The intrinsic second derivative

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1 Introduction

For a smooth mapping $\mathbf{F} : M \to N$ between two smooth finite-dimensional manifolds M and N, the second derivative is not intrinsic. In the overlap between charts an affine contribution appears which differs in different charts. However, in the case where the first derivative has a non-trivial kernel there is an intrinsic second derivative (and higher derivatives) when restricted to the kernel. PORTEOUS [5] discovered this intrinsic second derivative and proved the general case. It has been widely used in singularity theory (e.g. [4, 1]). It has recently been used in the study of degenerate relative equilibria [2] and in the study of degenerate conservation laws with dissipation [3]. In this report a self-contained proof of the intrinsic second derivative, for the case of mappings between vector spaces of the same dimension, is given as this is the special case needed in [2, 3].

2 Transformation of mappings

Let $\mathbb{X}$ and $\mathbb{Y}$ be $n-$dimensional vector spaces, and let $\mathbb{X}^*$ and $\mathbb{Y}^*$ be their respective dual spaces. Denote the respective pairings by $\langle\cdot,\cdot\rangle_{\mathbb{X}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{Y}}$. Identify the tangent space of $\mathbb{X}$ with $\mathbb{X}$, and the tangent space of $\mathbb{Y}$ with $\mathbb{Y}$.

Introduce the smooth mapping

$$\mathbf{F} \;:\; \mathbb{X} \to \mathbb{Y}\,, \tag{1}$$

The first derivative of $\mathbf{F}$ at a point $\mathbf{U} \in \mathbb{X}$ in the direction $\boldsymbol{\xi} \in \mathbb{X}$ is

$$\mathrm{D}\mathbf{F}(\mathbf{U})\boldsymbol{\xi} = \frac{d}{d\varepsilon}\mathbf{F}(\mathbf{U}+\varepsilon\boldsymbol{\xi})\bigg|_{\varepsilon=0}\,. \tag{2}$$

Similarly, for tangent vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ the second derivative at $\mathbf{U} \in \mathbb{X}$ is

$$\mathrm{D}^2\mathbf{F}(\mathbf{U})[\boldsymbol{\xi}_1,\boldsymbol{\xi}_2] = \frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\mathbf{F}(\mathbf{U}+\varepsilon_1\boldsymbol{\xi}_1+\varepsilon_2\boldsymbol{\xi}_2)\bigg|_{\varepsilon_1=\varepsilon_2=0}\,. \tag{3}$$

The problem of interest is how these derivatives tranform when $\mathbf{F}$ is transformed. Introduce additional $n-$dimensional vector spaces $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ with their duals and appropriate pairings. Introduce the diffeomorphisms

$$\Phi \;:\; \widetilde{\mathbb{X}} \to \mathbb{X} \quad \text{and} \quad \Psi \;:\; \mathbb{Y} \to \widetilde{\mathbb{Y}}\,.$$

Then the transformation of $\mathbf{F}$

$$\mathbf{G} = \Psi \circ \mathbf{F} \circ \Phi \,, \tag{4}$$

results in a mapping

$$\mathbf{G} \,:\, \widetilde{\mathbb{X}} \to \widetilde{\mathbb{Y}} \,. \tag{5}$$

3 Transformed first derivative

Write the transformation (4) as

$$\mathbf{G}(\mathbf{V}) = \Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big) \,, \quad \text{for} \quad \mathbf{V} \in \widetilde{\mathbb{X}} \,.$$

Then for any $\boldsymbol{\eta} \in \widetilde{\mathbb{X}}$,

$$\mathbf{G}(\mathbf{V} + \varepsilon\boldsymbol{\eta}) = \Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V} + \varepsilon\boldsymbol{\eta})\big)\Big) \,,$$

Differentiate with respect to ε and set $\varepsilon = 0$,

$$\mathrm{D}\mathbf{G}(\mathbf{V})\boldsymbol{\eta} = \mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\big[\mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}\big]\Big] \,. \tag{6}$$

3.1 Kernel of the first derivative

Suppose that $\mathrm{D}\mathbf{G}$ has a non-trivial kernel. For simplicity assume that the kernel of $\mathrm{D}\mathbf{G}$ and the kernel of $\mathrm{D}\mathbf{G}^*$ are one-dimensional.

Denote the kernel of $\mathrm{D}\mathbf{G}(\mathbf{V})$ for some fixed $\mathbf{V}$ by $\mathrm{span}\{\boldsymbol{\eta}\}$. Then, since Ψ is a diffeomorphism, $\mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}$ is in the kernel of $\mathrm{D}\mathbf{F}(\mathbf{U})$ for $\mathbf{U} = \Phi(\mathbf{V})$. Let $\boldsymbol{\xi} = \mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}$. Then, $\boldsymbol{\xi} \in \mathbb{X}$ and

$$\boldsymbol{\eta} \in \mathrm{Ker}(\mathrm{D}\mathbf{G}(\mathbf{V})) \quad \Leftrightarrow \quad \boldsymbol{\xi} \in \mathrm{Ker}(\mathrm{D}\mathbf{F}(\mathbf{U})) \,. \tag{7}$$

A similar relation holds for the adjoint eigenvector. To establish this, let $\boldsymbol{\eta}$ now be an arbitrary vector in $\widetilde{\mathbb{X}}$ – not necessarily in the kernel of $\mathrm{D}\mathbf{G}(\mathbf{V})$. But suppose $\boldsymbol{\zeta}$ is in the kernel of $\mathrm{D}\mathbf{G}(\mathbf{V})^*$ for some $\mathbf{V}$. Then acting on the first derivative in (6),

$$\langle\boldsymbol{\zeta}, \mathrm{D}\mathbf{G}(\mathbf{V})\boldsymbol{\eta}\rangle_{\widetilde{\mathbb{Y}}} = \Big\langle\boldsymbol{\zeta}, \mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\big[\mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}\big]\Big]\Big\rangle_{\widetilde{\mathbb{Y}}} \,,$$

where $\langle\cdot,\cdot\rangle_{\widetilde{\mathbb{Y}}}$ is a pairing on $\widetilde{\mathbb{Y}}$. The left-hand side vanishes. Now, define

$$\boldsymbol{\gamma} = \mathrm{D}\Psi(\mathbf{F})^*\boldsymbol{\zeta} \,.$$

Then, since Ψ is a diffeomorphism, $\boldsymbol{\gamma}$ is in the kernel of $\mathrm{D}\mathbf{F}(\mathbf{U})^*$ for $\mathbf{U} = \Phi(\mathbf{V})$. In summary, for $\boldsymbol{\zeta} \in \widetilde{\mathbb{Y}}^*$ and $\boldsymbol{\gamma} \in \mathbb{Y}^*$, we have

$$\boldsymbol{\zeta} \in \mathrm{Ker}\big(\mathrm{D}\mathbf{G}(\mathbf{V})^*\big) \quad \Leftrightarrow \quad \boldsymbol{\gamma} \in \mathrm{Ker}\big(\mathrm{D}\mathbf{F}(\mathbf{U})^*\big) \,. \tag{8}$$

4 Second derivative

For the second derivative, take

$$\mathbf{V} \mapsto \mathbf{V} + \varepsilon_1 \boldsymbol{\eta}_1 + \varepsilon_2 \boldsymbol{\eta}_2\,,$$

in (4) and form the second derivative,

$$\begin{aligned} \mathrm{D}^2\mathbf{G}(\mathbf{V})\,\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\} &= \mathrm{D}^2\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big\{\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\big[\mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}_1\big],\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\big[\mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}_2\big]\Big\}\,. \\ &\quad +\mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}^2\mathbf{F}\big(\Phi(\mathbf{V})\big)\big\{\mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}_1,\mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}_2\big\}\Big] \\ &\quad +\mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\big[\mathrm{D}^2\Phi(\mathbf{V})\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\}\big]\Big]\,. \end{aligned}$$

Now, define

$$\boldsymbol{\xi}_1 := \mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}_1 \quad \text{and} \quad \boldsymbol{\xi}_2 := \mathrm{D}\Phi(\mathbf{V})\boldsymbol{\eta}_2\,,$$

and substitute into the right-hand side

$$\begin{aligned} \mathrm{D}^2\mathbf{G}(\mathbf{V})\,\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\} &= \mathrm{D}^2\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big\{\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\boldsymbol{\xi}_1,\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\boldsymbol{\xi}_2\Big\}\,. \\ &\quad +\mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}^2\mathbf{F}\big(\Phi(\mathbf{V})\big)\big\{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\big\}\Big] \\ &\quad +\mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\big[\mathrm{D}^2\Phi(\mathbf{V})\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\}\big]\Big]\,. \end{aligned}$$

The form of the second derivative on the right-hand side is dramatically different from the left-hand side. However, suppose that one of the tangent vectors $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2$ is in the kernel of $\mathrm{D}\mathbf{G}(\mathbf{V})$. For definiteness suppose $\boldsymbol{\eta}_1$ is in the kernel and $\boldsymbol{\eta}_2$ is not. Then the second derivative expression simplifies to

$$\begin{aligned} \mathrm{D}^2\mathbf{G}(\mathbf{V})\,\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\} &= \mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}^2\mathbf{F}\big(\Phi(\mathbf{V})\big)\big\{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\big\}\Big] \\ &\quad +\mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\big[\mathrm{D}^2\Phi(\mathbf{V})\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\}\big]\Big]\,. \end{aligned} \tag{9}$$

4.1 The intrinsic second derivative

The second term on the right-hand side (9) is the problem term, since it involves the first derivative of $\mathbf{F}$ evaluated on a vector that does not vanish in general. However by using the adjoint eigenvector the second term can be eliminated.

Use the pairing on $\widetilde{\mathbb{Y}}$ and pair an adjoint eigenvector $\boldsymbol{\zeta}$ with the left hand side, noting that an adjoint eigenvector is generated for $\mathrm{D}\mathbf{F}$ via (8). The above expression (9) simplifies to

$$\begin{aligned} \big\langle \boldsymbol{\zeta}, \mathrm{D}^2\mathbf{G}(\mathbf{V})\,\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\}\big\rangle_{\widetilde{\mathbb{Y}}} &= \Big\langle \boldsymbol{\zeta}, \mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}^2\mathbf{F}\big(\Phi(\mathbf{V})\big)\big\{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\big\}\Big]\Big\rangle_{\widetilde{\mathbb{Y}}} \\ &\quad +\Big\langle \boldsymbol{\gamma}, \mathrm{D}\mathbf{F}\big(\Phi(\mathbf{V})\big)\big[\mathrm{D}^2\Phi(\mathbf{V})\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\}\big]\Big\rangle_{\mathbb{Y}}\,, \end{aligned}$$

noting that the second pairing is on $\mathbb{Y}$. Now, since $\boldsymbol{\gamma}$ is an adjoint eigenvector of $\mathrm{D}\mathbf{F}(\Phi)$, the second term on the right-hand side vanishes and we are left with

$$\langle \boldsymbol{\zeta}, \mathrm{D}^2\mathbf{G}(\mathbf{V})\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\}\rangle_{\widetilde{\mathbb{Y}}} = \left\langle \boldsymbol{\zeta}, \mathrm{D}\Psi\Big(\mathbf{F}\big(\Phi(\mathbf{V})\big)\Big)\Big[\mathrm{D}^2\mathbf{F}\big(\Phi(\mathbf{V})\big)\{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\}\Big]\right\rangle_{\widetilde{\mathbb{Y}}}$$

or

$$\langle \boldsymbol{\zeta}, \mathrm{D}^2\mathbf{G}(\mathbf{V})\{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\}\rangle_{\widetilde{\mathbb{Y}}} = \langle \boldsymbol{\gamma}, \mathrm{D}^2\mathbf{F}\big(\mathbf{U}\big)\{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\}\rangle_{\mathbb{Y}}$$

In summary, the second derivative has the same form in both coordinate systems when it is evaluated on the kernel and co-kernel of first derivative.

A similar argument carries over if the kernel has higher dimension (see [5]).

5 The case of mappings between manifolds

Suppose $\mathbf{F} : M \to N$ is a smooth mapping between smooth manifolds M and N, each of finite dimension, but not necessarily equal dimension. By restricting to charts on M and N, one encounters a composition of the form (4) in the overlap between charts. In general the second derivative is not intrinsic. On the other hand, if there is a nontrivial kernel of the first derivative, then there is a well-defined intrinsic second derivative of the mapping $\mathbf{F}$. A proof for the case of the second derivative is given in §4 of [4] and the general case is treated in [5].

References

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