Introduction to connections on principal fibre bundles

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1 Introduction

We recall the basic facts of bundle theory on which this thesis is based, and introduce nomenclature. Several pictures are presented, which should be thought of as analogies for the actual mathematical structures involved in the theory. The motivation for including quite so many illustrations here is that in general, diagrams are conspicuously absent from texts on the subject, yet are central to an intuitive, geometric understanding of the subject. This novel angle is important because many of the structures involved in bundles can be described in different but equivalent ways mathematically (dual representations), and this duality is often a lot easier to understand in the context of pictures of well-known geometrical structures.

2 Principal bundles

Let (P, M, G, π) be a principal fibre bundle with total space P over base manifold M, with Lie group G acting on the right on P, and projection π. Let $q \in P$, $x = \pi(q) \in M$, $p \in \pi^{-1}(x)$. Let the group right-action on P be denoted by Φ:

$$\Phi : G \times P \to P \tag{2.1}$$

$$\Phi(g, p) = \Phi_g(p) = p \cdot g \tag{2.2}$$

We can think of the group as being an action which pushes points in the bundle around the bundle *along* the fibres. Locally we can picture the situation as in Figure 1. Note that we only consider *principal* bundles in this work.

3 Horizontal spaces, vertical spaces and connections: the two viewpoints

3.1 The vector space viewpoint

At any point q, the tangent space T_qP to the bundle can be decomposed naturally in to two spaces, one parallel to the fibre, called the vertical subspace V_qP, and one transverse

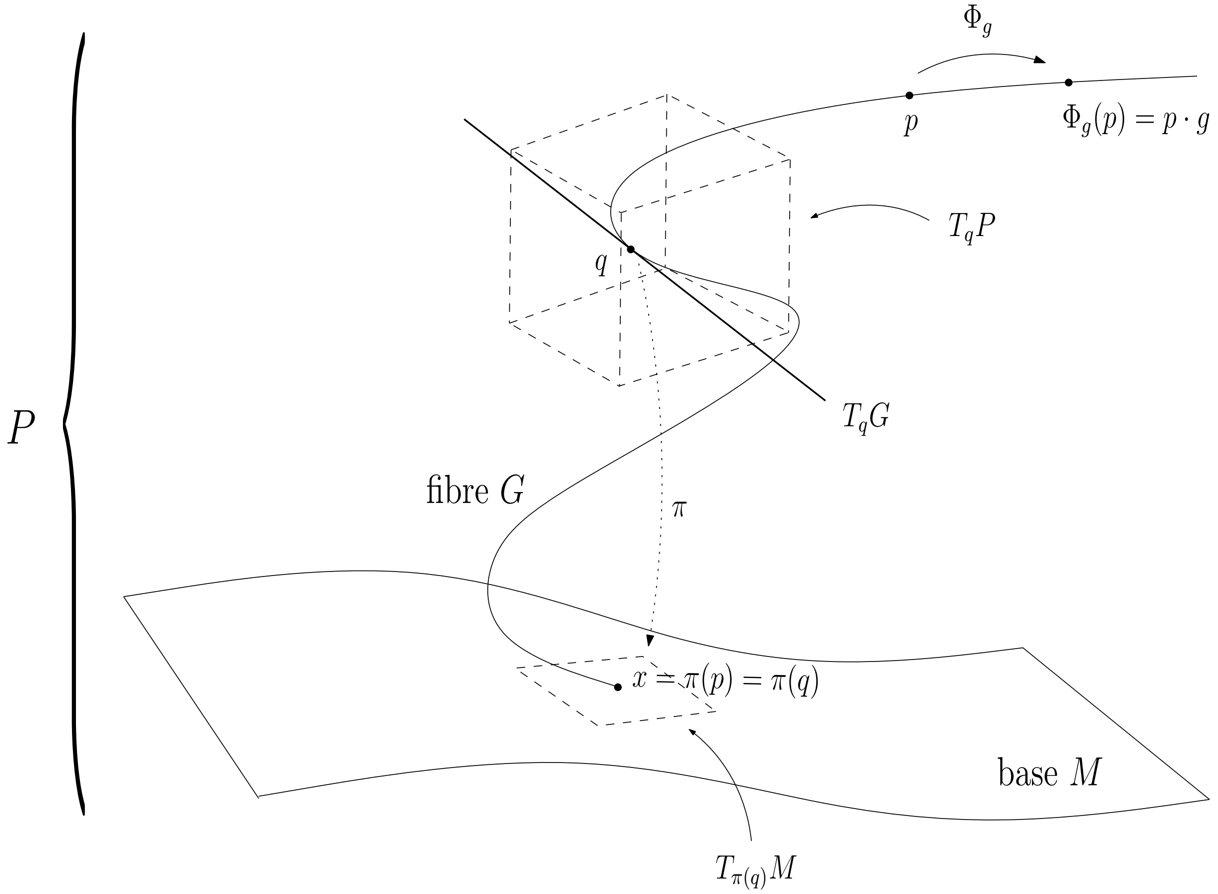

Figure 1: Visualizing a local representation of a principal bundle.

to the fibre, which, providing it satisfies certain conditions detailed below, we call the horizontal space H_qP. So we have the decomposition $T_qP = V_qP \oplus H_qP$. The vertical space is defined uniquely by

$$V_qP = \ker(\pi_*) \tag{3.3}$$

and it is clear how vertical spaces at different points in the bundle are related - they just transform smoothly with the fibre, since any parametrization of the fibre yields a parametrization of V_qP. However, there is in general no such unique way of describing the remaining space in T_qP once V_qP is taken out. But this is something we need to do in order to relate tangent spaces at different points in the bundle, and hence to define differentiation processes on the bundle. Horizontal spaces satisfy this need. The concept of the horizontal space H_qP is a way of describing these left-over spaces, of dimension $[\dim(T_qP) - \dim(T_qG)]$, in T_qP once the vertical space is taken out, which *varies smoothly* in the bundle, i.e. it gives us a consistent way of moving from fibre to fibre through the bundle. For principal bundles, in addition to being smoothly-varying, we require that H_qP is invariant under the group action. The assignment of such horizontal spaces is called a *connection* in a bundle:

Definition 3.1 *A connection in a principal bundle is a smoothly-varying assignment to each point $q \in P$ of a subspace H_qP of T_qP such that*

$$(i) \quad T_qP = V_qP \oplus H_qP \qquad \forall q \in P \tag{3.4}$$

$$(ii) \quad (\Phi_g)_*(H_qP) = H_{\Phi_g(q)}P \qquad \forall g \in G, q \in P \tag{3.5}$$

Figure 2 illustrates the situation. The first condition says that H_qP must be transverse to the fibre (which is necessary in order for it to span the rest of T_qP). The second condition says that if we use $(\Phi_g)_*$ to "push" $H_qP \subset T_qP$ *along* the fibre, then the result is the same as if we first push q along the fibre using Φ_g to the point $\Phi_g(q)$ and then form the subspace $H_{\Phi_g(q)}P$ at that point. (Note $(\Phi_g)_* : T_qP \to T_{\Phi_g(q)}P$ is the push-forward of Φ_g.) So a connection in a principal bundle is just a right-invariant distribution on the bundle which is transverse to the fibre at each point.

Remark 3.1 *An equivalent condition to condition (i) above is*

$$(i) \ \pi_*(H_qP) = T_{\pi(q)}M \qquad \forall q \in P \tag{3.6}$$

This just says that the dimension of H_qP has to be great enough to fill up the rest of T_qP completely once T_qG is taken out: since π_* maps T_qG to zero, and locally the bundle is a product space, the image of H_qP under π_* must have the same dimension as $T_{\pi(q)}M$ (i.e they are isomorphic).

3.2 The differential form viewpoint

Now, since the bundle group is a Lie group, there is a canonical identification of the tangent space to the fibre at each point q with the Lie algebra $\mathfrak{g}$ of G, so we can write $T_qG \cong \mathfrak{g}$, which leads to the following differential-form-based description of a connection.

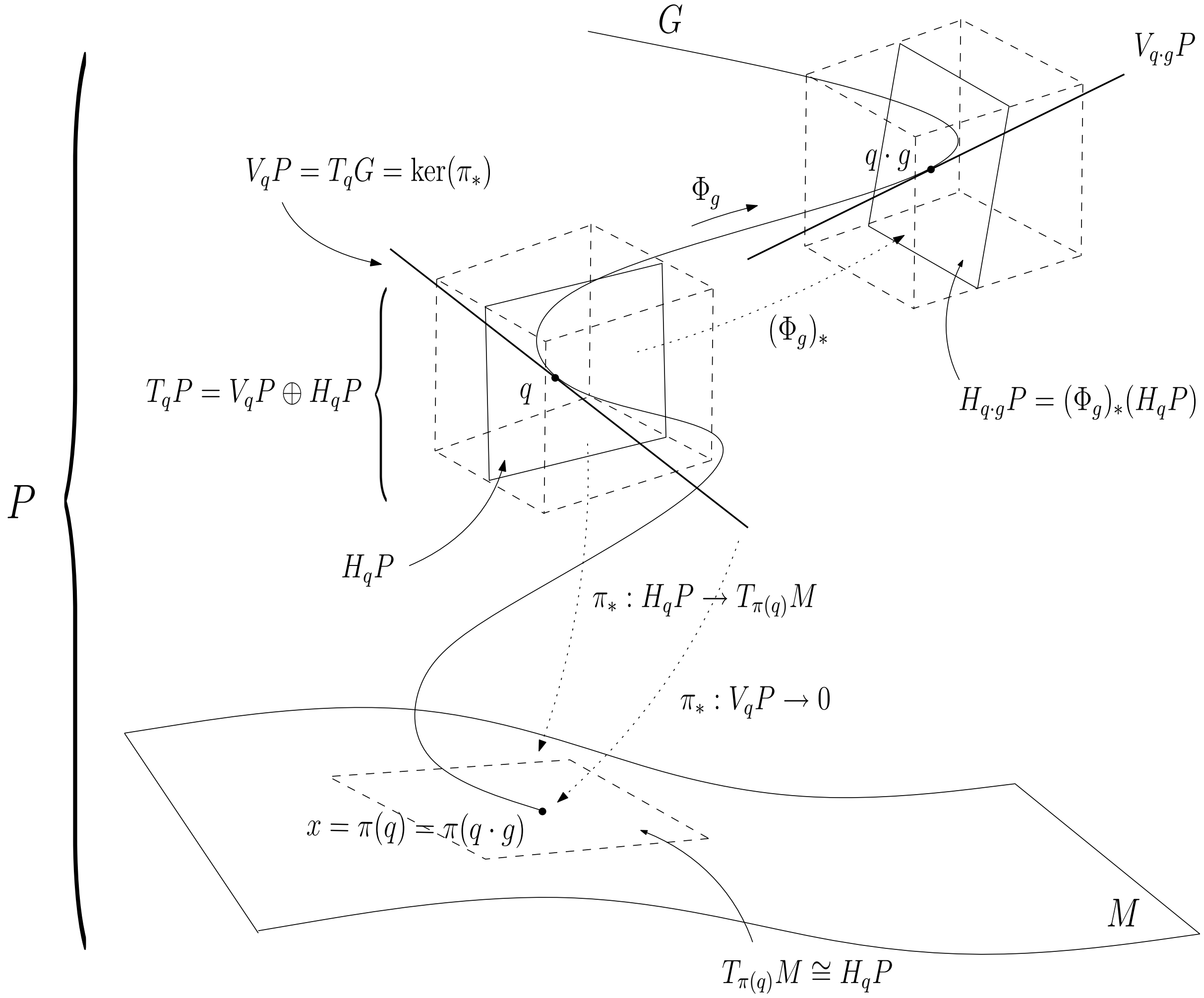

Figure 2: Horizontal spaces.

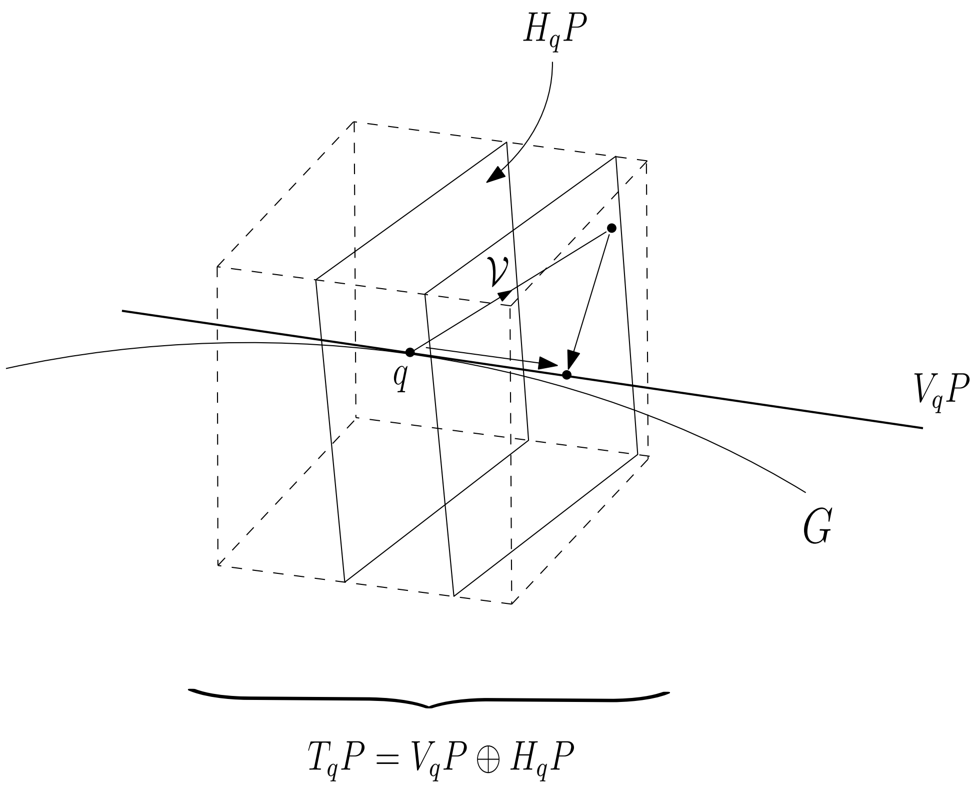

Figure 3: Using H_qP to project $\mathcal{V} \in T_qP$ on to V_qP.

From Figure 2 it is clear that a choice of H_qP in fact defines a projection of T_qP on to T_qG, and hence on to $\mathfrak{g}$. Figure 3 shows this explicitly: $\mathcal{V}$ is a vector in T_qP which is projected via H_qP to a vector in V_qP. This is also intuitively clear for higher dimension situations, not only that pictured, since we are just dealing with intersections of linear spaces. We thus have a linear map $T_qP \to \mathfrak{g}$, i.e. a $\mathfrak{g}$-valued differential 1-form on P, with kernel H_qP. This representation of a connection as a differential form is called a *connection 1-form*, but we need a few more definitions before we can describe it fully: let $\mathfrak{g} \cong T_eG$ be the Lie algebra of G, where e is the identity in G, let $\mathfrak{X}_L(G)$ be the set of all left-invariant vector fields on G. Take $\xi \in \mathfrak{g}$. Let $X_\xi \in \mathfrak{X}_L(G)$ be the unique left-invariant vector field on G corresponding to ξ, i.e such that $X_\xi(e) = \xi$. Each $\xi \in \mathfrak{g}$ induces a flow on P. Let $g_\xi(t)$ be the unique integral curve of X_ξ passing at $t = 0$ through $e \in G$, then $g_\xi(t)$ is a one-parameter subgroup of G.

Definition 3.2 *The exponential map* $\exp : \mathfrak{g} \to G$ *is defined by*

$$\exp(\xi) = g_\xi(1) \tag{3.7}$$

and it can be shown that

$$\exp(t\xi) = g_\xi(t) \in G \tag{3.8}$$

Figure 4 gives an idea of how these objects can be represented geometrically.

Definition 3.3 *The infinitesimal generator of the action* Φ*, corresponding to* ξ*, is a vector field* ξ_P *on* P *defined by*

$$\xi_P(q) = \frac{d}{dt}\Phi(g_\xi(t), q)\Big|_{t=0} \tag{3.9}$$

and so for $\Phi(g, q) = q \cdot g$ we get

$$\begin{aligned} \xi_P(q) &= \frac{d}{dt}\Phi(g_\xi(t), q)\Big|_{t=0} && (3.10)\\ &= \frac{d}{dt}\big(q \cdot g_\xi(t)\big)\Big|_{t=0} && (3.11)\\ &= \frac{d}{dt}\big(q \cdot \exp(t\xi)\big)\Big|_{t=0} && (3.12)\\ &= q\,\xi && (3.13) \end{aligned}$$

and we can now proceed to the formal definition of a connection 1-form on a principal bundle:

Definition 3.4 *A connection 1-form on* P *is a* $\mathfrak{g}$*-valued 1-form* $\omega_q : T_qP \to \mathfrak{g}$*, satisfying, for each* $q \in P$*,*

$$(i) \quad \omega_q(\xi_P(q)) = \xi \qquad \forall \xi \in \mathfrak{g} \tag{3.14}$$

$$(ii) \quad \omega_{\Phi_g(q)}([(\Phi_g)_*\big|_q]\mathcal{V}) = \mathrm{Ad}_g(\omega_q(\mathcal{V})) \qquad \forall \mathcal{V} \in T_qP,\ \forall g \in G \tag{3.15}$$

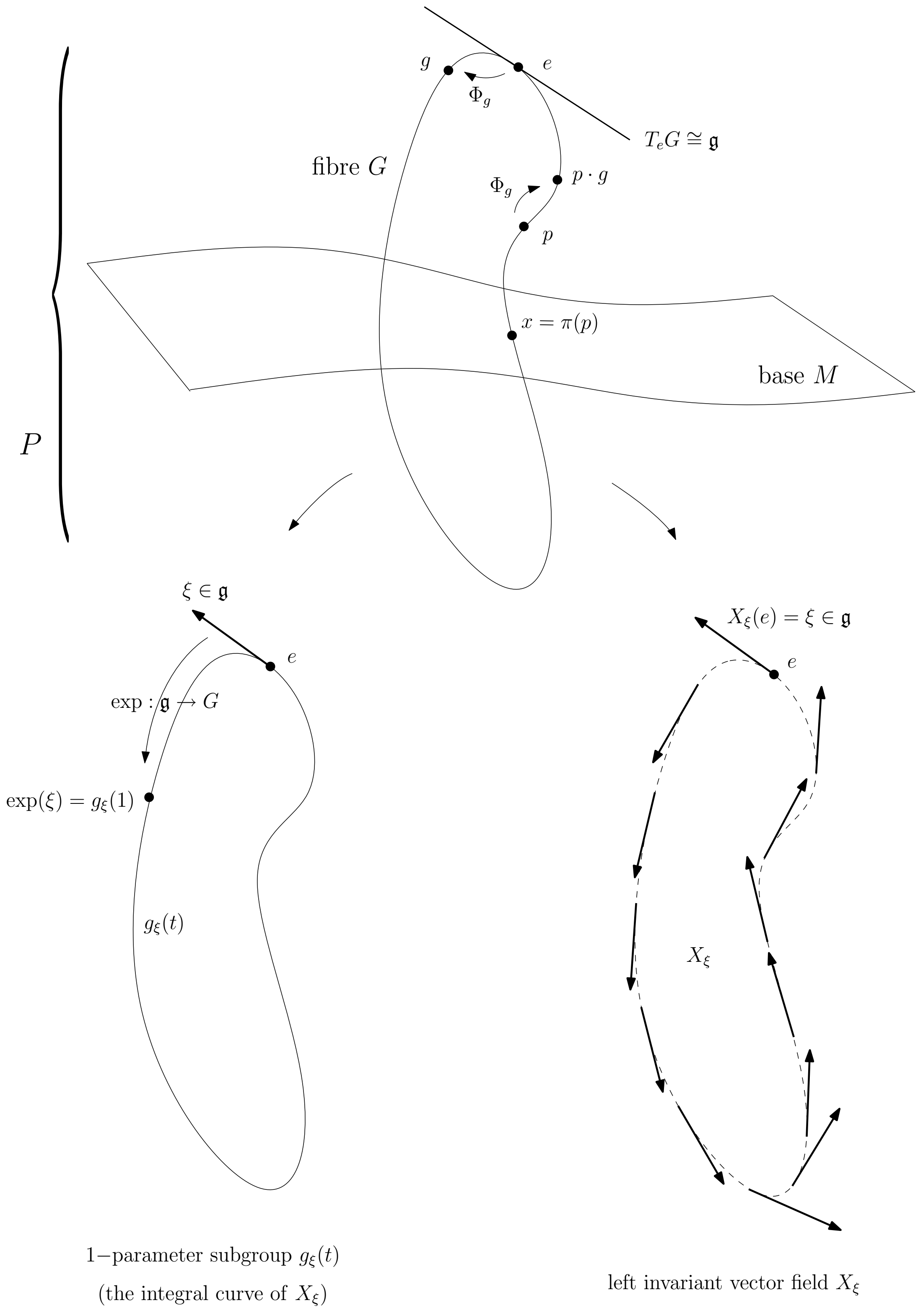

Figure 4: A right-action on a principal fibre bundle.

$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the adjoint action at $g \in G$ (i.e. the derivative of conjugation, evaluated at e), which in our case will only ever be given by

$$\mathrm{Ad}_g(\sigma) = g^{-1}\sigma g \qquad g \in G,\ \sigma \in \mathfrak{g}, \tag{3.16}$$

which holds for matrix groups. So condition (ii) becomes:

$$\omega_{\Phi_g(q)}([(\Phi_g)_*\big|_q]\mathcal{V}) = \Phi_g^{-1}(\omega_q(\mathcal{V}))\Phi_g \qquad \forall \mathcal{V} \in T_qP,\ \forall g \in G \tag{3.17}$$

We can visualize this as in Figure 5, which we can very loosely think of as: if you apply ω_{qg} to the tangent vector $((\Phi_g)_*\mathcal{V})$ at $q{\cdot}g \in G$, then this is the same as if you start at $q{\cdot}g \in G$, then transform *back* along the fibre under g^{-1} *then* apply ω_q to the tangent vector $\mathcal{V}$ at the original point q, then transform *forwards* along the fibre under g returning to $q \cdot g$. This characterization of the connection, which is essentially based on tangent covectors,

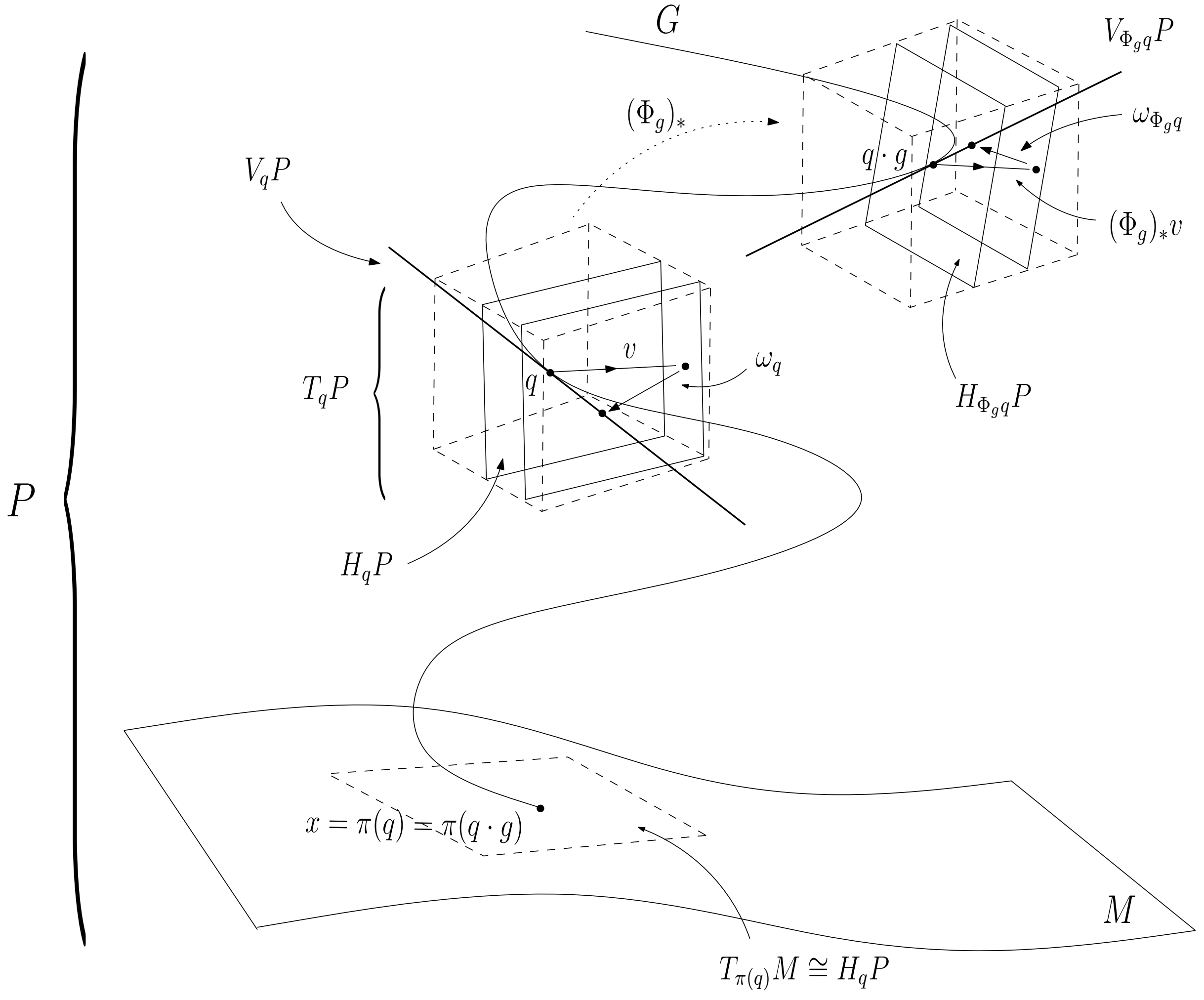

Figure 5: G-invariance of the connection form.

is just the representation dual to the previous one (Definition 3.1), which is essentially based on tangent vectors. Notice in particular the duality between equations (3.5) and (3.17).

Remark 3.2 *If we do not need to specify the point at which a connection form is evaluated, we may denote it by just* ω *instead of* ω_q.

References [3]-[5] provide good material on manifolds and bundles.

4 Parallel translation

In the context of all these diagrams depicting horizontal subspaces, the notions of horizontality and parallel translation become very easy to understand. Suppose we now have a path in the bundle given by $t \mapsto q(t) \in P$ with $t \in [c, d] \subset \mathbb{R}$.

Definition 4.1 *Path* $q(t) \in P$ *is horizontal w.r.t a given connection if the tangent vector* $\dot{q}(t)$ *lies in the horizontal subspace determined by the connection,* $H_{q(t)}P$, *for each* t.

Remark 4.1 *Recall, horizontal subspaces are by definition in the kernel of the associated connection 1-form* ω, *so path* $q(t)$ *is horizontal w.r.t* ω *iff* $\omega(\dot{q}(t)) = 0 \ \forall t$. *i.e. the projection of the tangent vector* $\dot{q}(t)$ *on to the vertical space at* $q(t)$ *is zero, at every point of the path.*

Figure 6 illustrates a horizontal path: $q(t)$ is a path through bundle P, and at each of the three points $q(t_1), q(t_2), q(t_3)$ the tangent space to the bundle is depicted (as a dashed-line box), along with the horizontal and vertical subspaces $H_{q(t_i)}$ and $V_{q(t_i)}$. Also shown are the tangent vectors to the path at the three points - these tangent vectors lie in the horizontal subspaces, representing the horizontal nature of the path. Now, since there is in general some flexibility in the choice of horizontal spaces in a bundle (subject to the conditions of the definition), a path can be horizontal w.r.t one connection while simultaneously being not horizontal w.r.t another. In figure 6, we would represent a connection w.r.t which $q(t)$ is not horizontal by tilting the horizontal space inside one or more of the "tangent space boxes" by some angle (or angles), since then the tangent vectors would not lie inside the horizontal spaces for every t.

Now, if path $q(t)$ is not horizontal w.r.t a given connection ω, then we can derive an expression which describes how far it deviates from being horizontal. We do this by considering the separation, in the fibre, between $q(t)$ and a second path, $\hat{q}(t)$, which starts at the same point $q(c) = \hat{q}(c)$ in the bundle, and *is* horizontal. Any path in bundle P which maps, under π, to the same path on base M as does $q(t)$ is called a *lift* of $q(t)$. If a lift of $q(t)$ is horizontal w.r.t ω then it is called a *horizontal lift* of $q(t)$, and translating (i.e. evaluating) some quantity along a horizontal path (as opposed to a non-horizontal path) is called *parallel translation.* So $\hat{q}(t)$ is a horizontal lift of $q(t)$ and we analyze it as follows. Using local coordinates for the local product structure in the bundle, every point on path $\hat{q}(t)$ can be written as the product of the point $q(t)$ and some element of the fibre, say $a \in G$. But since this is the case for *every* point along the path, we get a whole curve $a(t) \in G$ which measures the difference between the two curves in the bundle:

$$\hat{q}(t) = q(t)a(t) \tag{4.18}$$

We can think of $a(t)$ as a sort of "fine-tuning" variable, which, when varied by the right amount as $q(t)$ proceeds along its course, enables us to shift $\hat{q}(t)$ around the fibre exactly

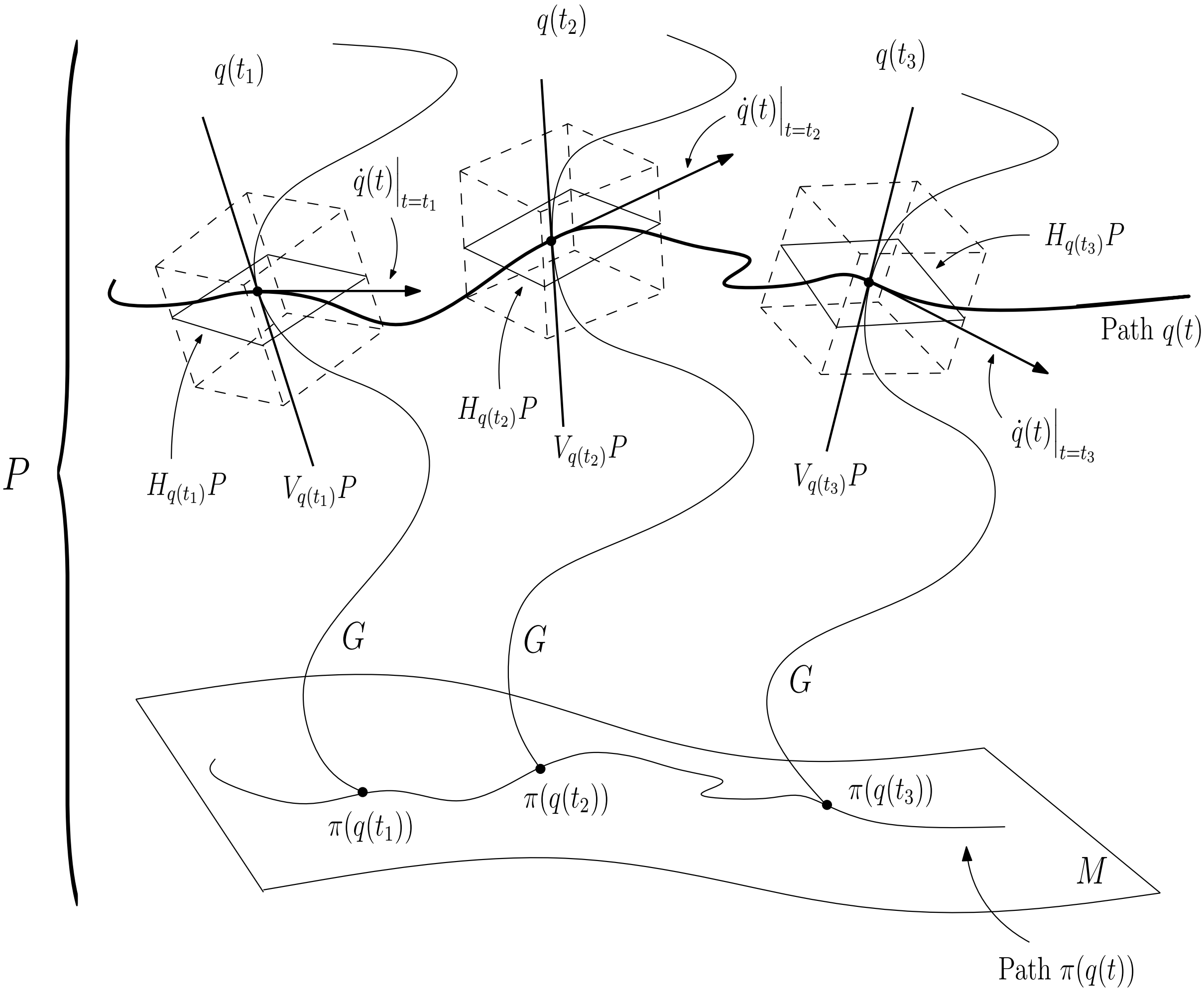

Figure 6: A horizontal path $q(t)$ through principal bundle (P, M, π, G).

the right amount necessary for it to remain horizontal. $a(t)$ will then be an expression describing how much $q(t)$ deviates from being horizontal. To obtain the equation which governs this required motion in the fibre, we differentiate (4.18) w.r.t t, apply the connection form ω, then use the horizontal and right-equivariance conditions, '$\omega(\dot{\hat{q}}(t)) = 0$' and (3.17) respectively. This yields the first-order differential equation for $a(t)$:

$$\dot{a}(t)a^{-1}(t) = -\omega(\dot{q}(t)) \tag{4.19}$$

with $a(c) = e$ required to satisfy $q(c) = \hat{q}(c)$. Detailed derivations of this equation are given in [2] (p.69), [1] (p.265) and [4] (p.364). This equation shows us precisely how $a(t)$ must vary as we move along $q(t)$ in order for $\hat{q}(t)$ to be the unique horizontal lift starting at $q(c)$. We summarize the concepts of horizontal lifts and parallel translation in figure 7.

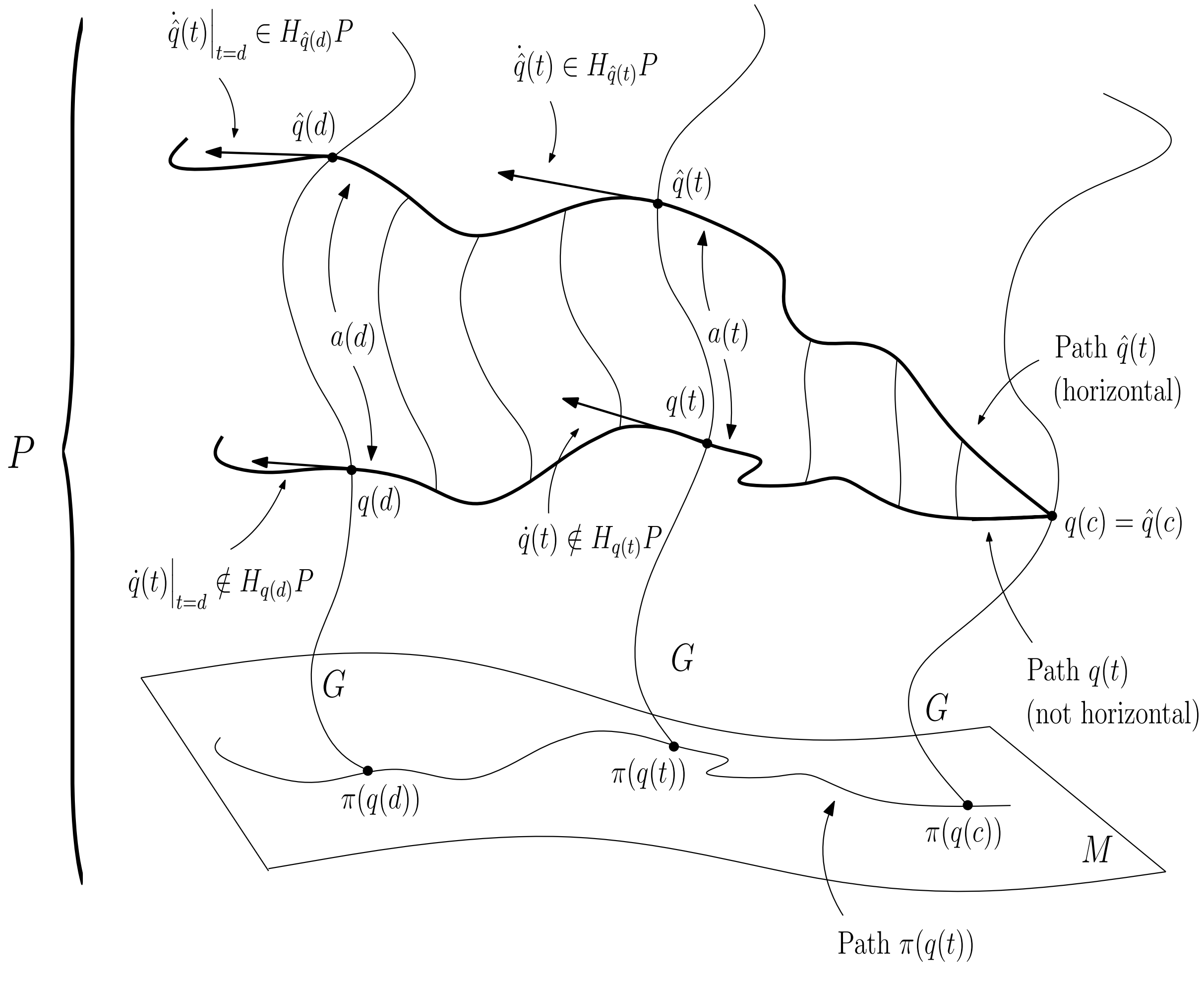

Figure 7: Generating a horizontal path $\hat{q}(t)$ from a non-horizontal path $q(t)$ by applying a shift $a(t)$ in the fibre.

Remark 4.2 *This method for generating horizontal paths also applies for paths $\hat{q}(t)$ which start at points in the fibre attached to $q(c)$, other than $q(c)$ itself. This just involves* *taking the initial condition $a(c)$, for (4.19), to be some value other than the identity in G, and would be represented on figure 7 by sliding the path $\hat{q}(t)$ up the fibre G so that $q(c)$ and $\hat{q}(c)$ are separated by a value $a(c)$ in the fibre.*

Remark 4.3 *It is important to remember that the diagrams in this chapter are merely pictorial analogies for the mathematical objects involved. In particular, fibres are always pictured here as being 1-dimensional, when in reality they can be of any dimension. For example, in Figure 7, the shift in the fibre is portrayed as being a simple shift up or down the fibre, but really the shift $a(t)$ is a path in the group G, whatever that may be.*

This concludes our review of fibre bundles, and our collection of very clear diagrams showing previously uncollated (if not unidentified) ways of thinking about the basic geometry of fibre bundles.

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