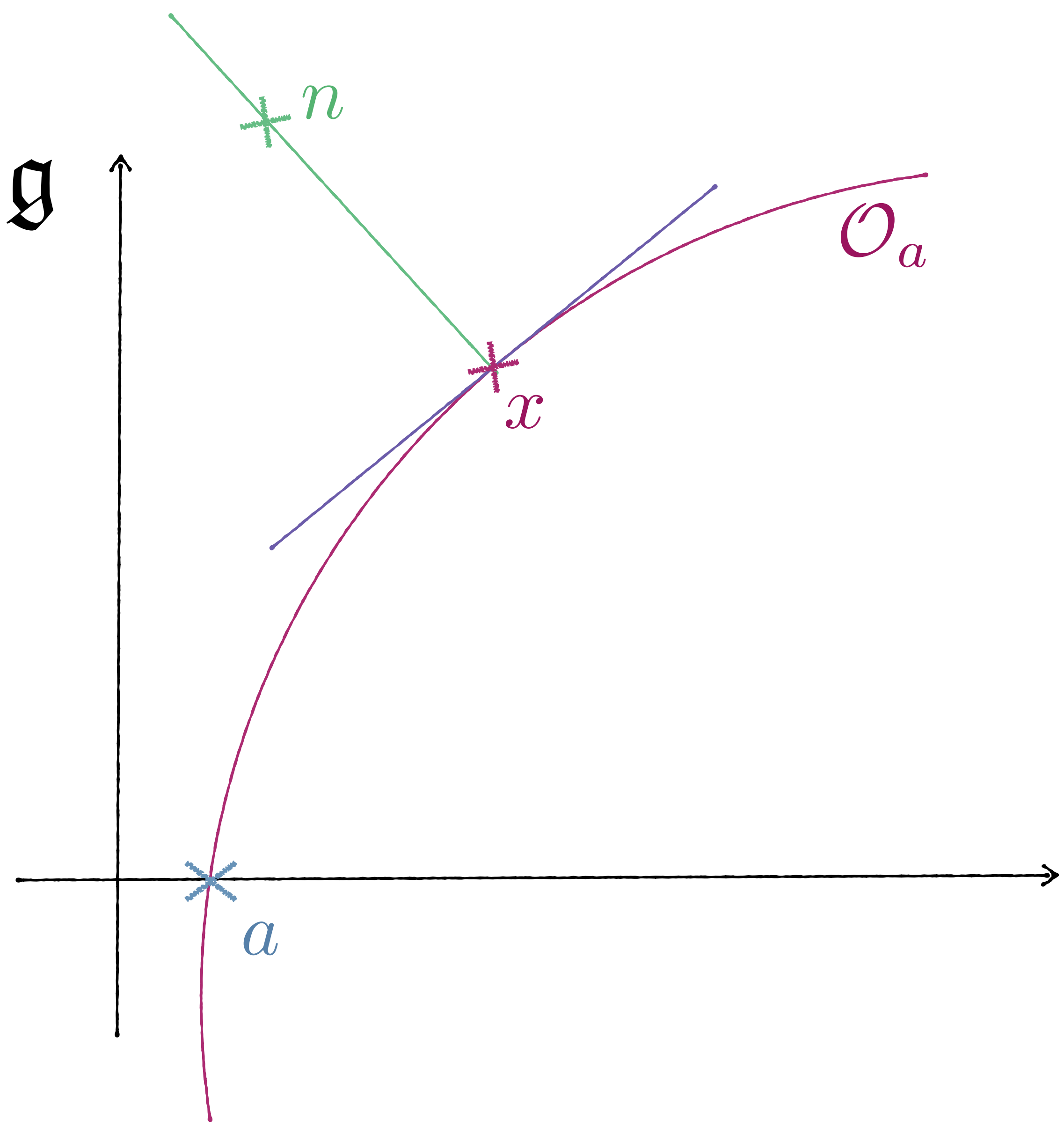


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Define the metric via $g\left(\hat{e}^{\pm}_a, \hat{e}^{\pm}_b\right) = \delta_{ab}$

This system is overdetermined!

Consistent iff $g\left(\hat{e}^v_a, \hat{e}^h_b\right) + g\left(\hat{e}^h_a, \hat{e}^v_b\right) = 0$

Obviously works in the Abelian case…

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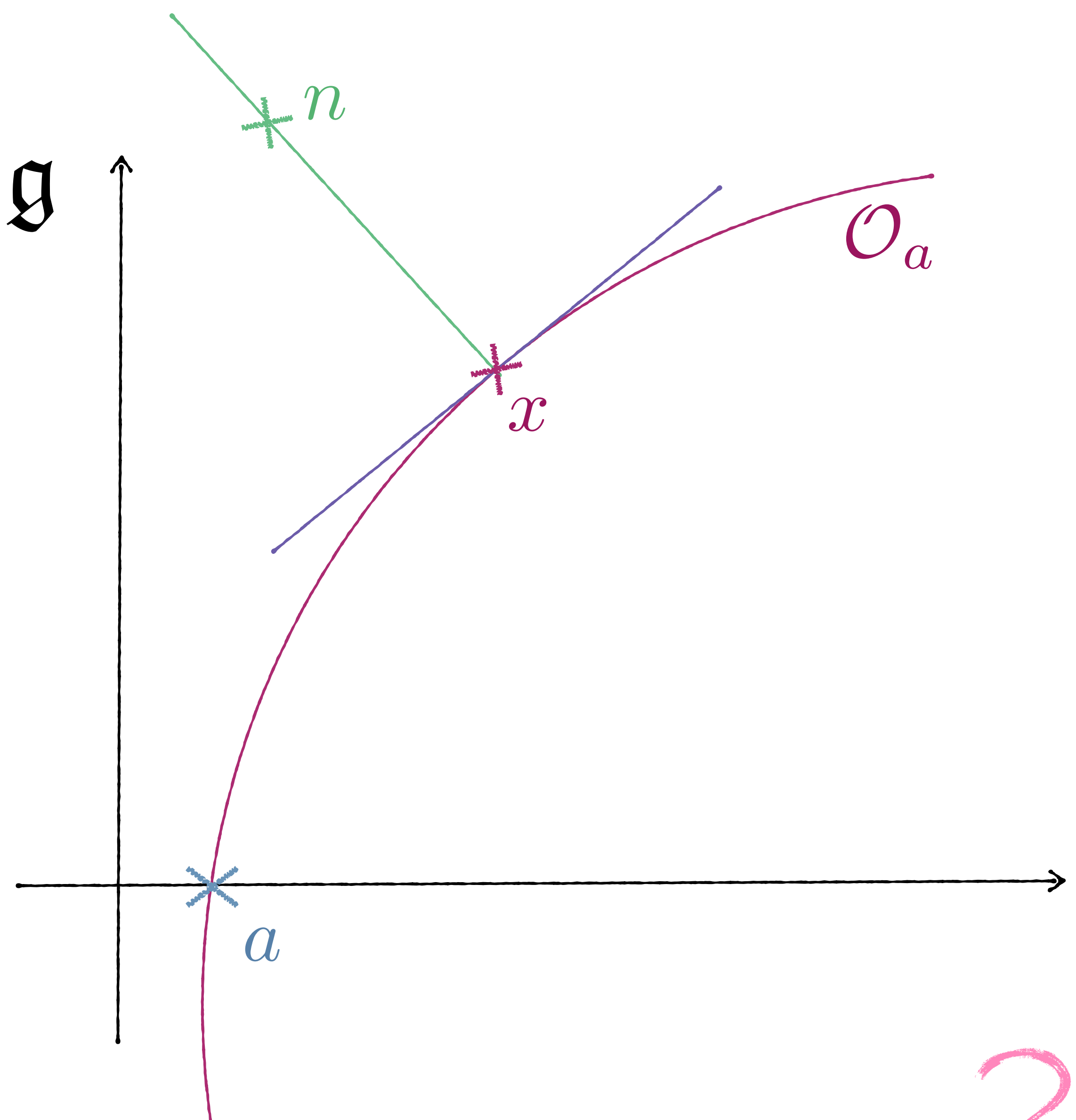


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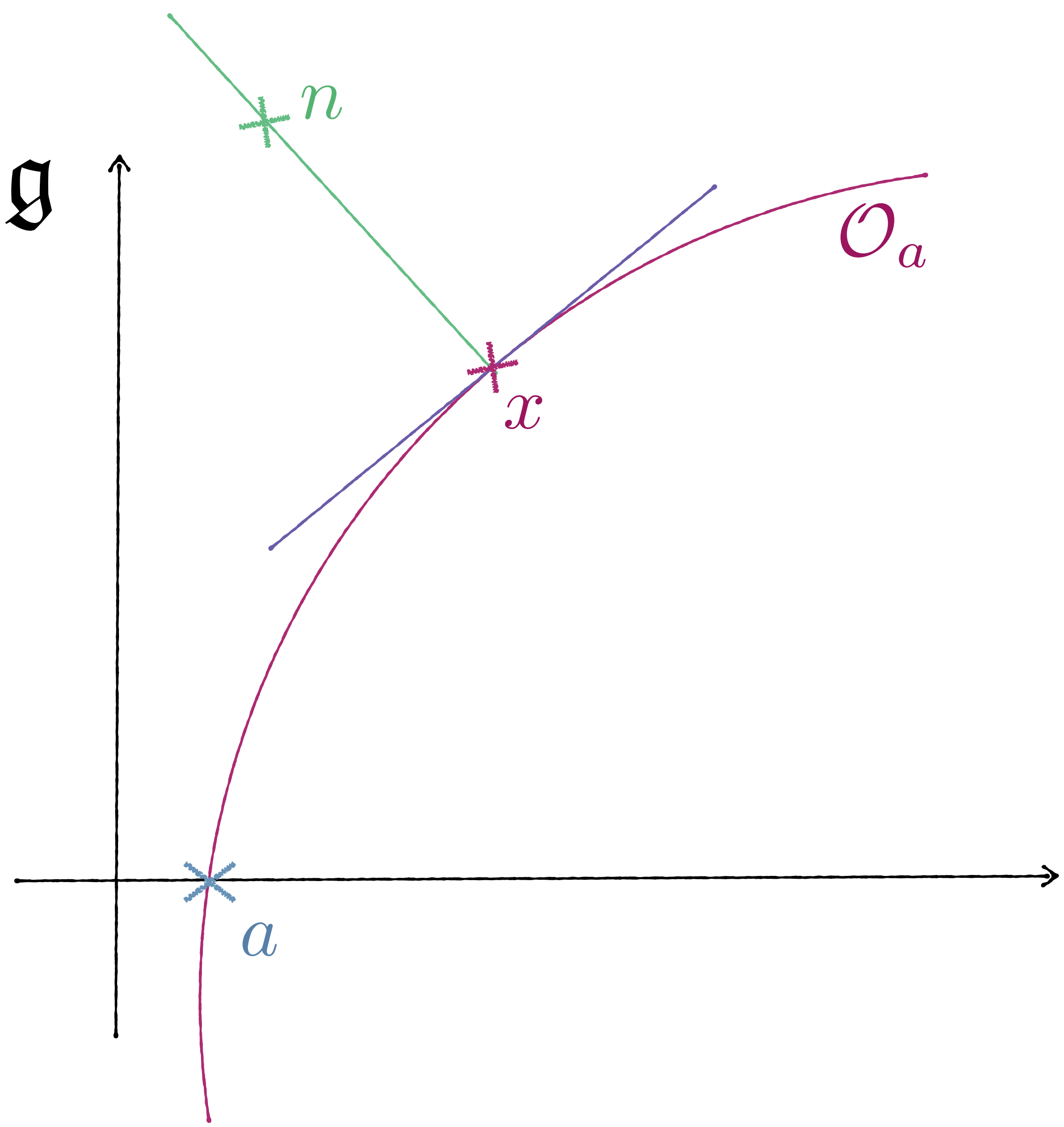
?

Generally: $N\mathcal{O}_a \hookrightarrow \mathfrak{g} \otimes \mathfrak{g}$, and metric block-diagonal the metric restricted to ann(x) must be $\mathrm{Ad}_H$-invariant.

# A generalised frame on $NO$

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Define the metric via $g\left(\hat{e}^{\pm}_a, \hat{e}^{\pm}_b\right) = \delta_{ab}$

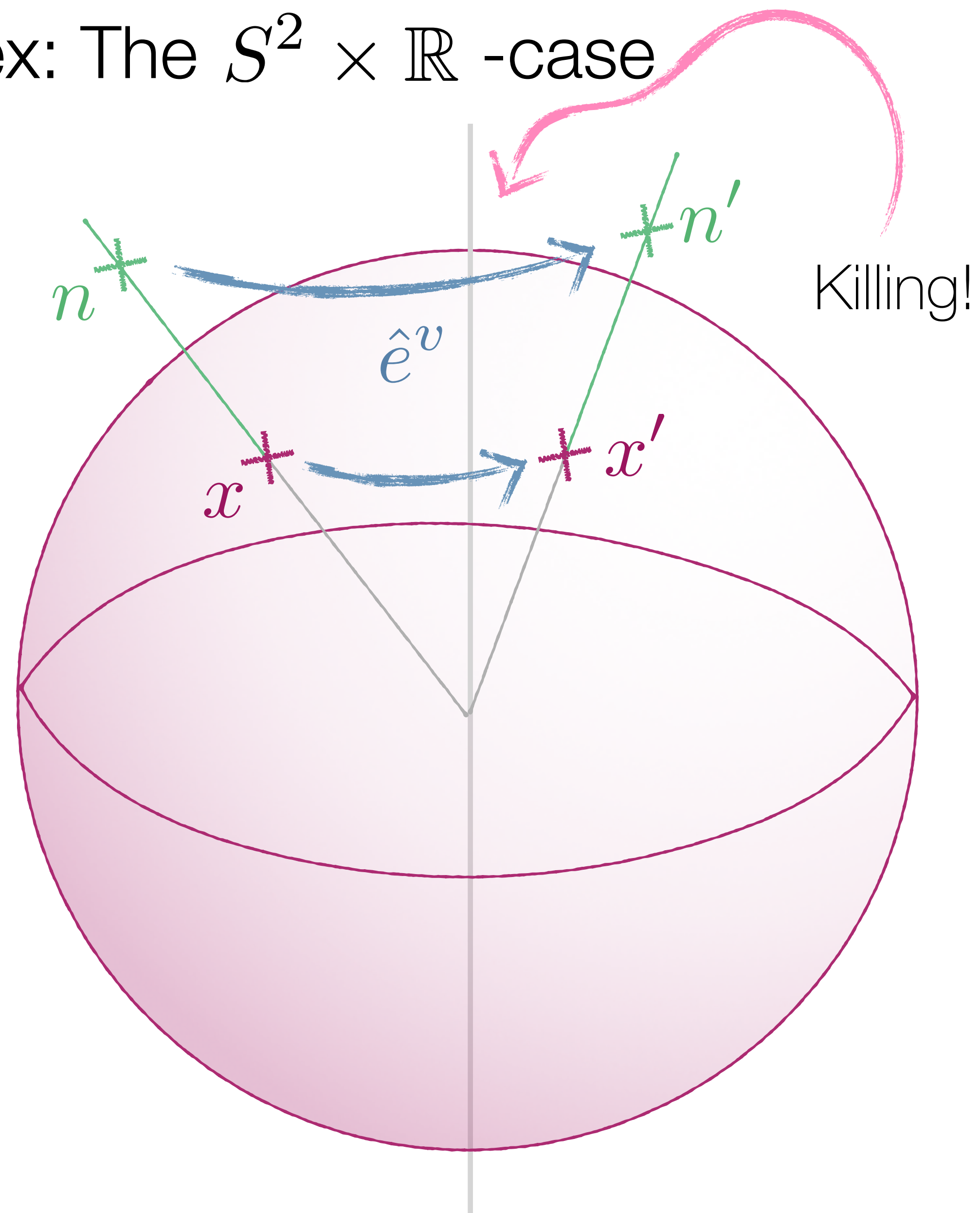
Note: the $\hat{e}^v$'s are Killing vectors of this metric!

$$0 = \mathcal{L}_{\hat{e}^v_c}\left(g\left(\hat{e}^v_a, \hat{e}^h_b\right)\right) = i_{\hat{e}^+_a} i_{\hat{e}^+_b} \mathcal{L}_{\hat{e}^v_c} g$$

$$\Longrightarrow \quad \mathcal{L}_{\hat{e}^v_c} g = 0$$

# A generalised frame on $N\mathcal{O}$

ex: The $S^2 \times \mathbb{R}$ -case

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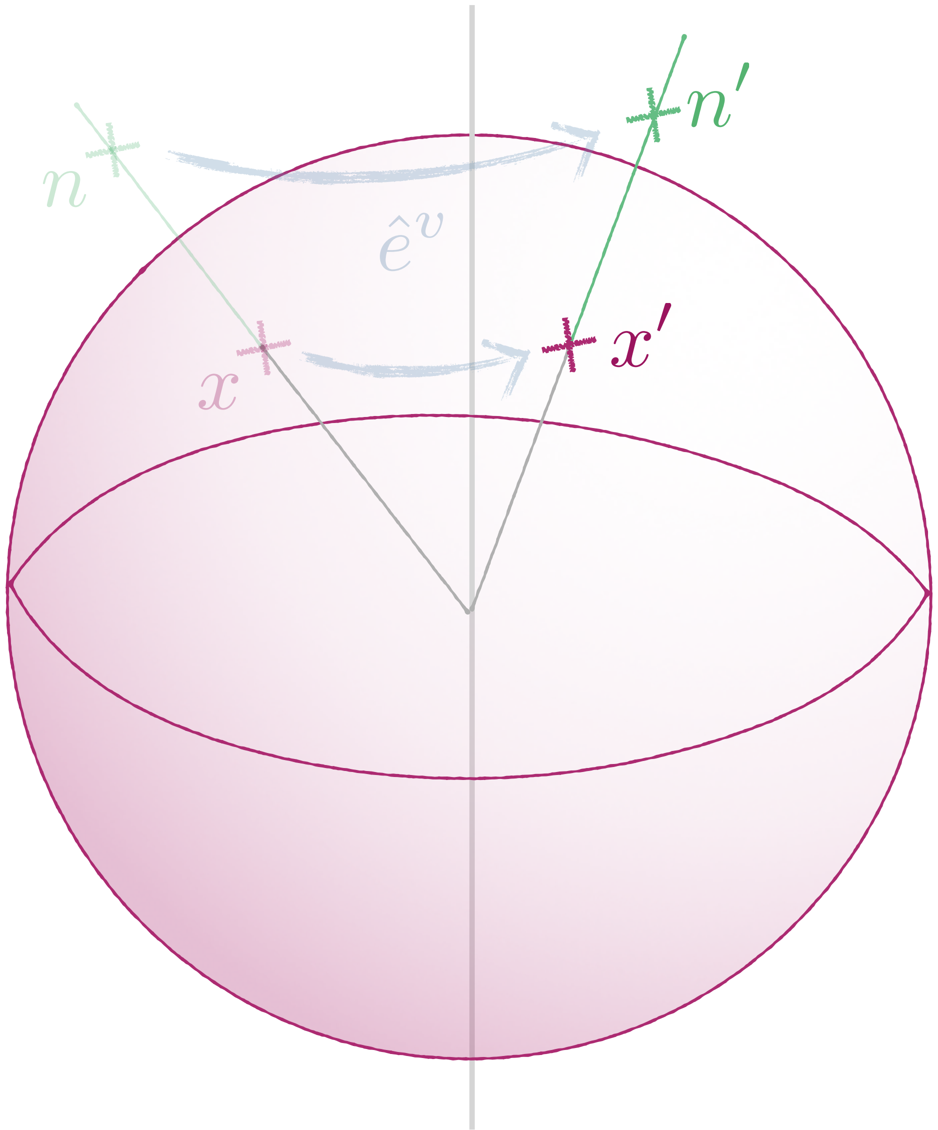


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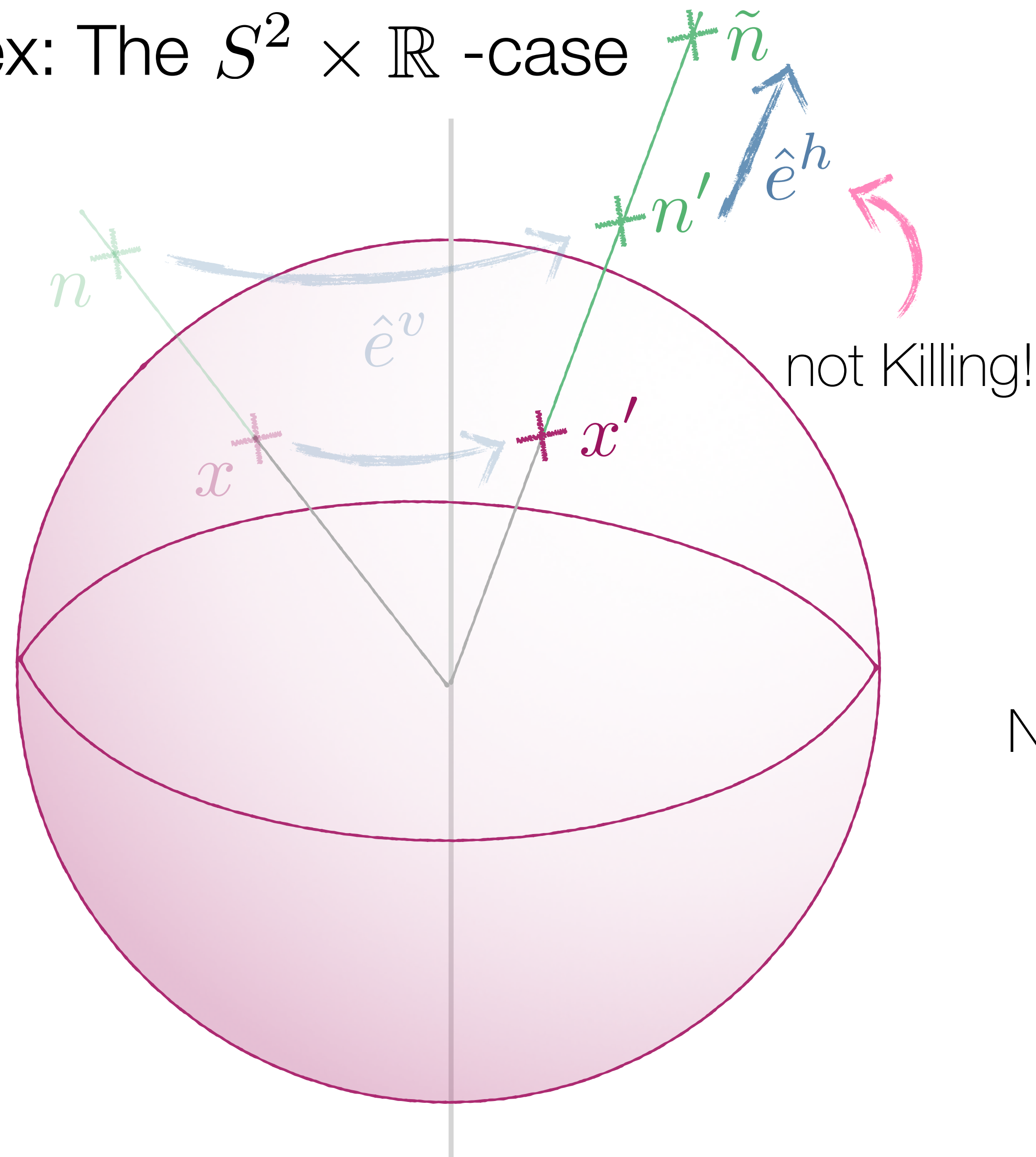


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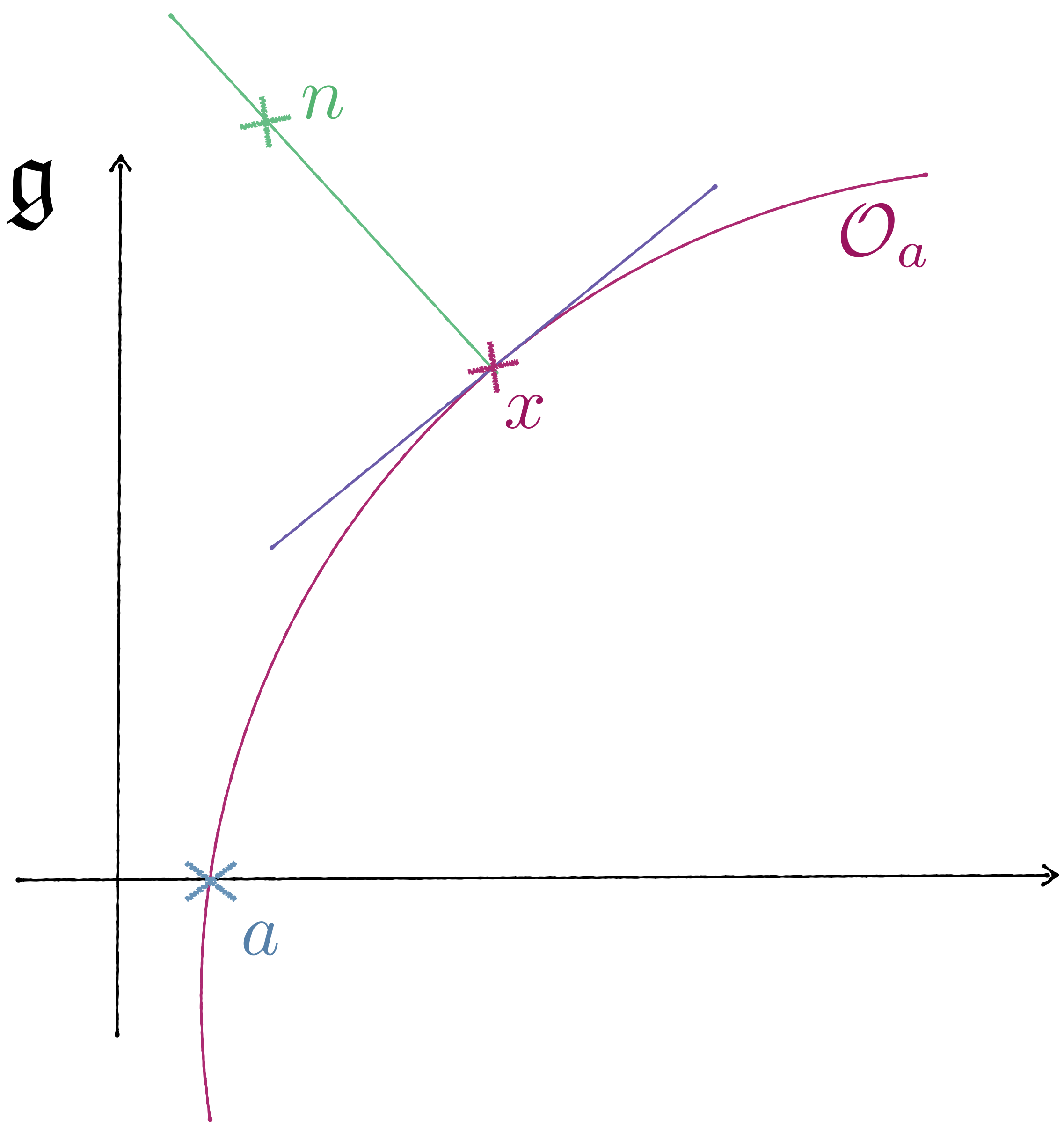


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**Need to find:**

- Suitable metric

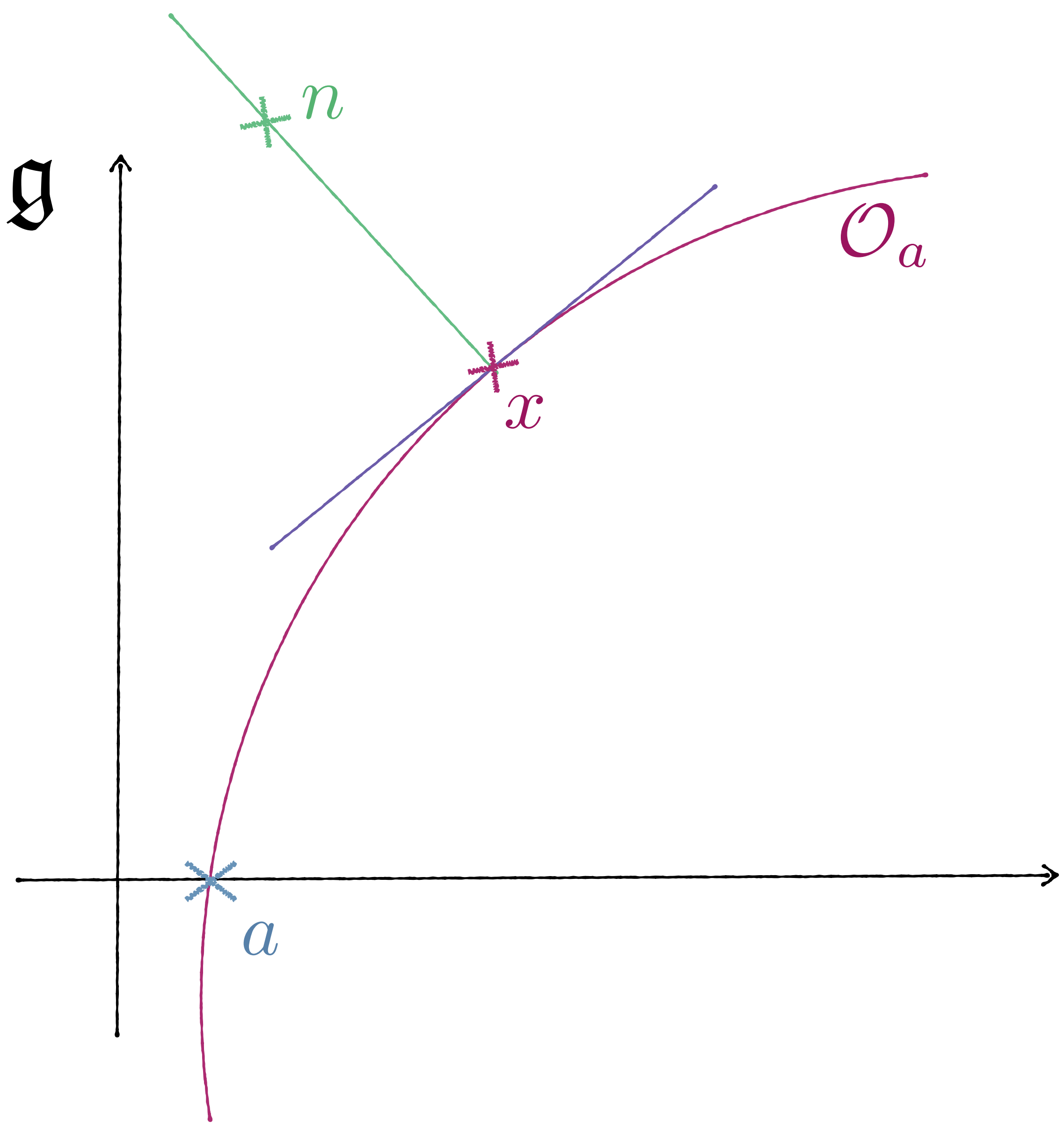
Define the metric via $g\left(\hat{e}_a^{\pm}, \hat{e}_b^{\pm}\right) = \delta_{ab}$

Note: we have an orthogonal matrix $P_a{}^b$

$\hat{e}_a^{+} = P_a{}^b \hat{e}_b^{-}$ $\qquad g\left(\hat{e}_a^{+}, \hat{e}_b^{-}\right) = P_{ab}$

symmetric for regular $\mathcal{O}_a$

# A generalised frame on $N\mathcal{O}$

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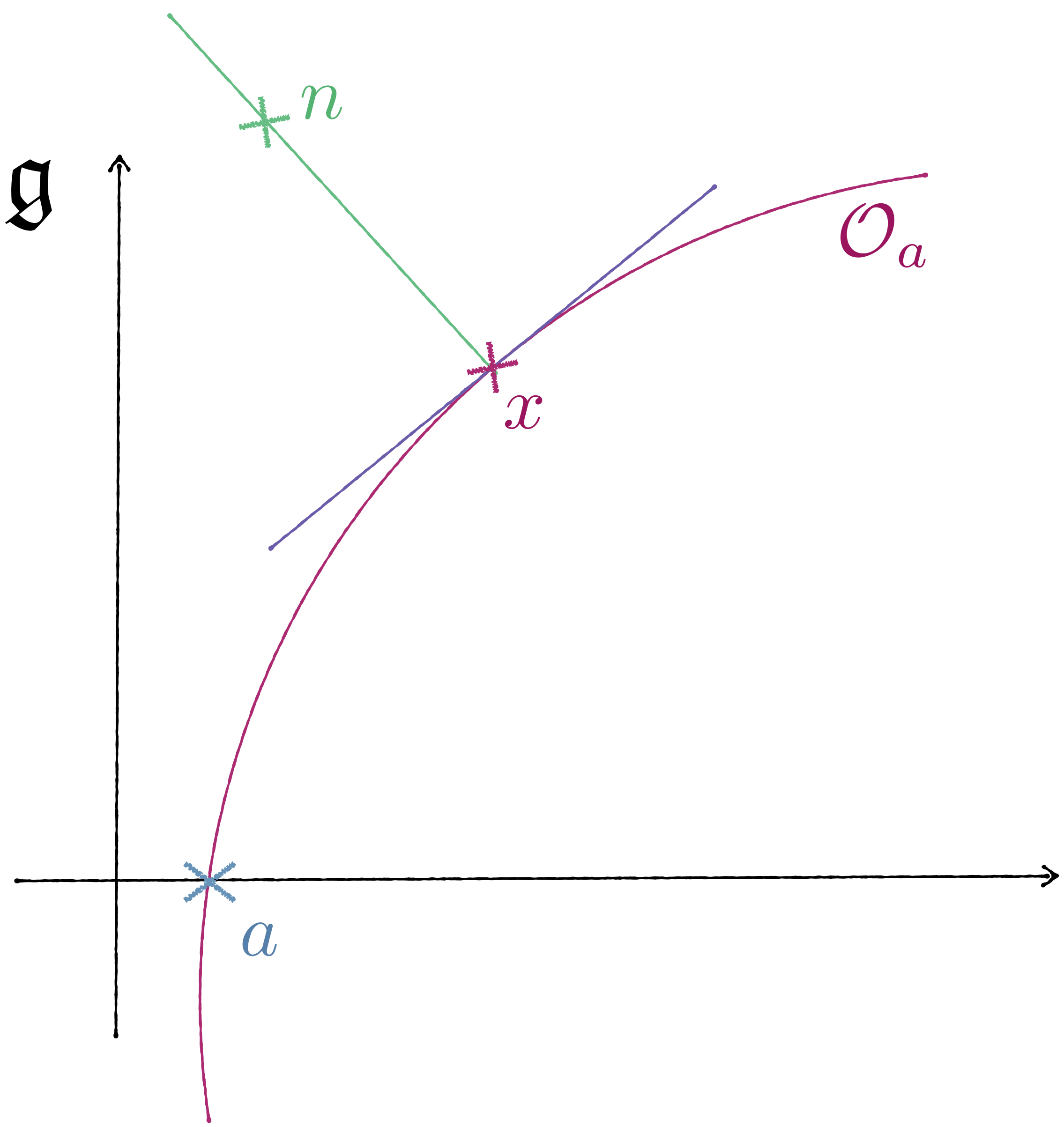
**Need to find:**

- Suitable metric
- 2-form B

Sort of automatic!

# A generalised frame on $N\mathcal{O}$

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Define a generalised frame by

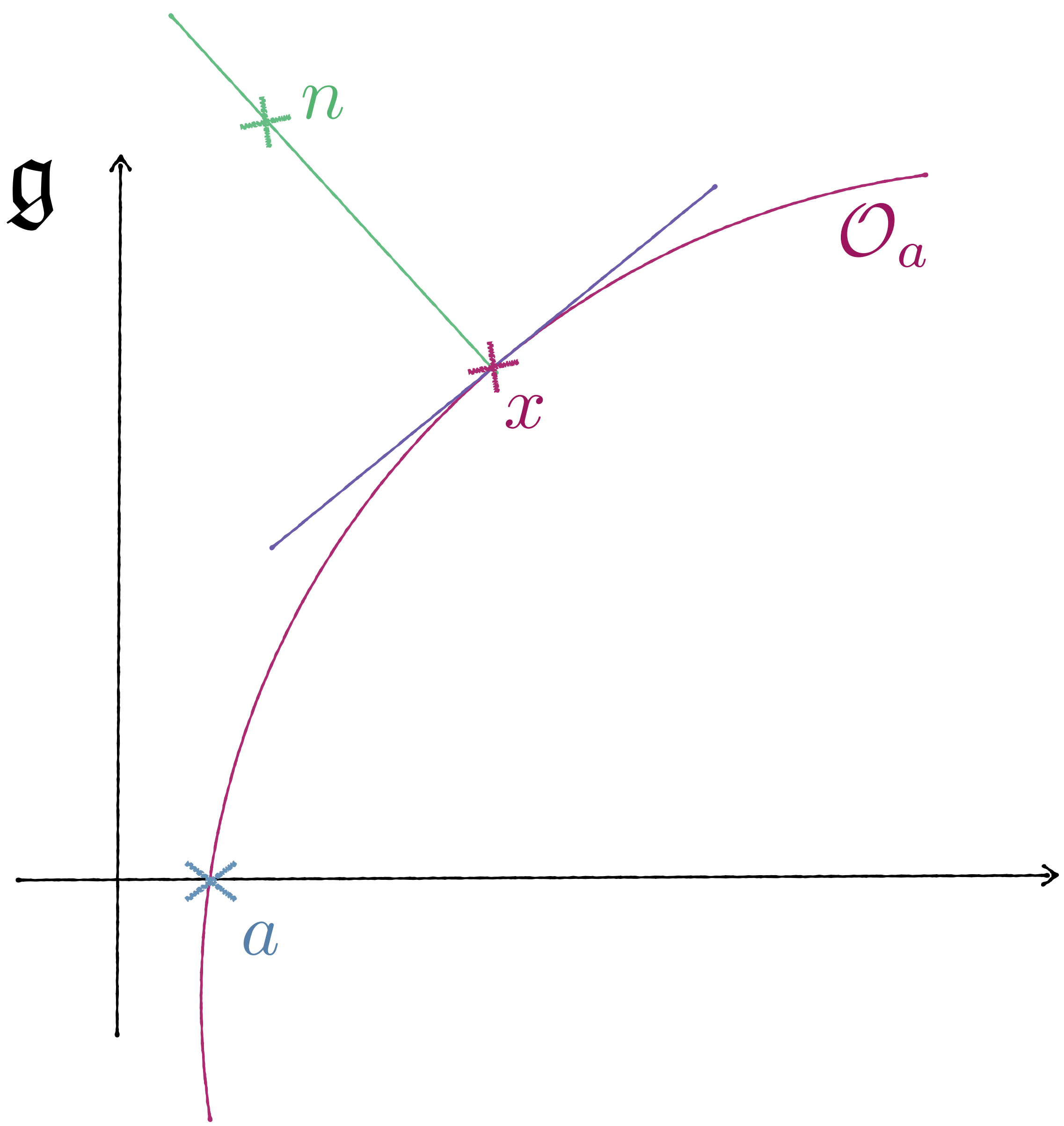
$$\hat{E}_A := \begin{pmatrix} \hat{E}^+ \\ \hat{E}^- \end{pmatrix} = e^B \begin{pmatrix} \hat{e}^+ + i_{\hat{e}^+} g \\ \hat{e}^- - i_{\hat{e}^-} g \end{pmatrix}$$

This is doubly orthonormal.

Generalised metric: $2\mathcal{G} = \left(e^{-B}\right)^T \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} e^{-B}$

$$\mathcal{G}\left(\hat{E}^{\pm}_a, \hat{E}^{\pm}_a\right) = \delta_{ab} \qquad \mathcal{G}\left(\hat{E}^{\pm}_a, \hat{E}^{\mp}_a\right) = 0$$

# A generalised frame on $N\mathcal{O}$

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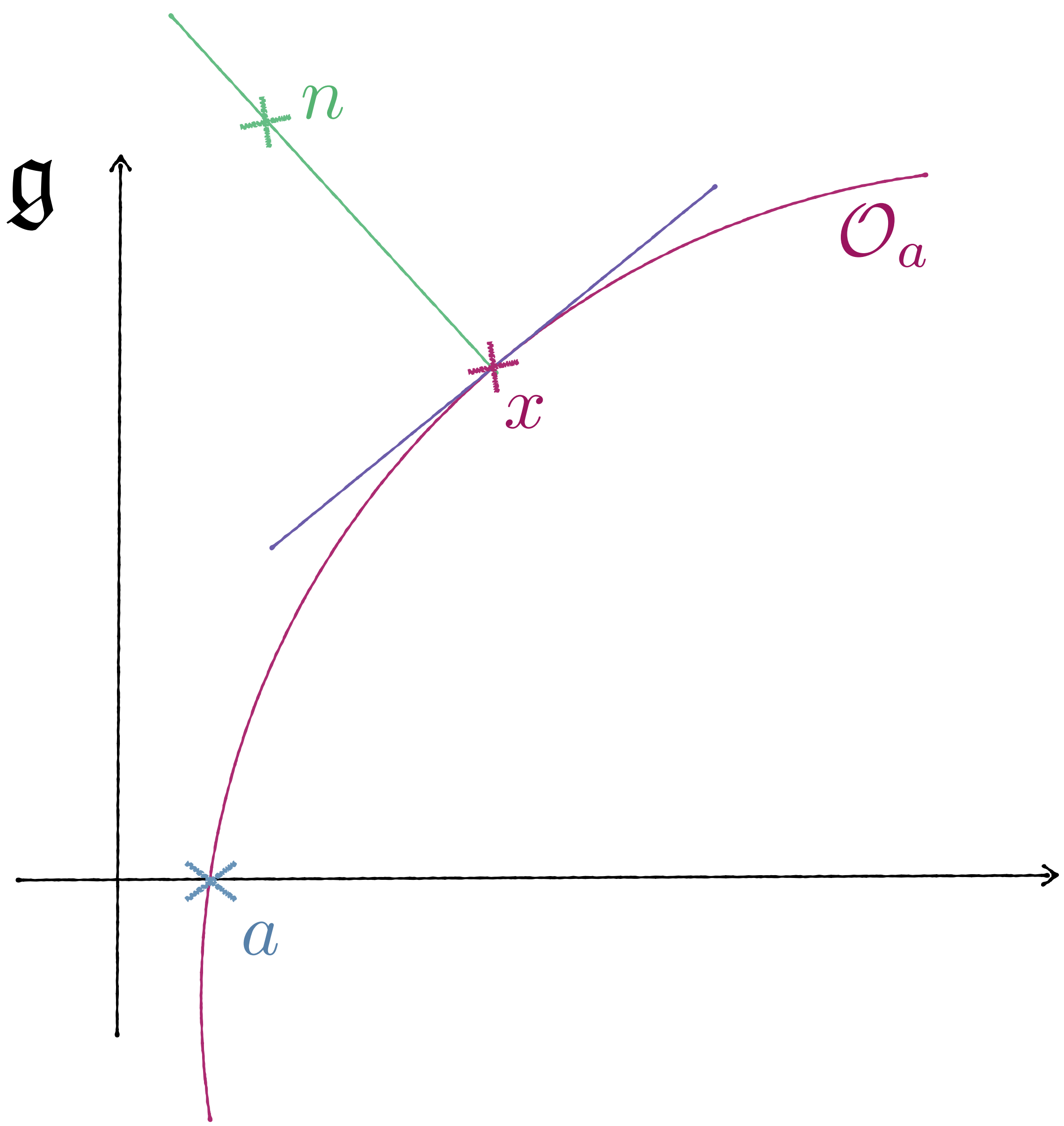
This is doubly orthonormal.

Natural metric: $\quad 2\eta\left(V, W\right) = i_v \mu + i_w \lambda$

$$\eta\left(\hat{E}^{\pm}_a, \hat{E}^{\pm}_a\right) = \pm\delta_{ab} \qquad \eta\left(\hat{E}^{\pm}_a, \hat{E}^{\mp}_a\right) = 0$$

# Finding $B$

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Fixing the 2-form B:

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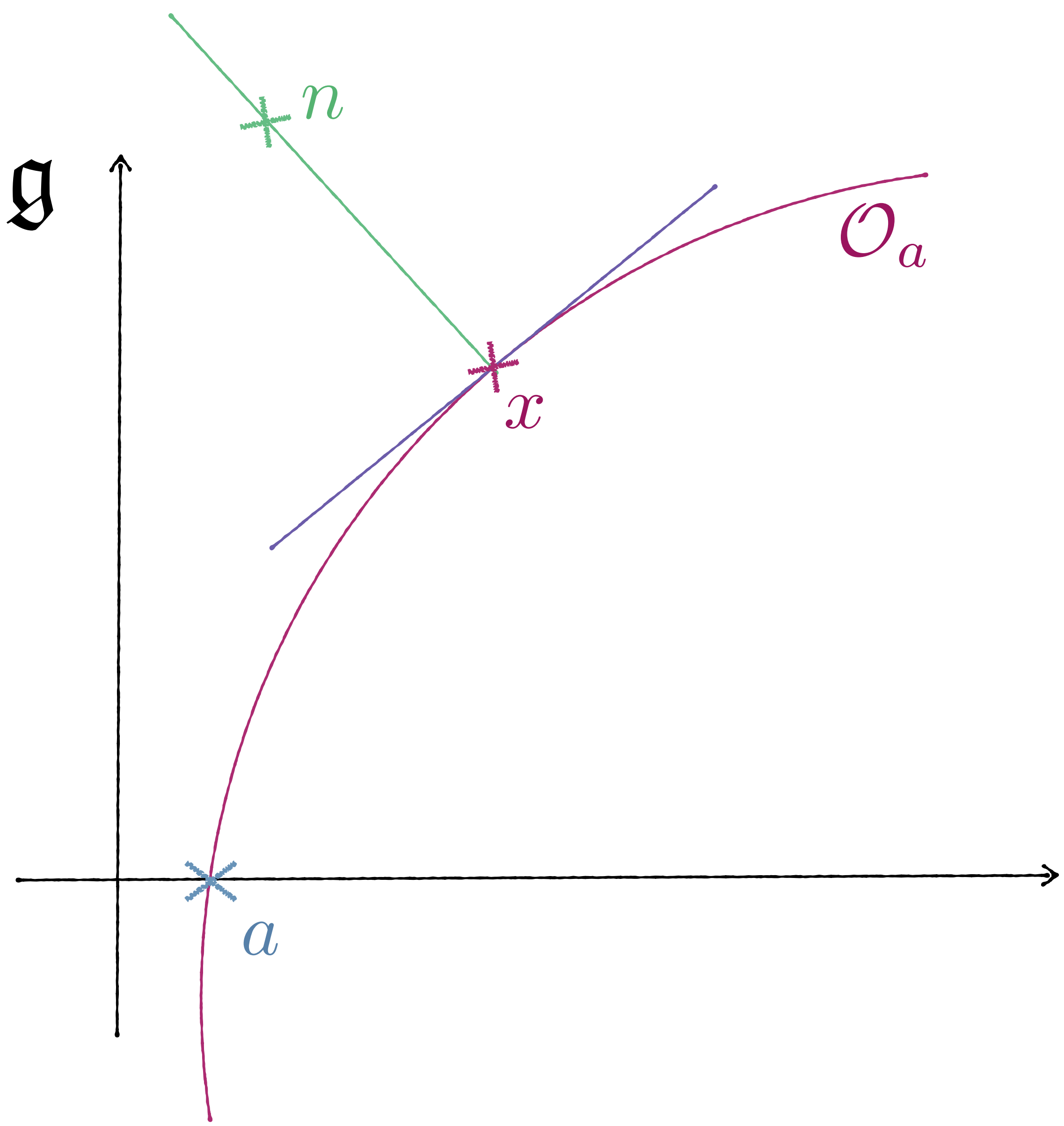
How? Our space is now gen. parallelisable…

We needed something more.

Leibniz condition!

# Finding $B$

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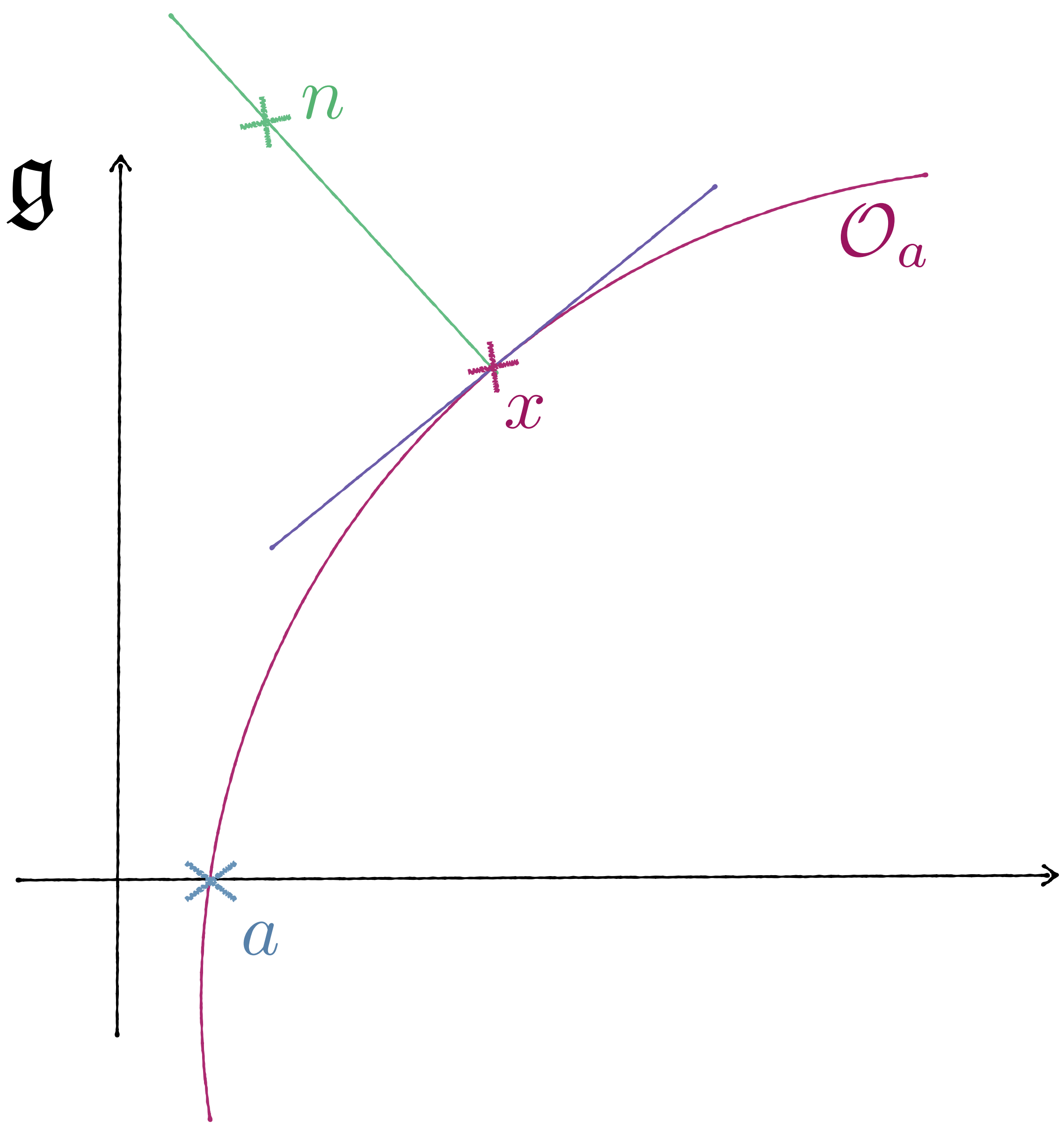
Leibniz condition!

$$L_{\hat{E}_a^{\pm}} \hat{E}_b^{\pm} = \frac{1}{2} f_{ab}{}^{c} \left( 3\hat{E}_c^{\pm} - \hat{E}_c^{\mp} \right)$$

$$L_{\hat{E}_a^{+}} \hat{E}_b^{-} = \frac{1}{2} f_{ab}{}^{c} \left( \hat{E}_c^{+} + \hat{E}_c^{-} \right)$$

from $\mathrm{Lie}\,(G \ltimes \mathfrak{g})$

# Finding $B$

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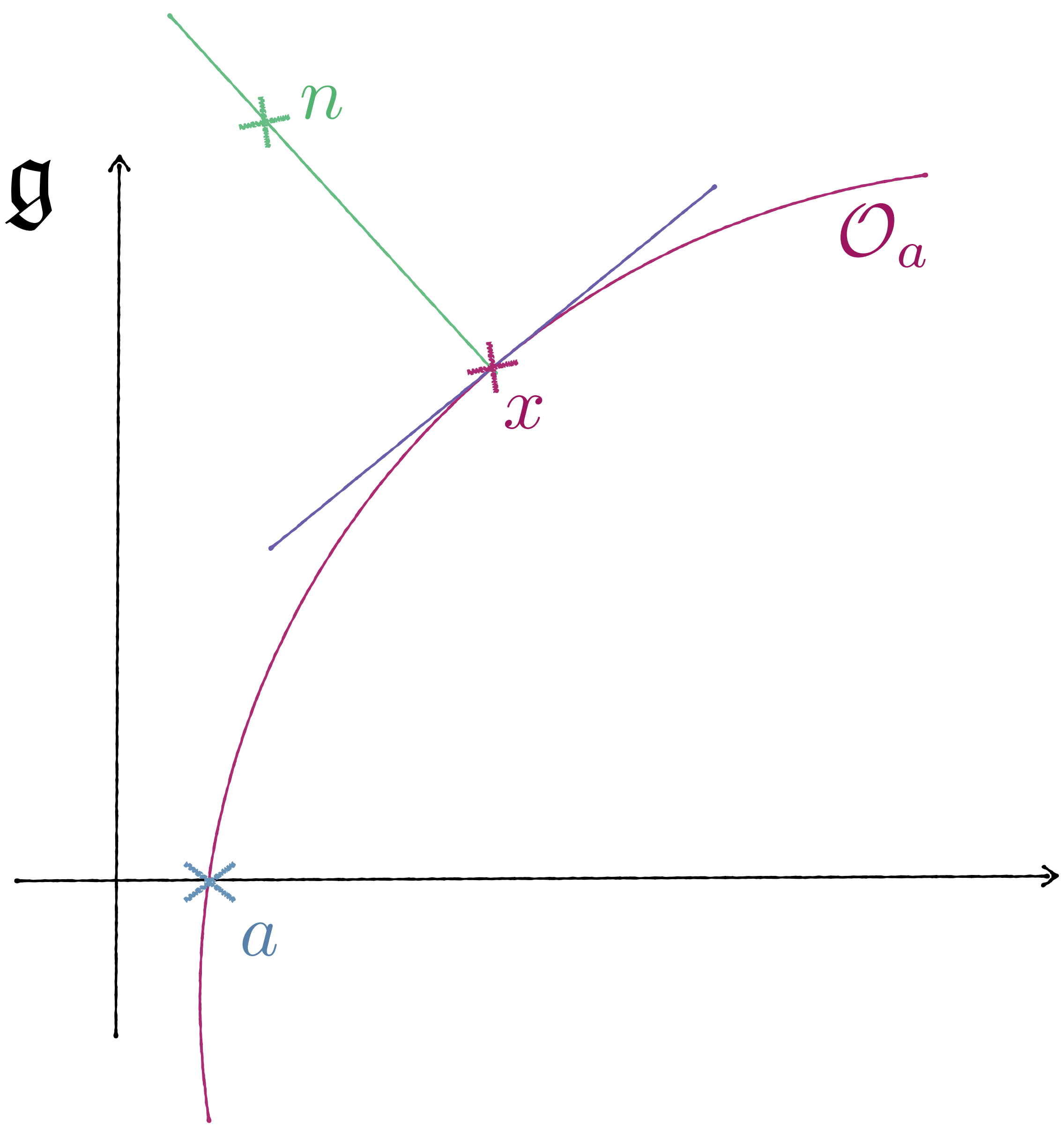
fixes $B$!

Using

$$L_{e^B V} e^B W = e^B \left(L_v W\right) - i_v i_w \left(dB\right)$$

# Finding $B$




Subtleties:

- All Leibniz conditions equivalent?

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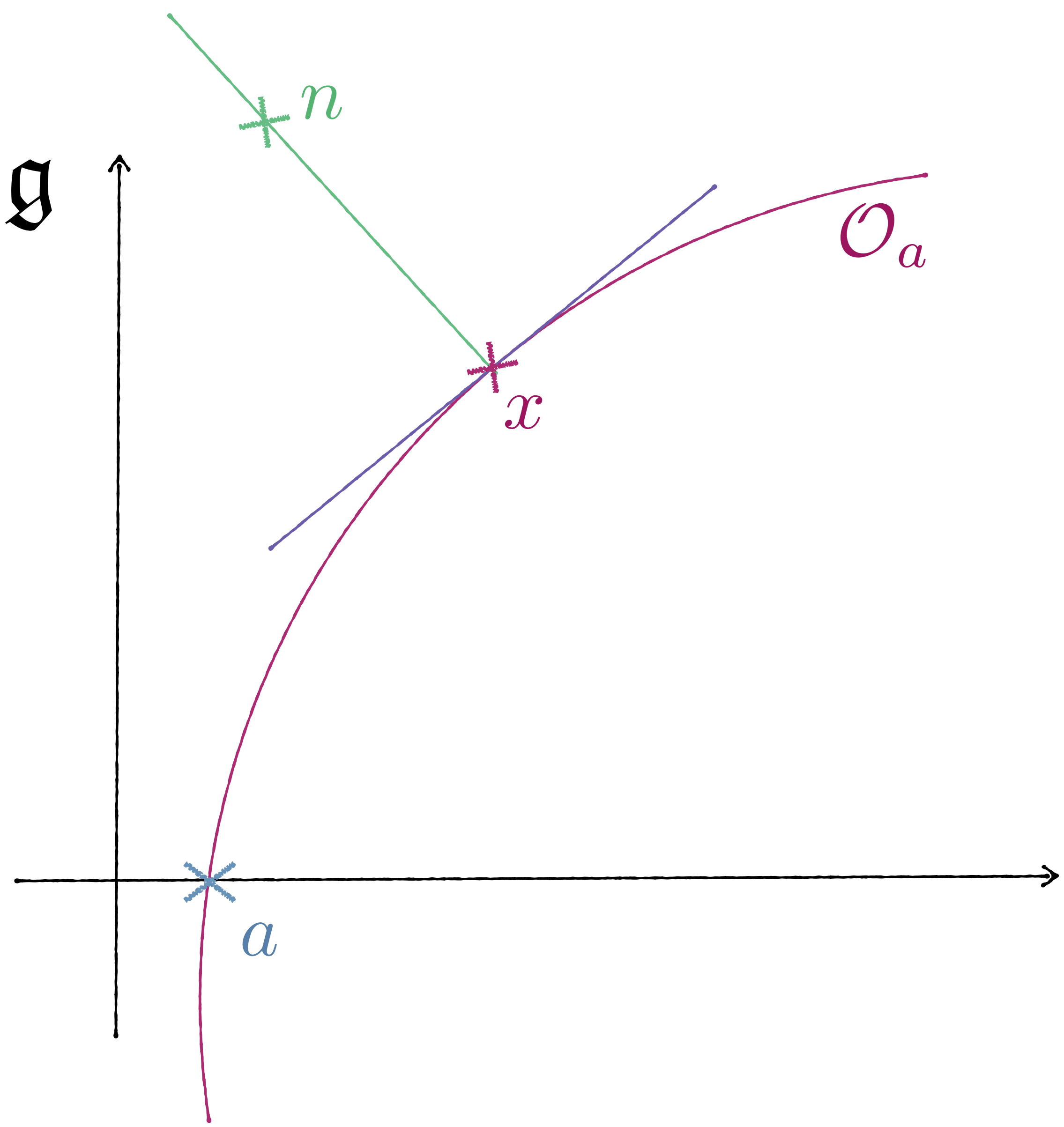


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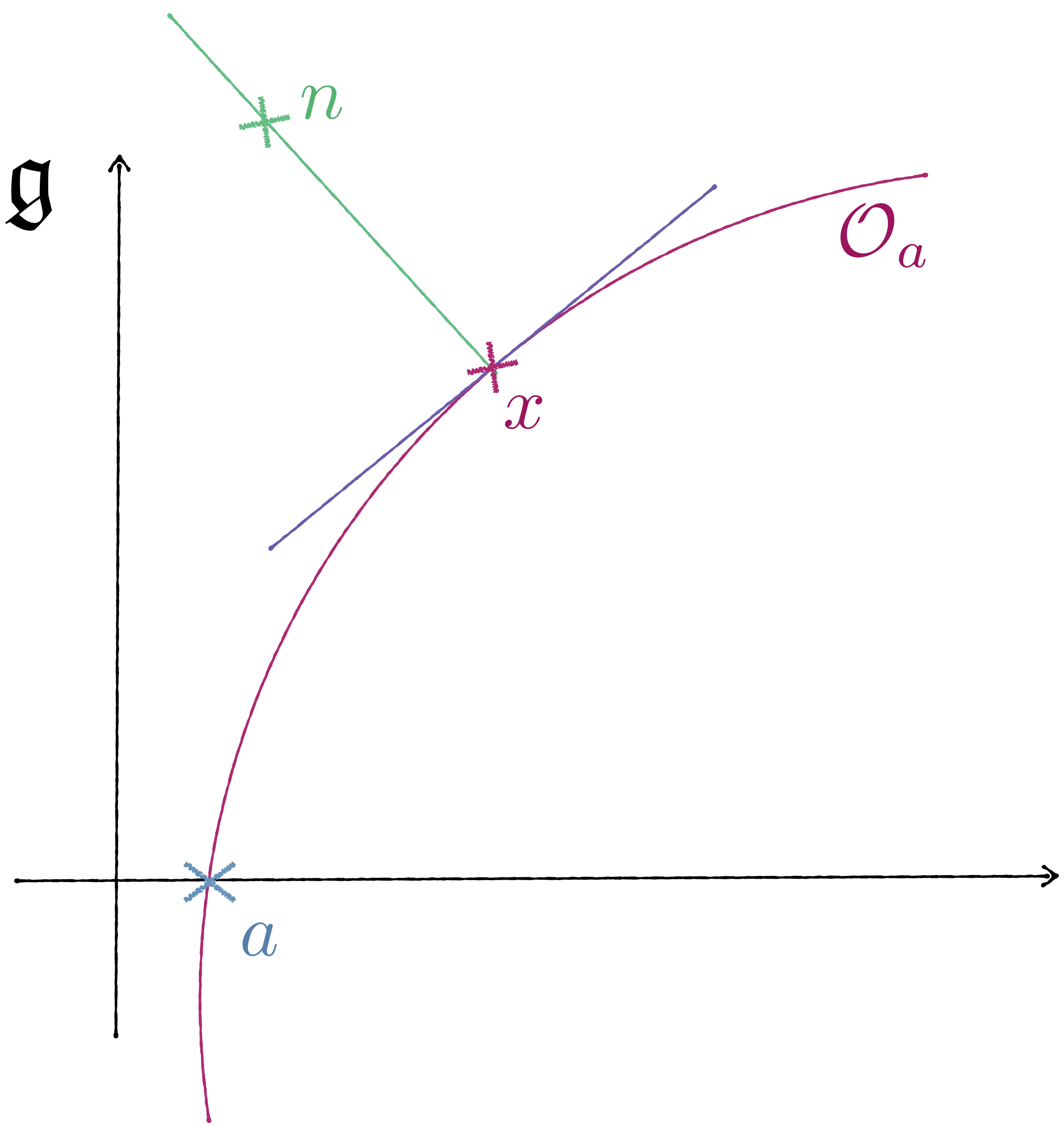
# Finding $B$

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  Is this closed?

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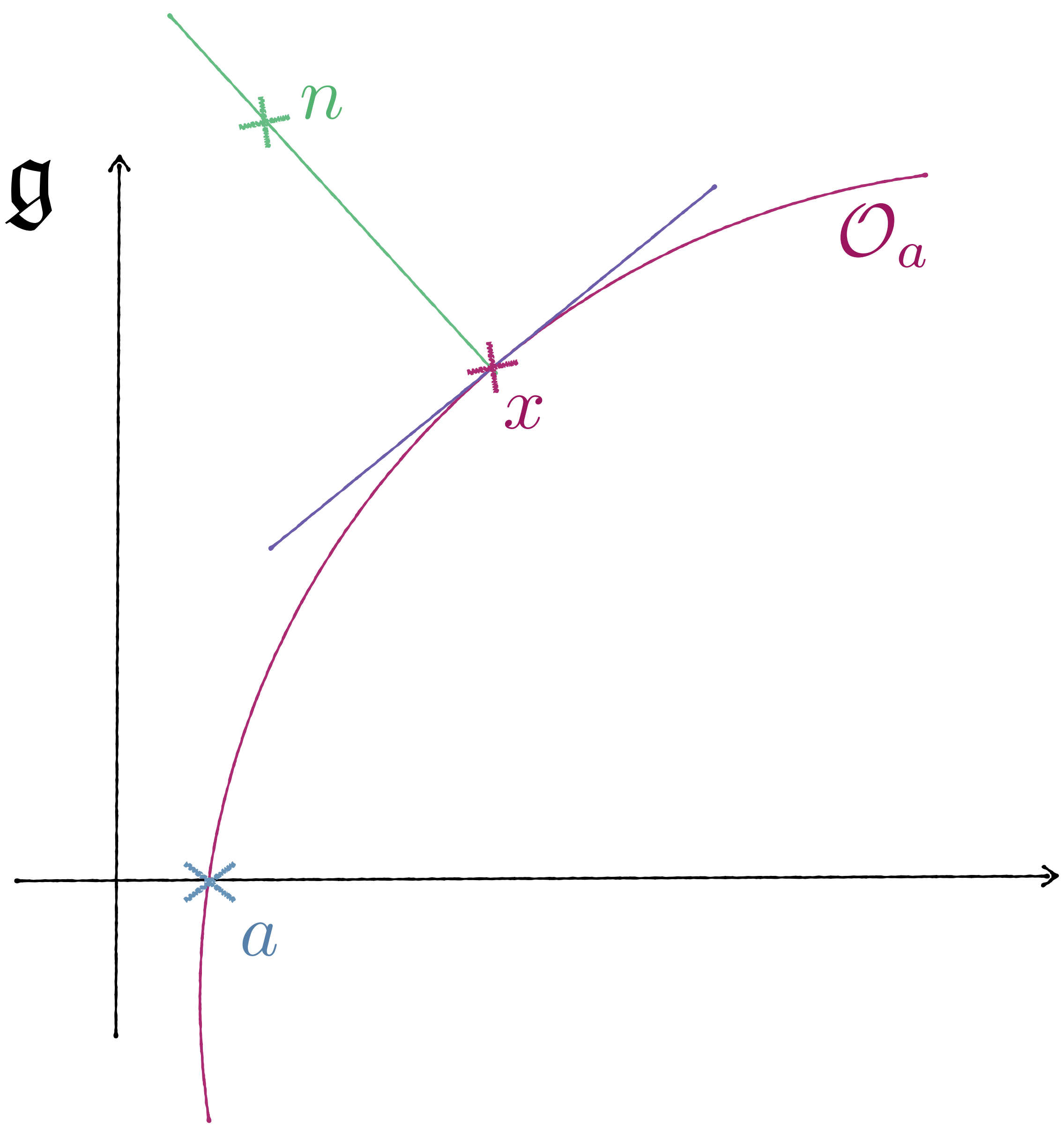


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- This actually only gives us a 3-form, $H$
  Is this closed?

$$\hat{E}_A := \begin{pmatrix} \hat{E}^+ \\ \hat{E}^- \end{pmatrix} = e^B \begin{pmatrix} \hat{e}^+ + i_{\hat{e}^+} g \\ \hat{e}^- - i_{\hat{e}^-} g \end{pmatrix}$$

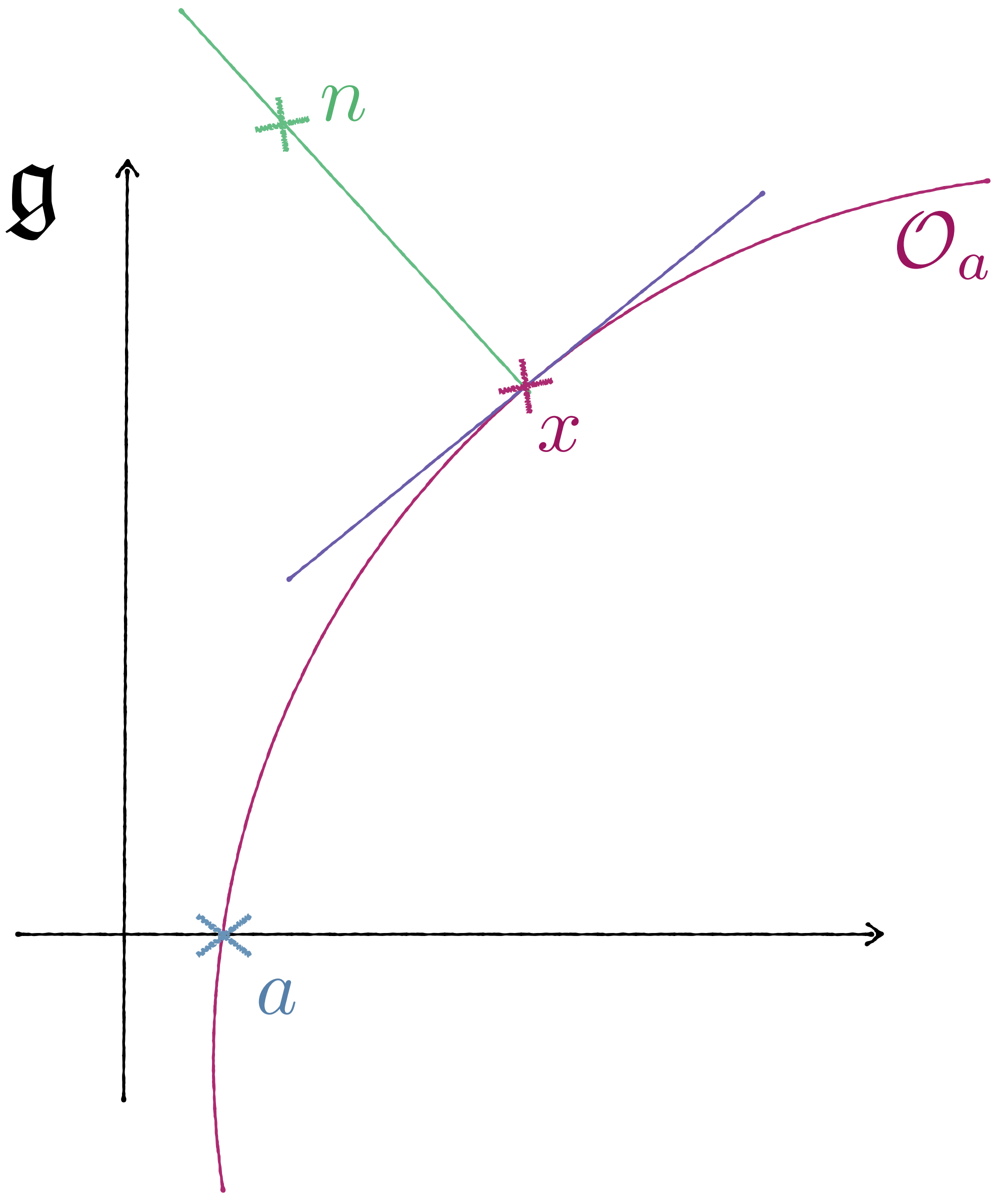
Leibniz condition!

$$L_{\hat{E}_a^\pm} \hat{E}_b^\pm = \frac{1}{2} f_{ab}{}^c \left( 3\hat{E}_c^\pm - \hat{E}_c^\mp \right) \qquad L_{\hat{E}_a^+} \hat{E}_b^- = \frac{1}{2} f_{ab}{}^c \left( \hat{E}_c^+ + \hat{E}_c^- \right)$$

Using

$$L_{e^B V} e^B W = e^B \left( L_v W \right) - i_v i_w \left( \; H \; \right)$$

# Finding $B$




Subtleties:

- This actually only gives us a 3-form, $H$

  Is this closed?

$$\hat{E}_A := \begin{pmatrix} \hat{E}^+ \\ \hat{E}^- \end{pmatrix} = e^B \begin{pmatrix} \hat{e}^+ + i_{\hat{e}^+} g \\ \hat{e}^- - i_{\hat{e}^-} g \end{pmatrix}$$

Leibniz condition!

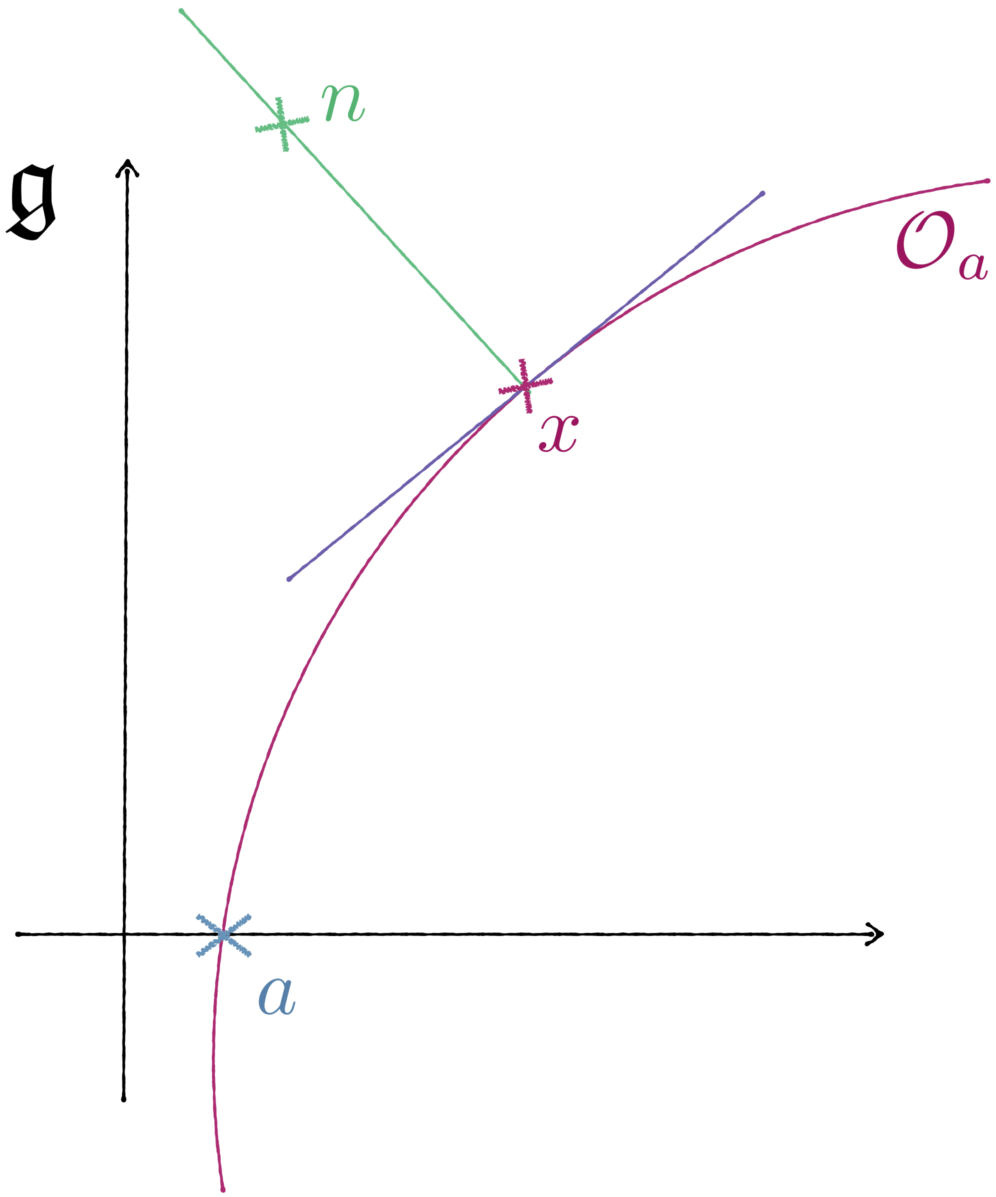
$$L_{\hat{E}_a^\pm} \hat{E}_b^\pm = \frac{1}{2} f_{ab}{}^c \left( 3\hat{E}_c^\pm - \hat{E}_c^\mp \right) \qquad L_{\hat{E}_a^+} \hat{E}_b^- = \frac{1}{2} f_{ab}{}^c \left( \hat{E}_c^+ + \hat{E}_c^- \right)$$

**ex:**

$$L_{\hat{E}_a^+} \hat{E}_b^+ = [\hat{e}_a^+, \hat{e}_b^+] + \mathcal{L}_{\hat{e}_a^+} i_{\hat{e}^+} g - i_{\hat{e}_b^+} d i_{\hat{e}_a^+} g - i_{\hat{e}_a^+} i_{\hat{e}_b^+} H$$

$$L_{e^B V} e^B W = e^B \left( L_v W \right) - i_v i_w \left( H \right)$$

# Finding $B$




Subtleties:

- This actually only gives us a 3-form, $H$
  Is this closed?

$$\hat{E}_A := \begin{pmatrix} \hat{E}^+ \\ \hat{E}^- \end{pmatrix} = e^B \begin{pmatrix} \hat{e}^+ + i_{\hat{e}^+} g \\ \hat{e}^- - i_{\hat{e}^-} g \end{pmatrix}$$

Leibniz condition!

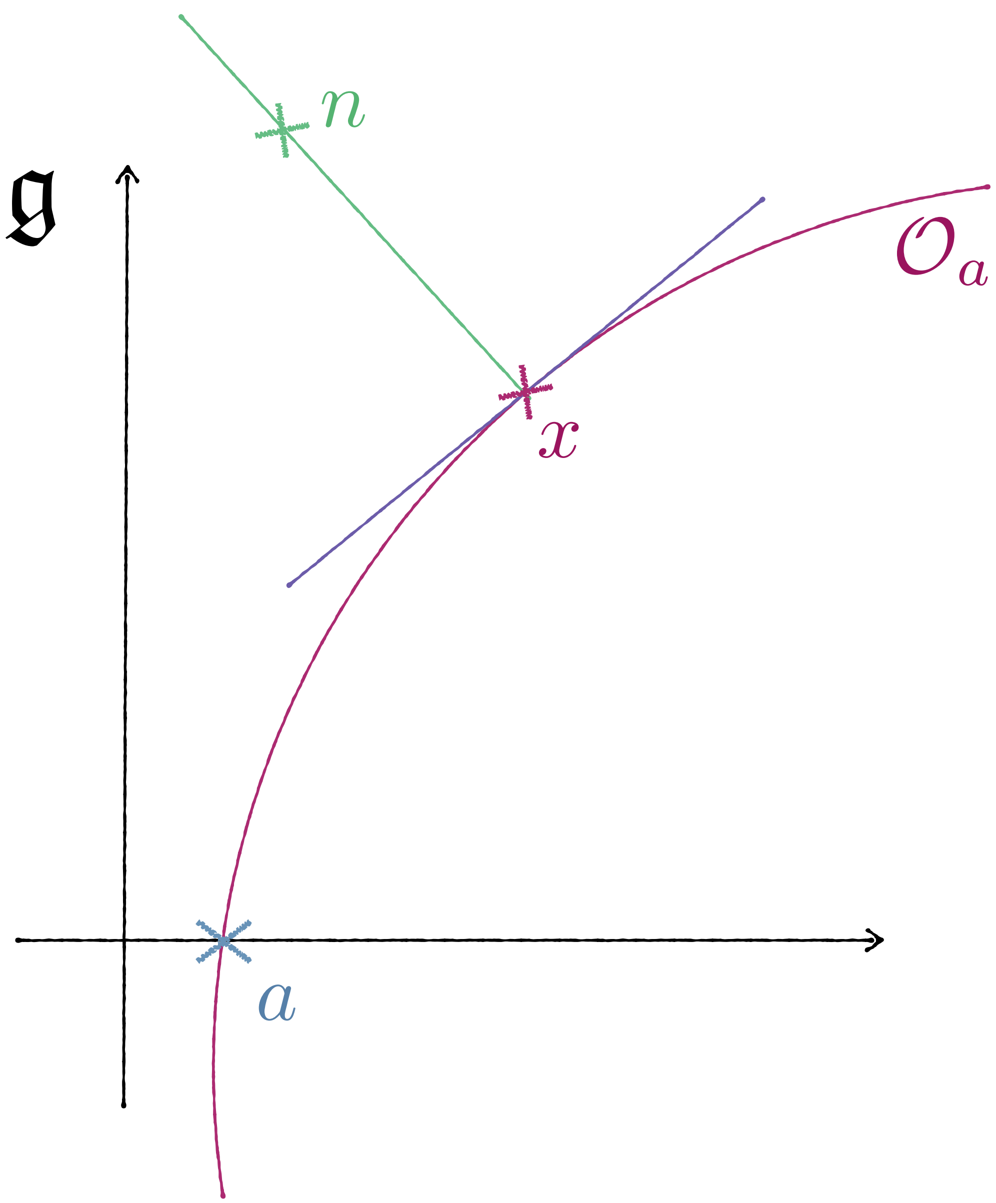
$$\boxed{L_{\hat{E}_a^\pm} \hat{E}_b^\pm = \frac{1}{2} f_{ab}{}^c \left( 3\hat{E}_c^\pm - \hat{E}_c^\mp \right)} \qquad L_{\hat{E}_a^+} \hat{E}_b^- = \frac{1}{2} f_{ab}{}^c \left( \hat{E}_c^+ + \hat{E}_c^- \right)$$

**ex:**

$$\begin{aligned} L_{\hat{E}_a^+} \hat{E}_b^+ &= [\hat{e}_a^+, \hat{e}_b^+] + \mathcal{L}_{\hat{e}_a^+} i_{\hat{e}^+} g - i_{\hat{e}_b^+} d i_{\hat{e}_a^+} g - i_{\hat{e}_a^+} i_{\hat{e}_b^+} H \\ &\overset{!}{=} [\hat{e}_a^+, \hat{e}_b^+] + \frac{1}{2} f_{ab}{}^c \left( 3 i_{\hat{e}_c^+} g + i_{\hat{e}_c^i} g \right) \end{aligned}$$

$$L_{e^B V} e^B W = e^B (L_v W) - i_v i_w ( H )$$

# Finding $B$




Subtleties:

- This actually only gives us a 3-form, $H$

  Is this closed?

$$\hat{E}_A := \begin{pmatrix} \hat{E}^+ \\ \hat{E}^- \end{pmatrix} = e^B \begin{pmatrix} \hat{e}^+ + i_{\hat{e}^+} g \\ \hat{e}^- - i_{\hat{e}^-} g \end{pmatrix}$$

Leibniz condition!

$$L_{\hat{E}_a^\pm} \hat{E}_b^\pm = \frac{1}{2} f_{ab}{}^c \left( 3\hat{E}_c^\pm - \hat{E}_c^\mp \right) \qquad L_{\hat{E}_a^+} \hat{E}_b^- = \frac{1}{2} f_{ab}{}^c \left( \hat{E}_c^+ + \hat{E}_c^- \right)$$

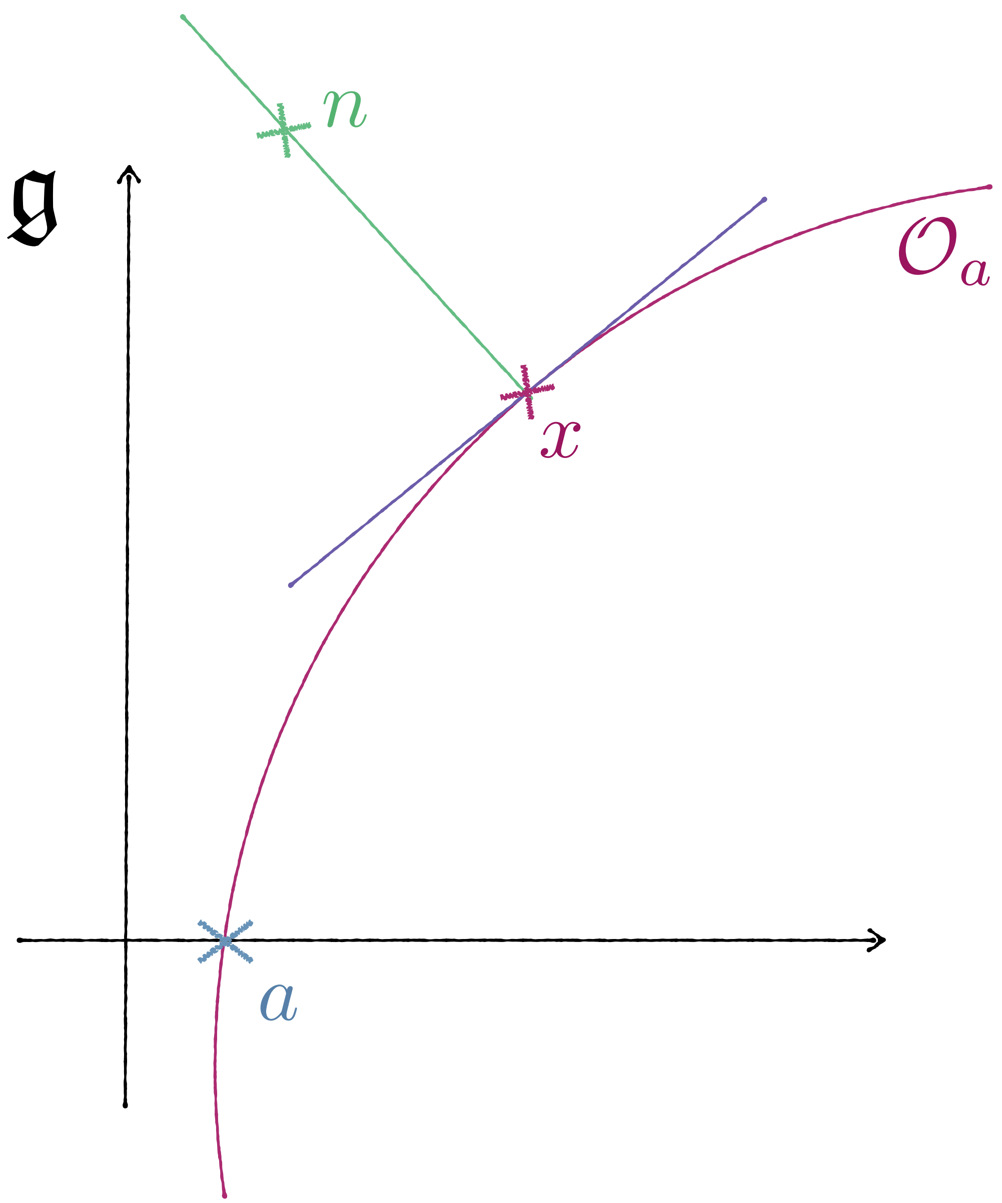
**ex:**

$$L_{\hat{E}_a^+} \hat{E}_b^+ = [\hat{e}_a^+, \hat{e}_b^+] + \mathcal{L}_{\hat{e}_a^+} i_{\hat{e}^+} g - i_{\hat{e}_b^+} d i_{\hat{e}_a^+} g - i_{\hat{e}_a^+} i_{\hat{e}_b^+} H$$

$$\overset{!}{=} [\hat{e}_a^+, \hat{e}_b^+] + \frac{1}{2} f_{ab}{}^c \left( 3 i_{\hat{e}_c^+} g + i_{\hat{e}_c^i} g \right)$$

$$L_{e^B V} e^B W = e^B \left( L_v W \right) - i_v i_w \left( H \right)$$

# Finding $B$




Subtleties:

- This actually only gives us a 3-form, $H$

  Is this closed? ✓

$$\hat{E}_A := \begin{pmatrix} \hat{E}^+ \\ \hat{E}^- \end{pmatrix} = e^B \begin{pmatrix} \hat{e}^+ + i_{\hat{e}^+} g \\ \hat{e}^- - i_{\hat{e}^-} g \end{pmatrix}$$

Leibniz condition!

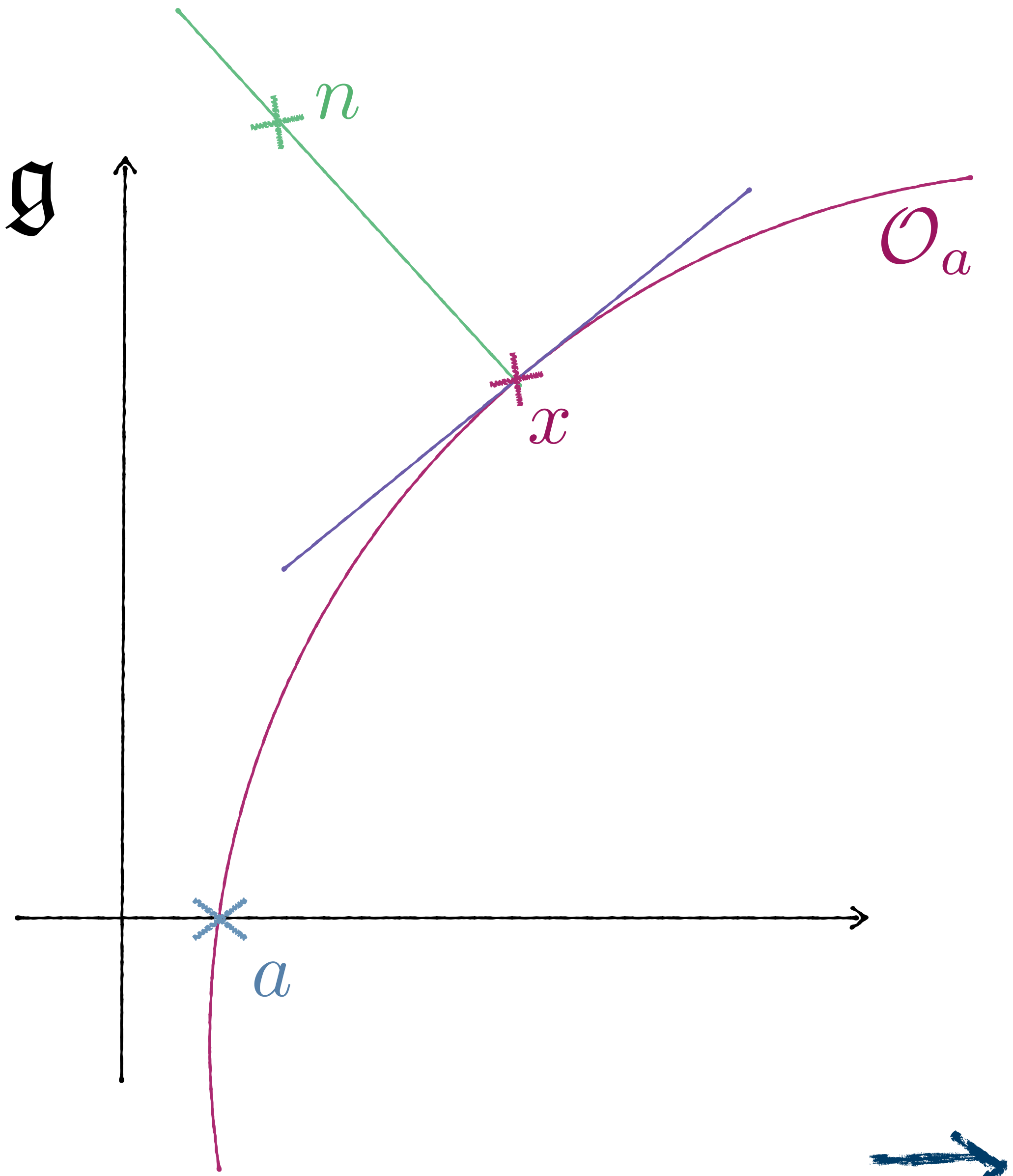
$$L_{\hat{E}^\pm_a} \hat{E}^\pm_b = \frac{1}{2} f_{ab}{}^c \left( 3\hat{E}^\pm_c - \hat{E}^\mp_c \right) \qquad L_{\hat{E}^+_a} \hat{E}^-_b = \frac{1}{2} f_{ab}{}^c \left( \hat{E}^+_c + \hat{E}^-_c \right)$$

$\Longrightarrow$

$$H = \frac{1}{12} \left[ -3 f_{abc} + f_{bc}{}^d P_{ad} + \text{cycl.} \right] i_{\hat{e}^+_a} g \wedge i_{\hat{e}^+_b} g \wedge i_{\hat{e}^+_c} g$$

After much Jacobi: $dH = 0$

# Summary




We have:

- Two sets of 2dim$G$ globally defined vectors
- A metric g defined via $g\left(\hat{e}_a^{\pm}, \hat{e}_b^{\pm}\right) = \delta_{ab}$

$$\Longrightarrow \quad \hat{E}_A := \begin{pmatrix} \hat{E}^+ \\ \hat{E}^- \end{pmatrix} = e^B \begin{pmatrix} \hat{e}^+ + i_{\hat{e}^+} g \\ \hat{e}^- - i_{\hat{e}^-} g \end{pmatrix}$$

- These satisfies Leibniz relations with the globally defined closed 3-form, $H$

$$H = \frac{1}{12}\left[-3f_{abc} + f_{bc}{}^{d} P_{ad} + \text{cycl.}\right] i_{\hat{e}_a^+} g \wedge i_{\hat{e}_b^+} g \wedge i_{\hat{e}_c^+} g$$

$\Longrightarrow$ These spaces are gen. Leibniz parallelisable!

# Some examples

$$G = SU(2)$$

- Only one kind of non-trivial orbit

$$N\mathcal{O}^{SU(2)} = \frac{SU(2)}{U(1)} \times \mathbb{R} \simeq S^2 \times \mathbb{R}$$

$$G = SU(3)$$

- Regular orbit: $\mathcal{O}^{SU(3)}_{reg.} = \dfrac{SU(3)}{U(1) \times U(1)} \simeq \mathbb{F}^3$ $\qquad N\mathcal{O}^{SU(3)}_{reg} = \mathbb{F}^3 \times \mathbb{R}^2$

- Deg. orbit: $\mathcal{O}^{SU(3)}_{deg.} = \dfrac{SU(3)}{SU(2) \times U(1)} \simeq \mathbb{CP}^2$ $\qquad N\mathcal{O}^{SU(3)}_{deg} = \pi : SU(3) \to \mathbb{CP}^2$

# Some examples

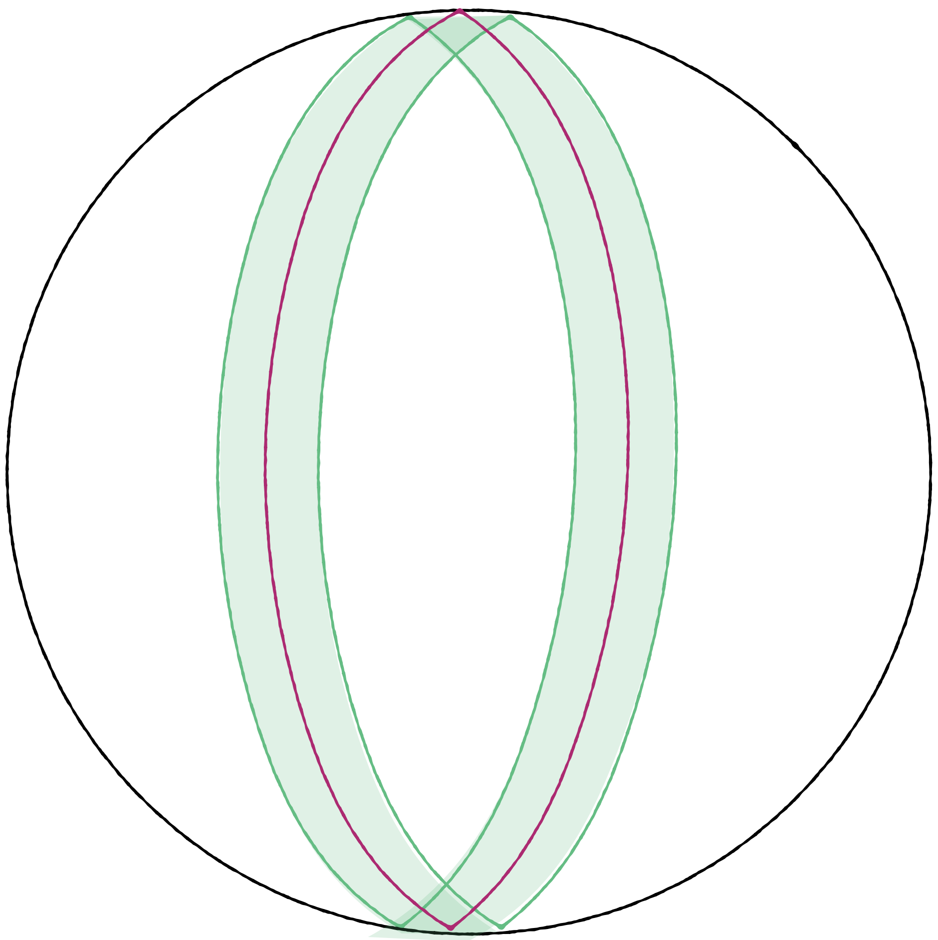
$G = SO(4)$

- Regular orbit: $N\mathcal{O}^{SO(4)}_{reg} = S^2 \times S^2 \times T^2$

$G = SO(5)$

- Regular orbit: $N\mathcal{O}^{SO(4)}_{reg} = \dfrac{SO(5)}{SO(2) \times SO(2)} \times T^2$
- Deg. orbit: $\mathcal{O}^{SO(5)}_{deg} = \dfrac{SO(5)}{SO(3) \times SO(2)} \simeq \mathbb{R}G_{3,2}$

# Summary



- Generalised geometry is a nice language for considering supergravity problems

  Algebraic $\longrightarrow$ Geometric

- Generalises Scherk-Schwartz algorithm for consistent truncations

- New generalised parallelisable spaces

  $\Rightarrow$ new consistent truncations

- Using adjoint orbits, we can explicitly construct such parallelisations

- Towards a classification!

Thank you