Shallow water sloshing in rotating vessels: details of the numerical algorithm

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1 Introduction

Details of the numerical algorithm used to solve the rotating shallow water equations (SWEs) in [3, 4] are recorded in this report. The scheme is a fully-implicit split-step scheme with second order accuracy in both space and time that has been widely used in hydraulics [5, 1, 2]. The nonlinearity is treated in the Eulerian representation using iteration. The new features in the algorithm here are (a) the inclusion of full time-dependent rotation due to the rigid body motion of the vessel, and (b) the use of exact boundary conditions in both steps of the split-step scheme.

First, in §2 the algorithm for the SWEs in one space dimension is outlined. This algorithm is a streamlined version of the algorithm used in [3]. The simplification introduced here is that the nonlinear term $Uh_x + hU_x$ is replaced by the linear term $-h_t$ in the $U-$momentum equation. The algorithm in §2 is also a special case of the algorithm for the SWEs in two space dimensions in §3 since that algorithm is based on splitting and one-dimensional sub-integrations.

2 Sloshing SWEs in one space dimension

The governing SWEs derived in [3] are

$$\begin{aligned} h_t + Uh_x + hU_x &= 0 \\ U_t + (\alpha(x,t) - \Omega^2 h)h_x + UU_x - 2\Omega h_t &= \beta(x,t) + \dot{\Omega} h \,, \end{aligned} \tag{2.1}$$

where $h(x,t)$ is the depth and $U(x,t)$ is the horizontal fluid velocity at the free surface. The tank has length L and the only boundary conditions are

$$U(0,t) = U(L,t) = 0 \,, \quad \forall\, t \,. \tag{2.2}$$

The functions $\alpha(x,t)$ and $\beta(x,t)$ are the terms due to the rotating-translating frame that do not depend on either h or U,

$$\begin{aligned} \alpha(x,t) &= g\cos\theta + \dot{\Omega}(x+d_1) - \Omega^2 d_2 - \ddot{q}_1 \sin\theta + \ddot{q}_2 \cos\theta, \\ \beta(x,t) &= -g\sin\theta + \dot{\Omega} d_2 + \Omega^2(x+d_1) - \ddot{q}_1\cos\theta - \ddot{q}_2 \sin\theta \,. \end{aligned} \tag{2.3}$$

The various terms and parameters are defined in [3].

The $x-$interval $0 \leq x \leq L$ is split into $JJ-1$ intervals of length $\Delta x = \frac{L}{JJ-1}$ and so

$$x_j := (j-1)\Delta x\,, \quad j = 1,\ldots,JJ \quad \text{and} \quad t_n = n\Delta t\,,$$

with $n = 0, 1, \ldots,$ and

$$h_j^n := h(x_j, t_n) \quad \text{and} \quad U_j^n := U(x_j, t_n)\,.$$

The first-order space derivatives are approximated by $3-$point centred differences and the first-order time derivatives by forward difference. The scheme is fully implicit.

The discretization of the equations (2.1) is then

$$\begin{aligned}
\frac{h_j^{n+1}-h_j^n}{\Delta t} + U_j^\star \frac{h_{j+1}^{n+1}-h_{j-1}^{n+1}}{2\Delta x} + h_j^\star \frac{U_{j+1}^{n+1}-U_{j-1}^{n+1}}{2\Delta x} &= 0\,,\\
\frac{U_j^{n+1}-U_j^n}{\Delta t} + \left(\alpha_j^{n+1} - (\Omega^{n+1})^2 h_j^\star\right) \frac{h_{j+1}^{n+1}-h_{j-1}^{n+1}}{2\Delta x} & \\
+U_j^\star \frac{U_{j+1}^{n+1}-U_{j-1}^{n+1}}{2\Delta x} - 2\Omega^{n+1}\frac{h_j^{n+1}-h_j^n}{\Delta t} &= \beta_j^{n+1} + \dot{\Omega}^{n+1} h_j^{n+1}\,,
\end{aligned} \tag{2.4}$$

where

$$\alpha_j^n := \alpha(x_j, t_n) \quad \text{and} \quad \beta_j^n := \beta(x_j, t_n)\,.$$

The starred variables are intermediate values for nonlinear coefficients. In order to treat the nonlinearity, an iteration scheme is used. First, the above equations are solved for one time step with $h_j^\star = h_j^n$ and $U_j^\star = U_j^n$ producing an approximation for h_j^{n+1} and U_j^{n+1}. The equations are then solved again with the starred variables replaced by the updates. This iteration step is repeated until the previous and current values of h and U are within a prescribed tolerance at all points.

The discrete system (2.4) can be put into block tridiagonal form. Setting

$$\mathbf{z}_j^n = \begin{bmatrix} h_j^n \\ U_j^n \end{bmatrix},$$

equation (2.4) can be expressed in the form

$$-{}_\star\mathbf{A}_j^{n+1}\mathbf{z}_{j-1}^{n+1} + \mathbf{B}^{n+1}\mathbf{z}_j^{n+1} + {}_\star\mathbf{A}_j^{n+1}\mathbf{z}_{j+1}^{n+1} = \begin{bmatrix} -2\Omega^{n+1} & 1 \\ 1 & 0 \end{bmatrix} \mathbf{z}_j^n + \Delta t\, \beta_j^{n+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2.5}$$

for $j = 2, \ldots, JJ-1$, with

$${}_\star\mathbf{A}_j^{n+1} = \frac{\Delta t}{2\Delta x} \begin{bmatrix} \alpha_j^{n+1} - (\Omega^{n+1})^2 h_j^\star & U_j^\star \\ U_j^\star & h_j^\star \end{bmatrix}, \tag{2.6}$$

and

$$\mathbf{B}^{n+1} = \begin{bmatrix} -2\Omega^{n+1} - \dot{\Omega}^{n+1}\Delta t & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.7}$$

The $\star$ left-subscript on ${}_\star\mathbf{A}_j^n$ is a reminder that the entries depend on nonlinear $\star-$terms. The equations at $j=1$ and $j=JJ$ are obtained from the boundary conditions. The only boundary condition at $x=0$ is $U=0$. The discrete version of this is

$$U_1^n = 0 \quad \text{and} \quad \frac{U_0^n + U_2^n}{2} = 0\,, \quad \text{for all} \quad n \in \mathbb{N}\,, \tag{2.8}$$

where U_0^n is a ficticious point Δx to the left of $x=0$. To obtain a boundary condition for h, use the mass equation at $x=0$

$$h_t + h^\star U_x = 0\,,$$

with discretization

$$\frac{h_1^{n+1} - h_1^n}{\Delta t} + h_1^\star \frac{U_2^{n+1}}{\Delta x} = 0\,. \tag{2.9}$$

Combining (2.8) and (2.9) gives the equation for $j=1$

$$\mathbf{z}_1^{n+1} + \frac{h_1^\star\,\Delta t}{\Delta x}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\mathbf{z}_2^{n+1} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathbf{z}_1^n\,. \tag{2.10}$$

Similarly at $x=L$,

$$U_{JJ}^n = 0 \quad \text{and} \quad \frac{U_{JJ-1}^n + U_{JJ+1}^n}{2} = 0\,, \quad \text{for all} \quad n \in \mathbb{N}\,, \tag{2.11}$$

where U_{JJ+1}^n is a ficticious point Δx to the right of $x=0$. The discrete mass equation at $j=JJ$ is

$$\frac{h_{JJ}^{n+1} - h_{JJ}^n}{\Delta t} - h_{JJ}^\star \frac{U_{JJ-1}^{n+1}}{\Delta x} = 0\,. \tag{2.12}$$

Combining these two equations gives the discretization at $j=JJ$,

$$-\frac{h_{JJ}^\star\,\Delta t}{\Delta x}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\mathbf{z}_{JJ-1}^{n+1} + \mathbf{z}_{JJ}^{n+1} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathbf{z}_{JJ}^n\,. \tag{2.13}$$

Hence, for fixed $h^\star$ and $U^\star$ the following block linear system of equations is to be solved,

$$\begin{aligned}
\mathbf{z}_1^{n+1} + h_1^\star\mathbf{N}\mathbf{z}_2^{n+1} &= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathbf{z}_1^n\\
-{}_\star\mathbf{A}_2^{n+1}\mathbf{z}_1^{n+1} + \mathbf{B}^{n+1}\mathbf{z}_2^{n+1} + {}_\star\mathbf{A}_2^{n+1}\mathbf{z}_3^{n+1} &= \begin{bmatrix}-2\Omega^{n+1} & 1\\ 1 & 0\end{bmatrix}\mathbf{z}_2^n + \Delta t\,\beta_2^{n+1}\begin{bmatrix}1\\ 0\end{bmatrix}\,,\\
-{}_\star\mathbf{A}_3^{n+1}\mathbf{z}_2^{n+1} + \mathbf{B}^{n+1}\mathbf{z}_3^{n+1} + {}_\star\mathbf{A}_3^{n+1}\mathbf{z}_4^{n+1} &= \begin{bmatrix}-2\Omega^{n+1} & 1\\ 1 & 0\end{bmatrix}\mathbf{z}_3^n + \Delta t\,\beta_3^{n+1}\begin{bmatrix}1\\ 0\end{bmatrix}\,,\\
\vdots \quad & \quad \vdots\\
-{}_\star\mathbf{A}_{JJ-1}^{n+1}\mathbf{z}_{JJ-2}^{n+1} + \mathbf{B}^{n+1}\mathbf{z}_{JJ-1}^{n+1} + {}_\star\mathbf{A}_{JJ-1}^{n+1}\mathbf{z}_{JJ}^{n+1} &= \begin{bmatrix}-2\Omega^{n+1} & 1\\ 1 & 0\end{bmatrix}\mathbf{z}_{JJ-1}^n + \Delta t\,\beta_{JJ-1}^{n+1}\begin{bmatrix}1\\ 0\end{bmatrix}\,,\\
-h_{JJ}^\star\mathbf{N}\mathbf{z}_{JJ-1}^{n+1} + \mathbf{z}_{JJ}^{n+1} &= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathbf{z}_{JJ}^n\,.
\end{aligned}$$

where

$$\mathbf{N} := \frac{\Delta t}{\Delta x}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

The right-hand side of the system is known, and for fixed h^* and U^* the system is a block tridiagonal linear system.

Numerical results using this scheme are reported in [3].

2.1 Structure of the numerical dissipation

The fully implicit scheme is dissipative. The dissipation eliminates transients and smooths the high-frequency oscillations near hydraulic jumps. In this subsection, the form of the numerical dissipation is identified. The form of the dissipation is similar to the action of viscosity, in that it is wavenumber dependent. An interesting feature of the dissipation is that it is Froude number dependent and becomes directional in the limit as the Froude number approaches unity.

To compute the truncation error, take the simplest case, where the SWEs are linear and the vessel is stationary,

$$h_t + u_0 h_x + h_0 u_x = 0 \quad \text{and} \quad u_t + u_0 u_x + g h_x = 0\,. \tag{2.14}$$

The fully implicit scheme is

$$\frac{h_j^{n+1} - h_j^n}{\Delta t} + u_0\left(\frac{h_{j+1}^{n+1} - h_{j-1}^{n+1}}{2\Delta x}\right) + h_0\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}\right) = 0\,,$$

and

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_0\left(\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}\right) + g\left(\frac{h_{j+1}^{n+1} - h_{j-1}^{n+1}}{2\Delta x}\right) = 0\,.$$

Expand each term in a Taylor series, e.g.

$$h_j^{n+1} = h_j^n + (h_j^n)_t \Delta t + \tfrac{1}{2}(h_j^n)_{tt}\Delta t^2 + \cdots\,,$$

and

$$\begin{aligned} h_{j+1}^{n+1} &= h_{j+1}^n + (h_{j+1}^n)_t\Delta t + \tfrac{1}{2}(h_{j+1}^n)_{tt}\Delta t^2 + \cdots\,,\\ &= h_j^n + (h_j^n)_x\Delta x + \tfrac{1}{2}(h_j^n)_{xx}\Delta x^2\\ &\quad +\Delta t\left((h_j^n)_t + (h_j^n)_{tx}\Delta x + \tfrac{1}{2}(h_j^n)_{xxt}\Delta x^2\right)\\ &\quad +\tfrac{1}{2}\Delta t^2\left((h_j^n)_{tt} + (h_j^n)_{ttx}\Delta x\right) + \cdots\,. \end{aligned}$$

Substitution of these expressions into (2.14) gives the leading order "modified equations"

$$\begin{aligned} h_t + u_0 h_x + h_0 u_x &= -\Delta t\left(u_0 h_{tx} + h_0 u_{tx} + \tfrac{1}{2}h_{tt}\right) + \cdots\,,\\ u_t + u_0 u_x + g h_x &= -\Delta t\left(u_0 u_{tx} + g h_{tx} + \tfrac{1}{2}u_{tt}\right) + \cdots\,. \end{aligned}$$

Background theory on "modified equations" and truncation error can be found in §11.1 of [6].

Use the leading order equation (2.14) to express h_{xt}, u_{xt}, h_{tt} and u_{tt} in terms of h_{xx} and u_{xx},

$$\begin{aligned} h_t + u_0 h_x + h_0 u_x &= \tfrac{1}{2}\Delta t\left((u_0^2 + gh_0)h_{xx} + 2h_0 u_0 u_{xx}\right) + \cdots, \\ u_t + u_0 u_x + gh_x &= \tfrac{1}{2}\Delta t\left((u_0^2 + gh_0)u_{xx} + 2gu_0 h_{xx}\right) + \cdots. \end{aligned}$$

The leading order dissipation has the viscous form; that is, the dissipation is of the same form as the heat equation,

$$\begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{bmatrix} u_0 & h_0 \\ g & u_0 \end{bmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x = \mathbf{D} \begin{pmatrix} h \\ u \end{pmatrix}_{xx} = \cdots,$$

where $\mathbf{D}$ is the dissipation matrix

$$\mathbf{D} = \tfrac{1}{2}\Delta t \begin{bmatrix} u_0^2 + gh_0 & 2h_0 u_0 \\ 2gu_0 & u_0^2 + gh_0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{D}$ are

$$\lambda_\pm = \tfrac{1}{2}\Delta t \left(u_0 \pm \sqrt{gh_0}\right)^2 = \tfrac{1}{2}\Delta t\, gh_0 (F \pm 1)^2,$$

where $F = u_0/\sqrt{gh_0}$. The eigenvalues are of order Δt and the dissipation matrix is positive definite away from criticality. This analysis shows that the leading order truncation error is dissipation.

In the limit as $F \to 1$ one of the eigenvalues of $\mathbf{D}$ vanishes. The eigenvector in this case is

$$\boldsymbol{\xi} = \begin{pmatrix} h_0 \\ -u_0 \end{pmatrix},$$

since

$$\mathbf{D}\boldsymbol{\xi} = (gh_0 - u_0^2) \begin{pmatrix} h_0 \\ u_0 \end{pmatrix}.$$

The eigenvector $\boldsymbol{\xi}$ is also an eigenvector of the Jacobian when $F = 1$ since

$$\begin{bmatrix} u_0 & h_0 \\ g & u_0 \end{bmatrix} \begin{pmatrix} h_0 \\ -u_0 \end{pmatrix} = (gh_0 - u_0^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence the dissipation matrix resonates with the structure of the SWEs; that is, the Jacobian and the dissipation matrix are both singular at criticality ($F^2 = 1$).

Carrying the modified equation analysis to the next order will reveal dispersive truncation error. By including dissipation and dispersion through truncation error, rather than explicitly, higher-order boundary conditions need not be imposed explicitly at $x = 0$ and $x = L$.

3 Sloshing SWEs in two space dimensions

The SWEs for fluid sloshing in a rotating vessel derived in [4] are

$$
\begin{aligned}
h_t + hU_x + Uh_x + hV_y + Vh_y &= 0 \\
U_t + UU_x + VU_y + 2\Omega_2 V h_y + 2\Omega_2 h_t - 2\Omega_3 V & \\
+ \left[\alpha(x,y,t) + 2\Omega_1 V - \left(\Omega_1^2 + \Omega_2^2\right) h\right] h_x &= -\left(\dot{\Omega}_2 + \Omega_1\Omega_3\right) h + \widehat{\beta}(x,y,t) \\
V_t + UV_x + VV_y - 2\Omega_1 U h_x - 2\Omega_1 h_t + 2\Omega_3 U & \\
+ \left[\alpha(x,y,t) - 2\Omega_2 U - \left(\Omega_1^2 + \Omega_2^2\right) h\right] h_y &= \left(\dot{\Omega}_1 - \Omega_2\Omega_3\right) h + \widetilde{\beta}(x,y,t)\,,
\end{aligned}
\tag{3.1}
$$

where $h(x,y,t)$ is the fluid depth and $(U(x,y,t),V(x,y,t))$ is the horizontal velocity field. The tank has length L_1 in the $x-$direction and length L_2 in the $y-$direction and the only boundary conditions are

$$
\begin{aligned}
U(0,y,t) = U(L_1,y,t) &= 0\,, \quad \text{for} \quad 0 \le y \le L_2 \quad \text{and} \quad \forall\ t\,, \\
V(x,0,t) = U(x,L_2,t) &= 0\,, \quad \text{for} \quad 0 \le x \le L_1 \quad \text{and} \quad \forall\ t\,.
\end{aligned}
\tag{3.2}
$$

The functions $\alpha(x,y,t)$, $\widehat{\beta}(x,y,t)$ and $\widetilde{\beta}(x,y,t)$ contain the terms from the rotating coordinate system that are independent of h,U,V,

$$
\begin{aligned}
\alpha(x,y,t) &= \left(\dot{\Omega}_1 + \Omega_2\Omega_3\right)(y+d_2) + \left(\Omega_1\Omega_3 - \dot{\Omega}_2\right)(x+d_1) \\
&\quad - (\Omega_1^2 + \Omega_2^2)\, d_3 + \mathbf{Q}\mathbf{e}_3 \cdot \ddot{\mathbf{q}} + g\mathbf{Q}\mathbf{e}_3 \cdot \mathbf{e}_3 \\
\widehat{\beta}(x,y,t) &= \left(\dot{\Omega}_3 - \Omega_1\Omega_2\right)(y+d_2) + (\Omega_2^2 + \Omega_3^2)(x+d_1) \\
&\quad - \left(\dot{\Omega}_2 + \Omega_1\Omega_3\right) d_3 - \mathbf{Q}\mathbf{e}_1 \cdot \ddot{\mathbf{q}} - g\mathbf{Q}\mathbf{e}_1 \cdot \mathbf{e}_3 \\
\widetilde{\beta}(x,y,t) &= -\left(\dot{\Omega}_3 + \Omega_1\Omega_2\right)(x+d_1) + (\Omega_1^2 + \Omega_3^2)(y+d_2) \\
&\quad + \left(\dot{\Omega}_1 - \Omega_2\Omega_3\right) d_3 - \mathbf{Q}\mathbf{e}_2 \cdot \ddot{\mathbf{q}} - g\mathbf{Q}\mathbf{e}_2 \cdot \mathbf{e}_3\,.
\end{aligned}
\tag{3.3}
$$

The terms and parameters are defined in [4].

An alternating direction implicit algorithm is used to solve this system of nonlinear equations numerically. The time step is split into two half steps, and in each half step the equations are solved in one-dimensional strips as shown in Figure 1. In the step $n \mapsto n+\frac{1}{2}$ the equations are solved in horizontal $x-$strips for fixed y (the green lines in Figure 1) and in the step $n+\frac{1}{2} \mapsto n+1$ the equations are solved in vertical $y-$strips for fixed x (the blue lines in Figure 1). Each of the one-dimensional systems has the form of linear equations with a block tridiagonal coefficient matrix as in §2.

The $x-$interval $0 \le x \le L_1$ is split into $II-1$ intervals

$$
x_i := (i-1)\Delta x\,, \quad i = 1,\dots,II\,, \quad \Delta x = \frac{L_1}{II-1}\,,
$$

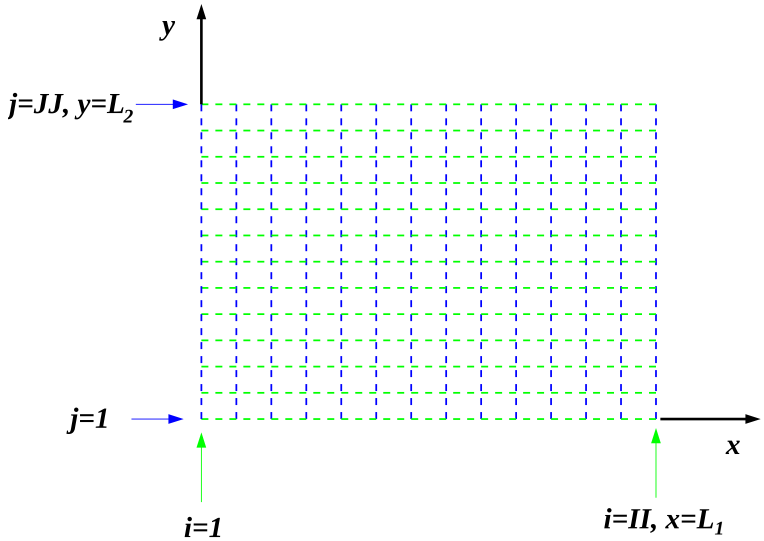

Figure 1: Schematic of the grid layout for the discretization.

and the $y-$interval $0 \le y \le L_2$ is split into $JJ-1$ intervals

$$y_j := (j-1)\Delta y\,, \quad j = 1,\dots,JJ\,, \quad \Delta y = \frac{L_2}{JJ-1}\,,$$

and the discretized values of h, U, V are represented by

$$h^n_{i,j} := h(x_i, y_j, t_n), \quad U^n_{i,j} := U(x_i, y_j, t_n) \quad \text{and} \quad V^n_{i,j} := V(x_i, y_j, t_n)\,,$$

where $t_n = n\Delta t$ with Δt the fixed time step.

Derivatives are discretized using centred difference. In the step $n \mapsto n + \frac{1}{2}$ the $x-$derivatives are treated implicitly and the $y-$derivatives are treated explicitly, and in the step $n+\frac{1}{2} \mapsto n+1$ the $y-$derivatives are treated implicitly and the $x-$derivatives are treated explicitly.

4 Algorithmic details for the half step $n \mapsto n + \frac{1}{2}$

Rewrite the governing equations in a form that emphasizes the nonlinearity

$$\begin{aligned}
h_t + h^\star U_x + U^\star h_x + hV_y + Vh_y &= 0 \\
U_t + U^\star U_x + VU_y + 2\Omega_2 V h_y + 2\Omega_2 h_t - 2\Omega_3 V & \\
+ \left[\alpha(x,y,t) + 2\Omega_1 V^\star - \left(\Omega_1^2+\Omega_2^2\right) h^\star\right] h_x &= -\left(\dot{\Omega}_2 + \Omega_1\Omega_3\right) h + \widehat{\beta}(x,y,t) \\
V_t + U^\star V_x + VV_y - 2\Omega_1 U^\star h_x - 2\Omega_1 h_t + 2\Omega_3 U & \\
+ \left[\alpha(x,y,t) - 2\Omega_2 U - \left(\Omega_1^2+\Omega_2^2\right) h\right] h_y &= +\left(\dot{\Omega}_1 - \Omega_2\Omega_3\right) h + \widetilde{\beta}(x,y,t)\,,
\end{aligned} \tag{4.1}$$

where $h^\star$, $U^\star$ and $V^\star$ are the current intermediate values of h, U and V. Note that only nonlinearites associated with $x-$derivatives are starred, as the nonlinear terms with $y-$derivatives are treated explicitly.

The discretization of the mass equation is

$$
\begin{aligned}
\frac{h_{i,j}^{n+\frac{1}{2}} - h_{i,j}^{n}}{\frac{1}{2}\Delta t} + h_{i,j}^{\star}\frac{U_{i+1,j}^{n+\frac{1}{2}} - U_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} + U_{i,j}^{\star}\frac{h_{i+1,j}^{n+\frac{1}{2}} - h_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x}& \\
+h_{i,j}^{n}\frac{V_{i,j+1}^{n} - V_{i,j-1}^{n}}{2\Delta y} + V_{i,j}^{n}\frac{h_{i,j+1}^{n} - h_{i,j-1}^{n}}{2\Delta y} &= 0\,.
\end{aligned}
\tag{4.2}
$$

The discretizations of the equations for U,V are

$$
\begin{aligned}
&\frac{U_{i,j}^{n+\frac{1}{2}} - U_{i,j}^{n}}{\frac{1}{2}\Delta t} + U_{i,j}^{\star}\frac{U_{i+1,j}^{n+\frac{1}{2}} - U_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} + V_{i,j}^{n}\frac{U_{i,j+1}^{n} - U_{i,j-1}^{n}}{2\Delta y} + 2\Omega_2^{n+\frac{1}{2}}V_{i,j}^{n}\frac{h_{i,j+1}^{n} - h_{i,j-1}^{n}}{2\Delta y} \\
&\quad + \left[\alpha_{i,j}^{n+\frac{1}{2}} + 2\Omega_1^{n+\frac{1}{2}}V_{i,j}^{\star} - \left(\left(\Omega_1^{n+\frac{1}{2}}\right)^2 + \left(\Omega_2^{n+\frac{1}{2}}\right)^2\right)h_{i,j}^{\star}\right]\frac{h_{i+1,j}^{n+\frac{1}{2}} - h_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} \\
&= 2\Omega_3^{n+\frac{1}{2}}V_{i,j}^{n+\frac{1}{2}} - \left(\dot{\Omega}_2^{n+\frac{1}{2}} + \Omega_1^{n+\frac{1}{2}}\Omega_3^{n+\frac{1}{2}}\right)h_{i,j}^{n+\frac{1}{2}} - 2\Omega_2^{n+\frac{1}{2}}\frac{h_{i,j}^{n+\frac{1}{2}} - h_{i,j}^{n}}{\frac{1}{2}\Delta t} + \widehat{\beta}_{i,j}^{n+\frac{1}{2}} \\
&\frac{V_{i,j}^{n+\frac{1}{2}} - V_{i,j}^{n}}{\frac{1}{2}\Delta t} + U_{i,j}^{\star}\frac{V_{i+1,j}^{n+\frac{1}{2}} - V_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} + V_{i,j}^{n}\frac{V_{i,j+1}^{n} - V_{i,j-1}^{n}}{2\Delta y} - 2\Omega_1^{n+\frac{1}{2}}U_{i,j}^{\star}\frac{h_{i+1,j}^{n+\frac{1}{2}} - h_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} \\
&+ \left[\alpha_{i,j}^{n+\frac{1}{2}} - 2\Omega_2^{n+\frac{1}{2}}U_{i,j}^{n} - \left(\left(\Omega_1^{n+\frac{1}{2}}\right)^2 + \left(\Omega_2^{n+\frac{1}{2}}\right)^2\right)h_{i,j}^{n}\right]\frac{h_{i,j+1}^{n} - h_{i,j-1}^{n}}{2\Delta y} \\
&= -2\Omega_3^{n+\frac{1}{2}}U_{i,j}^{n+\frac{1}{2}} + \left(\dot{\Omega}_1^{n+\frac{1}{2}} - \Omega_2^{n+\frac{1}{2}}\Omega_3^{n+\frac{1}{2}}\right)h_{i,j}^{n+\frac{1}{2}} + 2\Omega_1^{n+\frac{1}{2}}\frac{h_{i,j}^{n+\frac{1}{2}} - h_{i,j}^{n}}{\frac{1}{2}\Delta t} + \widetilde{\beta}_{i,j}^{n+\frac{1}{2}}\,,
\end{aligned}
\tag{4.3}
$$

where

$$
\alpha_{i,j}^{n} := \alpha(x_i, y_j, t_n), \quad \widehat{\beta}_{i,j}^{n} := \widehat{\beta}(x_i, y_j, t_n) \quad \text{and} \quad \widetilde{\beta}_{i,j}^{n} := \widetilde{\beta}(x_i, y_j, t_n)\,.
$$

By setting

$$
\mathbf{z}_{i,j}^{n} = \begin{bmatrix} h_{i,j}^{n} \\ U_{i,j}^{n} \\ V_{i,j}^{n} \end{bmatrix},
$$

equations (4.2)-(4.3) can be written in block tridiagonal form

$$
\begin{aligned}
-{}_{\star}\mathbf{A}_{i,j}^{n+\frac{1}{2}}\mathbf{z}_{i-1,j}^{n+\frac{1}{2}} + \mathbf{B}^{n+\frac{1}{2}}\mathbf{z}_{i,j}^{n+\frac{1}{2}} + {}_{\star}\mathbf{A}_{i,j}^{n+\frac{1}{2}}\mathbf{z}_{i+1,j}^{n+\frac{1}{2}} &= \mathbf{C}_{i,j}^{n+\frac{1}{2}}\mathbf{z}_{i,j-1}^{n} + \mathbf{D}^{n+\frac{1}{2}}\mathbf{z}_{i,j}^{n} \\
&\quad -\mathbf{C}_{i,j}^{n+\frac{1}{2}}\mathbf{z}_{i,j+1}^{n} + \boldsymbol{\beta}_{i,j}^{n+\frac{1}{2}}\,.
\end{aligned}
\tag{4.4}
$$

Matrices with a $\star$ left-subscript depend on $h^\star$, $U^\star$ and $V^\star$. The entries of the matrices are

$$
{}_{\star}\mathbf{A}^n_{i,j} = \frac{\Delta t}{4\Delta x}\begin{bmatrix} U^\star_{i,j} & h^\star_{i,j} & 0 \\ \widetilde{{}_{1\star}\alpha^n_{i,j}} & U^\star_{i,j} & 0 \\ -2\Omega_1^n U^\star_{i,j} & 0 & U^\star_{i,j}\end{bmatrix}, \quad \mathbf{D}^n = \begin{bmatrix} 1 & 0 & 0 \\ 2\Omega_2^n & 1 & 0 \\ -2\Omega_1^n & 0 & 1\end{bmatrix}
$$

$$
\mathbf{B}^n = \begin{bmatrix} 1 & 0 & 0 \\ 2\Omega_2^n + \frac{1}{2}\Delta t\left(\dot{\Omega}_2^n + \Omega_1^n\Omega_3^n\right) & 1 & -\Delta t\Omega_3^n \\ -2\Omega_1^n - \frac{1}{2}\Delta t\left(\dot{\Omega}_1^n - \Omega_2^n\Omega_3^n\right) & \Delta t\Omega_3^n & 1\end{bmatrix} \tag{4.5}
$$

$$
\mathbf{C}^n_{i,j} = \tfrac{\Delta t}{4\Delta y}\begin{bmatrix} V^{n-\frac{1}{2}}_{i,j} & 0 & h^{n-\frac{1}{2}}_{i,j} \\ 2\Omega_2^n V^{n-\frac{1}{2}}_{i,j} & V^{n-\frac{1}{2}}_{i,j} & 0 \\ \widetilde{{}_{2}\alpha^n_{i,j}} & 0 & V^{n-\frac{1}{2}}_{i,j}\end{bmatrix}, \quad \boldsymbol{\beta}^n_{i,j} = \tfrac{\Delta t}{2}\begin{bmatrix} 0 \\ \widehat{\beta}^n_{i,j} \\ \widetilde{\beta}^n_{i,j}\end{bmatrix},
$$

and

$$
\begin{aligned}
\widetilde{{}_{1\star}\alpha^n_{i,j}} &= \alpha^n_{i,j} + 2\Omega_1^n V^\star_{i,j} - \left((\Omega_1^n)^2 + (\Omega_2^n)^2\right) h^\star_{i,j} \\
\widetilde{{}_{2}\alpha^n_{i,j}} &= \alpha^n_{i,j} - 2\Omega_2^n U^{n-\frac{1}{2}}_{i,j} - \left((\Omega_1^n)^2 + (\Omega_2^n)^2\right) h^{n-\frac{1}{2}}_{i,j}.
\end{aligned}
$$

For fixed $j = 2, \ldots, JJ-1$ the equations (4.4) are applied for $i = 2, \ldots, II-1$. To complete the tridiagonal system equations are needed (for each fixed j) at $i = 1$ and $i = II$.

4.1 The equations at $i = 1$ and $i = II$ for $j = 2, \ldots, JJ-1$

The equations at $i = 1$ and $i = II$ are obtained from the boundary conditions at $x = 0$ and $x = L_1$ (3.2). The only boundary condition at $x = 0$ is $U(0, y, t) = 0$. The discrete version of this is

$$
U^n_{1,j} = 0 \quad \text{and} \quad \frac{U^n_{0,j} + U^n_{2,j}}{2} = 0\,, \quad \text{for each } j\,, \quad \text{and for all} \quad n \in \mathbb{N}\,. \tag{4.6}
$$

To obtain a boundary condition for h, use the mass equation at $x = 0$

$$
h_t + h^\star U_x + hV_y + Vh_y = 0\,,
$$

with discretization

$$
h^{n+\frac{1}{2}}_{1,j} + \tfrac{\Delta t}{2\Delta x} h^\star_{1,j} U^{n+\frac{1}{2}}_{2,j} = h^n_{1,j} - \tfrac{\Delta t}{4\Delta y} h^n_{1,j}\left(V^n_{1,j+1} - V^n_{1,j-1}\right) - \tfrac{\Delta t}{4\Delta y} V^n_{1,j}\left(h^n_{1,j+1} - h^n_{1,j-1}\right). \tag{4.7}
$$

To obtain a boundary condition for V, use the $y-$momentum equation at $x = 0$

$$
V_t + VV_y - 2\Omega_1 h_t + \left[\alpha(x,y,t) - \left(\Omega_1^2 + \Omega_2^2\right) h\right] h_y = \left(\dot{\Omega}_1 - \Omega_2\Omega_3\right) h + \widetilde{\beta}(x,y,t)\,,
$$

with discretization

$$V_{1,j}^{n+\frac{1}{2}} - \left[2\Omega_1^{n+\frac{1}{2}} + \tfrac{1}{2}\Delta t\left(\dot{\Omega}_1^{n+\frac{1}{2}} - \Omega_2^{n+\frac{1}{2}}\Omega_3^{n+\frac{1}{2}}\right)\right] h_{1,j}^{n+\frac{1}{2}} = V_{1,j}^n - \tfrac{\Delta t}{4\Delta y}V_{1,j}^n\left(V_{1,j+1}^n - V_{1,j-1}^n\right) - \tfrac{\Delta t}{4\Delta y}\widehat{\alpha_{1,j}^{n+\frac{1}{2}}}\left(h_{1,j+1}^n - h_{1,j-1}^n\right) - 2\Omega_1^{n+\frac{1}{2}}h_{1,j}^n + \tfrac{1}{2}\Delta t\widetilde{\beta}_{1,j}^{n+\frac{1}{2}}\,, \tag{4.8}$$

where

$$\widehat{\alpha_{i,j}^n} = \alpha_{i,j}^n - \left((\Omega_1^n)^2 + (\Omega_2^n)^2\right) h_{i,j}^{n-\frac{1}{2}}.$$

Combining equations (4.6), (4.7) and (4.8) gives the equation for $i = 1$

$$\mathbf{E}^{n+\frac{1}{2}}\mathbf{z}_{1,j}^{n+\frac{1}{2}} + {}_\star\mathbf{F}_{1,j}\mathbf{z}_{2,j}^{n+\frac{1}{2}} = \mathbf{G}_{1,j}^{n+\frac{1}{2}}\mathbf{z}_{1,j-1}^n + \mathbf{H}^{n+\frac{1}{2}}\mathbf{z}_{1,j}^n - \mathbf{G}_{1,j}^{n+\frac{1}{2}}\mathbf{z}_{1,j+1}^n + {}_1\boldsymbol{\beta}_{1,j}^{n+\frac{1}{2}}\,, \tag{4.9}$$

with

$$\mathbf{E}^{n+\frac{1}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2\Omega_1^{n+\frac{1}{2}} - \frac{1}{2}\Delta t\left(\dot{\Omega}_1^{n+\frac{1}{2}} - \Omega_2^{n+\frac{1}{2}}\Omega_3^{n+\frac{1}{2}}\right) & 0 & 1 \end{bmatrix},$$

$$\mathbf{G}_{i,j}^n = \tfrac{\Delta t}{4\Delta y}\begin{bmatrix} V_{i,j}^{n-\frac{1}{2}} & 0 & h_{i,j}^{n-\frac{1}{2}} \\ 0 & 0 & 0 \\ \widehat{\alpha_{i,j}^{n+\frac{1}{2}}} & 0 & V_{i,j}^{n-\frac{1}{2}} \end{bmatrix}, \quad {}_1\boldsymbol{\beta}_{i,j}^{n+\frac{1}{2}} = \tfrac{\Delta t}{2}\begin{bmatrix} 0 \\ 0 \\ \widetilde{\beta}_{i,j}^{n+\frac{1}{2}} \end{bmatrix} \tag{4.10}$$

$$\mathbf{H}^{n+\frac{1}{2}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -2\Omega_1^{n+\frac{1}{2}} & 0 & 1 \end{bmatrix} \quad {}_\star\mathbf{F}_{i,j} = \tfrac{\Delta t}{2\Delta x}\begin{bmatrix} 0 & h_{i,j}^\star & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A similar strategy is used to construct the discrete equations at $x = L_1$. The velocity boundary condition is

$$U_{II,j}^n = 0 \quad \text{and} \quad \frac{U_{II-1,j}^n + U_{II+1,j}^n}{2} = 0\,, \quad \text{for each } j\,, \quad \text{and for all} \quad n \in \mathbb{N}\,. \tag{4.11}$$

The discrete mass equation is

$$h_{II,j}^{n+\frac{1}{2}} - \tfrac{\Delta t}{2\Delta x}h_{II,j}^\star U_{II-1,j}^{n+\frac{1}{2}} = h_{II,j}^n - \tfrac{\Delta t}{4\Delta y}h_{II,j}^n\left(V_{II,j+1}^n - V_{II,j-1}^n\right) - \tfrac{\Delta t}{4\Delta y}V_{II,j}^n\left(h_{II,j+1}^n - h_{II,j-1}^n\right)\,, \tag{4.12}$$

and the discrete $y-$momentum equation is

$$V_{II,j}^{n+\frac{1}{2}} - \left[2\Omega_1^{n+\frac{1}{2}} + \tfrac{1}{2}\Delta t\left(\dot{\Omega}_1^{n+\frac{1}{2}} - \Omega_2^{n+\frac{1}{2}}\Omega_3^{n+\frac{1}{2}}\right)\right] h_{II,j}^{n+\frac{1}{2}} = V_{II,j}^n - \tfrac{\Delta t}{4\Delta y}V_{II,j}^n\left(V_{II,j+1}^n - V_{II,j-1}^n\right) - \tfrac{\Delta t}{4\Delta y}\widehat{\alpha_{II,j}^{n+\frac{1}{2}}}\left(h_{II,j+1}^n - h_{II,j-1}^n\right) - 2\Omega_1^{n+\frac{1}{2}}h_{II,j}^n + \tfrac{1}{2}\Delta t\widetilde{\beta}_{II,j}^{n+\frac{1}{2}}\,. \tag{4.13}$$

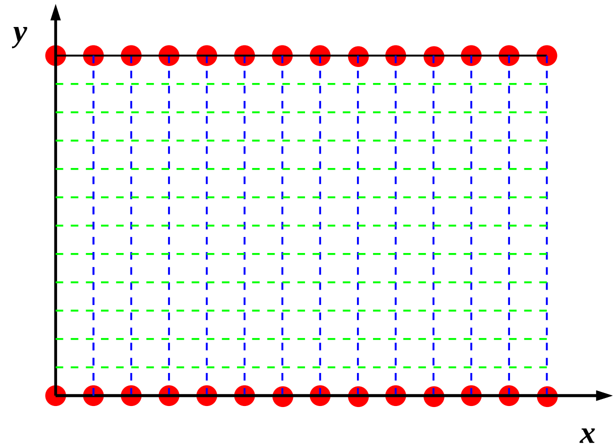

Figure 2: The grid layout with the horizontal lines $j = 1$ and $j = JJ$ highlighted.

Combining equations (4.11), (4.12) and (4.13) gives the equation for $i = II$

$$-{}_\star\mathbf{F}_{II,j}\mathbf{z}_{II-1,j}^{n+\frac{1}{2}} + \mathbf{E}^{n+\frac{1}{2}}\mathbf{z}_{II,j}^{n+\frac{1}{2}} \;=\; \mathbf{G}_{II,j}^{n+\frac{1}{2}}\mathbf{z}_{II,j-1}^{n} + \mathbf{H}^{n+\frac{1}{2}}\mathbf{z}_{II,j}^{n} - \mathbf{G}_{II,j}^{n+\frac{1}{2}}\mathbf{z}_{II,j+1}^{n} +_1 \boldsymbol{\beta}_{II,j}^{n+\frac{1}{2}}\,. \tag{4.14}$$

This completes the construction of the block tridiagonal system at $j-$interior points. For each fixed $j = 2, \ldots, JJ-1$, and fixed $h^\star$, $U^\star$ and $V^\star$, we solve the following system

$$\begin{aligned}
\mathbf{E}^{n+\frac{1}{2}}\mathbf{z}_{1,j}^{n+\frac{1}{2}} + {}_\star\mathbf{F}_{1,j}\mathbf{z}_{2,j}^{n+\frac{1}{2}} &= \mathbf{G}_{1,j}^{n+\frac{1}{2}}\mathbf{z}_{1,j-1}^{n} + \mathbf{H}^{n+\frac{1}{2}}\mathbf{z}_{1,j}^{n} \\
&\qquad -\mathbf{G}_{1,j}^{n+\frac{1}{2}}\mathbf{z}_{1,j+1}^{n} +_1 \boldsymbol{\beta}_{1,j}^{n+\frac{1}{2}}\,, \\
-{}_\star\mathbf{A}_{2,j}^{n+\frac{1}{2}}\mathbf{z}_{1,j}^{n+\frac{1}{2}} + \mathbf{B}^{n+\frac{1}{2}}\mathbf{z}_{2,j}^{n+\frac{1}{2}} + {}_\star\mathbf{A}_{2,j}^{n+\frac{1}{2}}\mathbf{z}_{3,j}^{n+\frac{1}{2}} &= \mathbf{C}_{2,j}^{n+\frac{1}{2}}\mathbf{z}_{2,j-1}^{n} + \mathbf{D}^{n+\frac{1}{2}}\mathbf{z}_{2,j}^{n} \\
&\qquad -\mathbf{C}_{2,j}^{n+\frac{1}{2}}\mathbf{z}_{2,j+1}^{n} + \boldsymbol{\beta}_{2,j}^{n+\frac{1}{2}}\,, \\
-{}_\star\mathbf{A}_{3,j}^{n+\frac{1}{2}}\mathbf{z}_{2,j}^{n+\frac{1}{2}} + \mathbf{B}^{n+\frac{1}{2}}\mathbf{z}_{3,j}^{n+\frac{1}{2}} + {}_\star\mathbf{A}_{3,j}^{n+\frac{1}{2}}\mathbf{z}_{4,j}^{n+\frac{1}{2}} &= \mathbf{C}_{3,j}^{n+\frac{1}{2}}\mathbf{z}_{3,j-1}^{n} + \mathbf{D}^{n+\frac{1}{2}}\mathbf{z}_{3,j}^{n} \\
&\qquad -\mathbf{C}_{3,j}^{n+\frac{1}{2}}\mathbf{z}_{3,j+1}^{n} + \boldsymbol{\beta}_{3,j}^{n+\frac{1}{2}}\,, \\
&\vdots \\
-{}_\star\mathbf{F}_{II,j}\mathbf{z}_{II-1,j}^{n+\frac{1}{2}} + \mathbf{E}^{n+\frac{1}{2}}\mathbf{z}_{II,j}^{n+\frac{1}{2}} &= \mathbf{G}_{II,j}^{n+\frac{1}{2}}\mathbf{z}_{II,j-1}^{n} + \mathbf{H}^{n+\frac{1}{2}}\mathbf{z}_{II,j}^{n} \\
&\qquad -\mathbf{G}_{II,j}^{n+\frac{1}{2}}\mathbf{z}_{II,j+1}^{n} +_1 \boldsymbol{\beta}_{II,j}^{n+\frac{1}{2}}\,.
\end{aligned} \tag{4.15}$$

This system is iterated until $h^\star \to h^{n+\frac{1}{2}}$ and $U^\star \to U^{n+\frac{1}{2}}$. The equations along the grid lines $j = 1$ and $j = JJ$, highlighted in Figure 2, are solved separately.

4.2 Grid lines $(i,1)$ and (i,JJ) for $i=1,\ldots,II$

The equations for grid lines $(i,1)$ and (i,JJ) are obtained from the boundary conditions. The only boundary condition at grid line $(i,1)$ is $V(i,1)=0$. The discrete version of this is

$$V_{i,1}^{n}=0 \quad \text{and} \quad \frac{V_{i,0}^{n}+V_{i,2}^{n}}{2}=0\,, \quad \text{for each } i\,, \quad \text{and for all} \quad n\in\mathbb{N}\,. \tag{4.16}$$

To obtain a boundary condition for h, use the mass equation at $(i,1)$

$$h_t+h^{\star}U_x+U^{\star}h_x+hV_y=0\,,$$

with discretization

$$h_{i,1}^{n+\frac{1}{2}}+\tfrac{\Delta t}{4\Delta x}h_{i,1}^{\star}\left(U_{i+1,1}^{n+\frac{1}{2}}-U_{i-1,1}^{n+\frac{1}{2}}\right)+\tfrac{\Delta t}{4\Delta x}U_{i,1}^{\star}\left(h_{i+1,1}^{n+\frac{1}{2}}-h_{i-1,1}^{n+\frac{1}{2}}\right) \;=\; \left(1-\tfrac{\Delta t}{2\Delta y}V_{i,2}^{n}\right)h_{i,1}^{n}\,. \tag{4.17}$$

To obtain a boundary condition for U, use the $x-$momentum equation at $(i,1)$

$$U_t+U^{\star}U_x+2\Omega_2h_t+\left[\alpha(x,0,t)-\left(\Omega_1^2+\Omega_2^2\right)h^{\star}\right]h_x=-\left(\dot{\Omega}_2+\Omega_1\Omega_3\right)h+\widehat{\beta}(x,0,t)\,,$$

with discretization

$$\begin{aligned}&U_{i,1}^{n+\frac{1}{2}}+\tfrac{\Delta t}{4\Delta x}U_{i,1}^{\star}\left(U_{i+1,1}^{n+\frac{1}{2}}-U_{i-1,1}^{n+\frac{1}{2}}\right)+\left[2\Omega_2^{n+\frac{1}{2}}+\tfrac{1}{2}\Delta t\left(\dot{\Omega}_2^{n+\frac{1}{2}}+\Omega_1^{n+\frac{1}{2}}\Omega_3^{n+\frac{1}{2}}\right)\right]h_{i,1}^{n+\frac{1}{2}}\\&+\tfrac{\Delta t}{4\Delta x}\widehat{\alpha_{i,1}^{n+\frac{1}{2}\star}}\left(h_{i+1,1}^{n+\frac{1}{2}}-h_{i-1,1}^{n+\frac{1}{2}}\right)=U_{i,1}^{n}+2\Omega_2^{n+\frac{1}{2}}h_{i,1}^{n}+\tfrac{1}{2}\Delta t\widehat{\beta}_{i,1}^{n+\frac{1}{2}}\,,\end{aligned} \tag{4.18}$$

where

$$_{\star}\widehat{\alpha_{i,j}^{n}} \;=\; \alpha_{i,j}^{n}-\left((\Omega_1^n)^2+(\Omega_2^n)^2\right)h_{i,j}^{\star}.$$

Combining equations (4.16), (4.17) and (4.18) gives the equation for $(i,1)$

$$-{}_{\star}\mathbf{M}_{i,1}^{n+\frac{1}{2}}\mathbf{z}_{i-1,1}^{n+\frac{1}{2}}+\mathbf{N}^{n+\frac{1}{2}}\mathbf{z}_{i,1}^{n+\frac{1}{2}}+{}_{\star}\mathbf{M}_{i,1}^{n+\frac{1}{2}}\mathbf{z}_{i+1,1}^{n+\frac{1}{2}} \;=\; \mathbf{O}_{i,2}^{n+\frac{1}{2}}\mathbf{z}_{i,1}^{n}+{}_2\,\boldsymbol{\beta}_{i,1}^{n+\frac{1}{2}}\,, \tag{4.19}$$

with

$$\begin{aligned}&{}_{\star}\mathbf{M}_{i,j}^{n}=\tfrac{\Delta t}{4\Delta x}\begin{bmatrix}U_{i,j}^{\star} & h_{i,j}^{\star} & 0\\ {}_{\star}\widehat{\alpha_{i,j}^{n}} & U_{i,j}^{\star} & 0\\ 0 & 0 & 0\end{bmatrix}\,,\quad \mathbf{O}_{i,j}^{n}=\begin{bmatrix}1-\tfrac{\Delta t}{2\Delta y}V_{i,j}^{n-\frac{1}{2}} & 0 & 0\\ 2\Omega_2^n & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\,,\\ &\mathbf{N}^{n}=\begin{bmatrix}1 & 0 & 0\\ 2\Omega_2^n+\tfrac{1}{2}\Delta t\left(\dot{\Omega}_2^n+\Omega_1^n\Omega_3^n\right) & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\,,\quad {}_2\boldsymbol{\beta}_{i,j}^{n}=\tfrac{\Delta t}{2}\begin{bmatrix}0\\ \widehat{\beta}_{i,j}^{n}\\ 0\end{bmatrix}\,.\end{aligned} \tag{4.20}$$

The mass equation at grid point $(1,1)$ is

$$h_t+h^{\star}U_x+hV_y=0\,,$$

with discretization

$$h_{1,1}^{n+\frac{1}{2}} + \tfrac{\Delta t}{2\Delta x} h^{\star}_{1,1} U_{2,1}^{n+\frac{1}{2}} \quad = \quad \left(1 - \tfrac{\Delta t}{2\Delta y} V^n_{1,2}\right) h^n_{1,1}\,. \tag{4.21}$$

Combining equation (4.21) with $V_{1,1}^{n+\frac{1}{2}} = 0$ and $U_{1,1}^{n+\frac{1}{2}} = 0$ gives

$$\mathbf{I}\mathbf{z}_{1,1}^{n+\frac{1}{2}} + {}_{\star}\mathbf{Q}_{1,1}\mathbf{z}_{2,1}^{n+\frac{1}{2}} \quad = \quad \mathbf{R}^n_{1,2}\mathbf{z}^n_{1,1}\,, \tag{4.22}$$

with

$${}_{\star}\mathbf{Q}_{i,j} = \tfrac{\Delta t}{2\Delta x}\begin{bmatrix} 0 & h^{\star}_{i,j} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{R}^n_{i,j} = \begin{bmatrix} 1 - \frac{\Delta t}{2\Delta y} V^n_{i,j} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.23}$$

and $\mathbf{I}$ is a 3×3 identity matrix. Similarly the discrete mass equation at grid point $(II, 1)$ is

$$h_{II,1}^{n+\frac{1}{2}} - \tfrac{\Delta t}{2\Delta x} h^{\star}_{II,1} U_{II-1,1}^{n+\frac{1}{2}} \quad = \quad \left(1 - \tfrac{\Delta t}{2\Delta y} V^n_{II,2}\right) h^n_{II,1}\,. \tag{4.24}$$

Combining equation (4.24) with $V_{II,1}^{n+\frac{1}{2}} = 0$ and $U_{II,1}^{n+\frac{1}{2}} = 0$ gives

$$-{}_{\star}\mathbf{Q}_{II,1}\mathbf{z}_{II-1,1}^{n+\frac{1}{2}} + \mathbf{I}\mathbf{z}_{II,1}^{n+\frac{1}{2}} \quad = \quad \mathbf{R}^n_{II,2}\mathbf{z}^n_{II,1}\,. \tag{4.25}$$

For fixed $h^\star$ and $U^\star$ the following block linear system of equations is to be solved along the grid line $(i, 1)$ for $i = 1, \ldots, II$,

$$\begin{aligned}
\mathbf{I}\mathbf{z}_{1,1}^{n+\frac{1}{2}} + {}_{\star}\mathbf{Q}_{1,1}\mathbf{z}_{2,1}^{n+\frac{1}{2}} \quad &= \quad \mathbf{R}^n_{1,2}\mathbf{z}^n_{1,1}\,, \\
-{}_{\star}\mathbf{M}_{2,1}^{n+\frac{1}{2}}\mathbf{z}_{1,1}^{n+\frac{1}{2}} + \mathbf{N}^{n+\frac{1}{2}}\mathbf{z}_{2,1}^{n+\frac{1}{2}} + {}_{\star}\mathbf{M}_{2,1}^{n+\frac{1}{2}}\mathbf{z}_{3,1}^{n+\frac{1}{2}} \quad &= \quad \mathbf{O}_{2,2}^{n+\frac{1}{2}}\mathbf{z}^n_{2,1} +_2 \boldsymbol{\beta}_{2,1}^{n+\frac{1}{2}}\,, \\
-{}_{\star}\mathbf{M}_{3,1}^{n+\frac{1}{2}}\mathbf{z}_{2,1}^{n+\frac{1}{2}} + \mathbf{N}^{n+\frac{1}{2}}\mathbf{z}_{3,1}^{n+\frac{1}{2}} + {}_{\star}\mathbf{M}_{3,1}^{n+\frac{1}{2}}\mathbf{z}_{4,1}^{n+\frac{1}{2}} \quad &= \quad \mathbf{O}_{3,2}^{n+\frac{1}{2}}\mathbf{z}^n_{3,1} +_2 \boldsymbol{\beta}_{3,1}^{n+\frac{1}{2}}\,, \\
\vdots \quad &\quad \vdots \\
-{}_{\star}\mathbf{Q}_{II,1}\mathbf{z}_{II-1,1}^{n+\frac{1}{2}} + \mathbf{I}\mathbf{z}_{II,1}^{n+\frac{1}{2}} \quad &= \quad \mathbf{R}^n_{II,2}\mathbf{z}^n_{II,1}\,.
\end{aligned} \tag{4.26}$$

A similar system is derived along the upper boundary grid line (i, JJ) for $i = 1, \ldots, II$,

$$V^n_{i,JJ} = 0 \quad \text{and} \quad \frac{V^n_{i,JJ-1} + V^n_{i,JJ+1}}{2} = 0\,, \quad \text{for each } i\,, \quad \text{and for all} \quad n \in \mathbb{N}\,. \tag{4.27}$$

The discrete mass equation is

$$\begin{aligned}
&h_{i,JJ}^{n+\frac{1}{2}} + \tfrac{\Delta t}{4\Delta x} h^{\star}_{i,JJ}\left(U_{i+1,JJ}^{n+\frac{1}{2}} - U_{i-1,JJ}^{n+\frac{1}{2}}\right) + \tfrac{\Delta t}{4\Delta x} U^{\star}_{i,JJ}\left(h_{i+1,JJ}^{n+\frac{1}{2}} - h_{i-1,JJ}^{n+\frac{1}{2}}\right) \\
&= \left(1 + \tfrac{\Delta t}{2\Delta y} V^n_{i,JJ-1}\right) h^n_{i,JJ}\,,
\end{aligned} \tag{4.28}$$

and the discrete $x-$momentum equation is

$$U_{i,JJ}^{n+\frac{1}{2}} + \tfrac{\Delta t}{4\Delta x} U_{i,JJ}^{\star}\left(U_{i+1,JJ}^{n+\frac{1}{2}} - U_{i-1,JJ}^{n+\frac{1}{2}}\right) + \left[2\Omega_2^{n+\frac{1}{2}} + \tfrac{1}{2}\Delta t\left(\dot{\Omega}_2^{n+\frac{1}{2}} + \Omega_1^{n+\frac{1}{2}}\Omega_3^{n+\frac{1}{2}}\right)\right] h_{i,JJ}^{n+\frac{1}{2}}$$
$$+ \tfrac{\Delta t}{4\Delta x}\widehat{{}_\star\alpha_{i,JJ}^{n+\frac{1}{2}}}\left(h_{i+1,JJ}^{n+\frac{1}{2}} - h_{i-1,JJ}^{n+\frac{1}{2}}\right) = U_{i,JJ}^{n} + 2\Omega_2^{n+\frac{1}{2}} h_{i,JJ}^{n} + \tfrac{1}{2}\Delta t \widehat{\beta}_{i,JJ}^{n+\frac{1}{2}}\,. \tag{4.29}$$

Combining equations (4.27), (4.28) and (4.29) gives the equation for (i, JJ)

$$-{}_\star\mathbf{M}_{i,JJ}^{n+\frac{1}{2}}\mathbf{z}_{i-1,JJ}^{n+\frac{1}{2}} + \mathbf{N}^{n+\frac{1}{2}}\mathbf{z}_{i,JJ}^{n+\frac{1}{2}} + {}_\star\mathbf{M}_{i,JJ}^{n+\frac{1}{2}}\mathbf{z}_{i+1,JJ}^{n+\frac{1}{2}} \;=\; \mathbf{S}_{i,JJ-1}^{n+\frac{1}{2}}\mathbf{z}_{i,JJ}^{n} +_2 \boldsymbol{\beta}_{i,JJ}^{n+\frac{1}{2}}\,, \tag{4.30}$$

where

$$\mathbf{S}_{i,j}^{n} = \begin{bmatrix} 1 + \frac{\Delta t}{2\Delta y} V_{i,j}^{n-\frac{1}{2}} & 0 & 0 \\ 2\Omega_2^{n} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.31}$$

The discrete mass equation at grid point $(1, JJ)$ is

$$h_{1,JJ}^{n+\frac{1}{2}} + \tfrac{\Delta t}{2\Delta x} h_{1,JJ}^{\star} U_{2,JJ}^{n+\frac{1}{2}} \;=\; \left(1 + \tfrac{\Delta t}{2\Delta y} V_{1,JJ-1}^{n}\right) h_{1,JJ}^{n}\,. \tag{4.32}$$

Combining equation (4.32) with $V_{1,JJ}^{n+\frac{1}{2}} = 0$ and $U_{1,JJ}^{n+\frac{1}{2}} = 0$ gives

$$\mathbf{I}\mathbf{z}_{1,JJ}^{n+\frac{1}{2}} + {}_\star\mathbf{Q}_{1,JJ}\mathbf{z}_{2,JJ}^{n+\frac{1}{2}} \;=\; \mathbf{T}_{1,JJ-1}^{n}\mathbf{z}_{1,JJ}^{n}\,, \tag{4.33}$$

with

$$\mathbf{T}_{i,j}^{n} = \begin{bmatrix} 1 + \frac{\Delta t}{2\Delta y} V_{i,j}^{n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.34}$$

Similarly the discrete mass equation at grid point (II, JJ) is

$$h_{II,JJ}^{n+\frac{1}{2}} - \tfrac{\Delta t}{2\Delta x} h_{II,JJ}^{\star} U_{II-1,JJ}^{n+\frac{1}{2}} \;=\; \left(1 + \tfrac{\Delta t}{2\Delta y} V_{II,JJ-1}^{n}\right) h_{II,JJ}^{n}\,. \tag{4.35}$$

Combining equation (4.35) with $V_{II,JJ}^{n+\frac{1}{2}} = 0$ and $U_{II,JJ}^{n+\frac{1}{2}} = 0$ gives

$$-{}_\star\mathbf{Q}_{II,JJ}\mathbf{z}_{II-1,JJ}^{n+\frac{1}{2}} + \mathbf{I}\mathbf{z}_{II,JJ}^{n+\frac{1}{2}} \;=\; \mathbf{T}_{II,JJ-1}^{n}\mathbf{z}_{II,JJ}^{n}\,. \tag{4.36}$$

For fixed $h^\star$ and $U^\star$ the following block linear system of equations is to be solved for the grid line (i, JJ),

$$\begin{aligned}
\mathbf{I}\mathbf{z}_{1,JJ}^{n+\frac{1}{2}} + {}_\star\mathbf{Q}_{1,JJ}\mathbf{z}_{2,JJ}^{n+\frac{1}{2}} &= \mathbf{T}_{1,JJ-1}^{n}\mathbf{z}_{1,JJ}^{n}\,, \\
-{}_\star\mathbf{M}_{2,JJ}^{n+\frac{1}{2}}\mathbf{z}_{1,JJ}^{n+\frac{1}{2}} + \mathbf{N}^{n+\frac{1}{2}}\mathbf{z}_{2,JJ}^{n+\frac{1}{2}} + {}_\star\mathbf{M}_{2,JJ}^{n+\frac{1}{2}}\mathbf{z}_{3,JJ}^{n+\frac{1}{2}} &= \mathbf{S}_{2,JJ-1}^{n+\frac{1}{2}}\mathbf{z}_{2,JJ}^{n} +_2 \boldsymbol{\beta}_{2,JJ}^{n+\frac{1}{2}}\,, \\
-{}_\star\mathbf{M}_{3,JJ}^{n+\frac{1}{2}}\mathbf{z}_{2,JJ}^{n+\frac{1}{2}} + \mathbf{N}^{n+\frac{1}{2}}\mathbf{z}_{3,JJ}^{n+\frac{1}{2}} + {}_\star\mathbf{M}_{3,JJ}^{n+\frac{1}{2}}\mathbf{z}_{4,JJ}^{n+\frac{1}{2}} &= \mathbf{S}_{3,JJ-1}^{n+\frac{1}{2}}\mathbf{z}_{3,JJ}^{n} +_2 \boldsymbol{\beta}_{3,JJ}^{n+\frac{1}{2}}\,, \\
\vdots \quad & \quad \vdots \\
-{}_\star\mathbf{Q}_{II,JJ}\mathbf{z}_{II-1,JJ}^{n+\frac{1}{2}} + \mathbf{I}\mathbf{z}_{II,JJ}^{n+\frac{1}{2}} &= \mathbf{T}_{II,JJ-1}^{n}\mathbf{z}_{II,JJ}^{n}\,.
\end{aligned} \tag{4.37}$$

This completes the algorithm details for the first half step $n \mapsto n+\frac{1}{2}$. For each fixed $h^\star$ and $U^\star$, it involves solving a linear block tridiagonal system for each $j = 1, \ldots, JJ$. Then the process is repeated with updates of $h^\star$ and $U^\star$ till convergence $h^\star \to h^{n+\frac{1}{2}}$ and $U^\star \to U^{n+\frac{1}{2}}$.

5 Algorithmic details for the half step $n+\frac{1}{2} \mapsto n+1$

Rewrite the governing equations in a form that emphasizes the nonlinearity

$$
\begin{aligned}
h_t + hU_x + Uh_x + h^\star V_y + V^\star h_y &= 0 \\
U_t + UU_x + V^\star U_y + 2\Omega_2 V^\star h_y + 2\Omega_2 h_t - 2\Omega_3 V & \\
+ \left[\alpha(x,y,t) + 2\Omega_1 V - \left(\Omega_1^2 + \Omega_2^2\right) h\right] h_x &= -\left(\dot{\Omega}_2 + \Omega_1\Omega_3\right) h + \widehat{\beta}(x,y,t) \\
V_t + UV_x + V^\star V_y - 2\Omega_1 U h_x - 2\Omega_1 h_t + 2\Omega_3 U & \\
+ \left[\alpha(x,y,t) - 2\Omega_2 U^\star - \left(\Omega_1^2 + \Omega_2^2\right) h^\star\right] h_y &= +\left(\dot{\Omega}_1 - \Omega_2\Omega_3\right) h + \widetilde{\beta}(x,y,t)\,,
\end{aligned}
\tag{5.1}
$$

where $h^\star$ and $V^\star$ are the current intermediate values of h and V. In the second half step the $y-$derivatives are discretized implicitly and the $x-$derivatives are discretized explicitly. Only implicit nonlinear terms are starred.

The discretizations of the equations (5.1) for the second half of the time step is

$$
\begin{aligned}
&\frac{h_{i,j}^{n+1} - h_{i,j}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} + h_{i,j}^{n+\frac{1}{2}} \frac{U_{i+1,j}^{n+\frac{1}{2}} - U_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} + U_{i,j}^{n+\frac{1}{2}} \frac{h_{i+1,j}^{n+\frac{1}{2}} - h_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} \\
&\qquad + h_{i,j}^\star \frac{V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}}{2\Delta y} + V_{i,j}^\star \frac{h_{i,j+1}^{n+1} - h_{i,j-1}^{n+1}}{2\Delta y} = 0 \\
&\frac{U_{i,j}^{n+1} - U_{i,j}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} + U_{i,j}^{n+\frac{1}{2}} \frac{U_{i+1,j}^{n+\frac{1}{2}} - U_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} + V_{i,j}^\star \frac{U_{i,j+1}^{n+1} - U_{i,j-1}^{n+1}}{2\Delta y} + 2\Omega_2^{n+1} V_{i,j}^\star \frac{h_{i,j+1}^{n+1} - h_{i,j-1}^{n+1}}{2\Delta y} \\
&+ \left[\alpha_{i,j}^{n+1} + 2\Omega_1^{n+1} V_{i,j}^{n+\frac{1}{2}} - \left(\left(\Omega_1^{n+1}\right)^2 + \left(\Omega_2^{n+1}\right)^2\right) h_{i,j}^{n+\frac{1}{2}}\right] \frac{h_{i+1,j}^{n+\frac{1}{2}} - h_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} \\
&\quad = 2\Omega_3^{n+1} V_{i,j}^{n+1} - \left(\dot{\Omega}_2^{n+1} + \Omega_1^{n+1}\Omega_3^{n+1}\right) h_{i,j}^{n+1} - 2\Omega_2^{n+1} \frac{h_{i,j}^{n+1} - h_{i,j}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} + \widehat{\beta}_{i,j}^{n+1} \\
&\frac{V_{i,j}^{n+1} - V_{i,j}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} + U_{i,j}^{n+\frac{1}{2}} \frac{V_{i+1,j}^{n+\frac{1}{2}} - V_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} + V_{i,j}^\star \frac{V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}}{2\Delta y} - 2\Omega_1^{n+1} U_{i,j}^{n+\frac{1}{2}} \frac{h_{i+1,j}^{n+\frac{1}{2}} - h_{i-1,j}^{n+\frac{1}{2}}}{2\Delta x} \\
&+ \left[\alpha_{i,j}^{n+1} - 2\Omega_2^{n+1} U_{i,j}^\star - \left(\left(\Omega_1^{n+1}\right)^2 + \left(\Omega_2^{n+1}\right)^2\right) h_{i,j}^\star\right] \frac{h_{i,j+1}^{n+1} - h_{i,j-1}^{n+1}}{2\Delta y} \\
&\quad = -2\Omega_3^{n+1} U_{i,j}^{n+1} + \left(\dot{\Omega}_1^{n+1} - \Omega_2^{n+1}\Omega_3^{n+1}\right) h_{i,j}^{n+1} + 2\Omega_1^{n+1} \frac{h_{i,j}^{n+1} - h_{i,j}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} + \widetilde{\beta}_{i,j}^{n+1}\,.
\end{aligned}
\tag{5.2}
$$

Equation (5.2) can be expressed in block tridiagonal form

$$\begin{aligned} -{}_\star\mathbf{C}^{n+1}_{i,j}\mathbf{z}^{n+1}_{i,j-1} + \mathbf{B}^{n+1}\mathbf{z}^{n+1}_{i,j} + {}_\star\mathbf{C}^{n+1}_{i,j}\mathbf{z}^{n+1}_{i,j+1} &= \mathbf{A}^{n+1}_{i,j}\mathbf{z}^{n+\frac{1}{2}}_{i-1,j} + \mathbf{D}^{n+1}\mathbf{z}^{n+\frac{1}{2}}_{i,j} \\ &\qquad -\mathbf{A}^{n+1}_{i,j}\mathbf{z}^{n+\frac{1}{2}}_{i+1,j} + \boldsymbol{\beta}^{n+1}_{i,j} \,, \end{aligned} \tag{5.3}$$

with

$$\begin{aligned} \mathbf{A}^n_{i,j} &= \tfrac{\Delta t}{4\Delta x}\begin{bmatrix} U^{n-\frac{1}{2}}_{i,j} & h^{n-\frac{1}{2}}_{i,j} & 0 \\ \widetilde{{}_1\alpha^n_{i,j}} & U^{n-\frac{1}{2}}_{i,j} & 0 \\ -2\Omega^n_1 U^{n-\frac{1}{2}}_{i,j} & 0 & U^{n-\frac{1}{2}}_{i,j} \end{bmatrix}, \\ {}_\star\mathbf{C}^n_{i,j} &= \tfrac{\Delta t}{4\Delta y}\begin{bmatrix} V^\star_{i,j} & 0 & h^\star_{i,j} \\ 2\Omega^n_2 V^\star_{i,j} & V^\star_{i,j} & 0 \\ \widetilde{{}_{2\star}\alpha^n_{i,j}} & 0 & V^\star_{i,j} \end{bmatrix}, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \widetilde{{}_1\alpha^n_{i,j}} &= \alpha^n_{i,j} + 2\Omega^n_1 V^{n-\frac{1}{2}}_{i,j} - \left((\Omega^n_1)^2 + (\Omega^n_2)^2\right) h^{n-\frac{1}{2}}_{i,j} \\ \widetilde{{}_2\alpha^n_{i,j}} &= \alpha^n_{i,j} - 2\Omega^n_2 U^\star_{i,j} - \left((\Omega^n_1)^2 + (\Omega^n_2)^2\right) h^\star_{i,j}. \end{aligned}$$

For fixed $i = 2, \ldots, II-1$ the equations (5.3) are applied for $j = 2, \ldots, JJ-1$. To complete the tridiagonal system equations are needed (for each fixed i) at $j = 1$ and $j = JJ$.

5.1 The equations at $j = 1$ and $j = JJ$ for $i = 2, \ldots, II-1$

The equations at $j = 1$ and $j = JJ$ are obtained from the boundary conditions at $y = 0$ and $y = L_2$. The only boundary condition at $y = 0$ is $V = 0$. The discrete version of this is equation (4.16). To obtain a boundary condition for h, use the mass equation evaluated at $y = 0$

$$h_t + hU_x + Uh_x + h^\star V_y = 0 \,,$$

with discretization

$$\begin{aligned} h^{n+1}_{i,1} + \tfrac{\Delta t}{2\Delta y} h^\star_{i,1} V^{n+1}_{i,2} &= h^{n+\frac{1}{2}}_{i,1} - \tfrac{\Delta t}{4\Delta x} h^{n+\frac{1}{2}}_{i,1}\left(U^{n+\frac{1}{2}}_{i+1,1} - U^{n+\frac{1}{2}}_{i-1,1}\right) \\ &\qquad - \tfrac{\Delta t}{4\Delta x} U^{n+\frac{1}{2}}_{i,1}\left(h^{n+\frac{1}{2}}_{i+1,1} - h^{n+\frac{1}{2}}_{i-1,1}\right) . \end{aligned} \tag{5.5}$$

To obtain a boundary condition for U, use the $x-$momentum equation at $y = 0$

$$U_t + UU_x + 2\Omega_2 h_t + \left[\alpha(x,y,t) - \left(\Omega^2_1 + \Omega^2_2\right) h\right] h_x = -\left(\dot\Omega_2 + \Omega_1\Omega_3\right) h + \widehat\beta(x,y,t) \,,$$

with discretization

$$\begin{aligned} U^{n+1}_{i,1} + \left[2\Omega^{n+1}_2 + \tfrac{1}{2}\Delta t\left(\dot\Omega^{n+1}_2 + \Omega^{n+1}_1\Omega^{n+1}_3\right)\right] h^{n+1}_{i,1} &= U^{n+\frac{1}{2}}_{i,1} \\ &\quad - \tfrac{\Delta t}{4\Delta x} U^{n+\frac{1}{2}}_{i,1}\left(U^{n+\frac{1}{2}}_{i+1,1} - U^{n+\frac{1}{2}}_{i-1,1}\right) \\ &\quad - \tfrac{\Delta t}{4\Delta x}\widehat{\alpha^{n+1}_{i,1}}\left(h^{n+\frac{1}{2}}_{i+1,1} - h^{n+\frac{1}{2}}_{i-1,1}\right) \\ &\quad + 2\Omega^{n+1}_2 h^{n+\frac{1}{2}}_{i,1} + \tfrac{1}{2}\Delta t\widehat\beta^{n+1}_{i,1} \,. \end{aligned} \tag{5.6}$$

Combining equations (4.16), (5.5) and (5.6) gives the equation for $j=1$

$$\mathbf{P}^{n+1}\mathbf{z}_{i,1}^{n+1} + {}_\star\mathbf{J}_{i,1}\mathbf{z}_{i,2}^{n+1} = \mathbf{K}_{i,1}^{n+1}\mathbf{z}_{i-1,1}^{n+\frac{1}{2}} + \mathbf{L}^{n+1}\mathbf{z}_{i,1}^{n+\frac{1}{2}} - \mathbf{K}_{i,1}^{n+1}\mathbf{z}_{i+1,1}^{n+\frac{1}{2}} +_2 \boldsymbol{\beta}_{i,1}^{n+1}\,, \tag{5.7}$$

with

$$\mathbf{P}^n = \begin{bmatrix} 1 & 0 & 0 \\ 2\Omega_2^n + \frac{1}{2}\Delta t\left(\dot{\Omega}_2^n + \Omega_1^n\Omega_3^n\right) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{L}^n = \begin{bmatrix} 1 & 0 & 0 \\ 2\Omega_2^n & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\,,$$
$$\mathbf{K}_{i,j}^n = \tfrac{\Delta t}{4\Delta x}\begin{bmatrix} U_{i,j}^{n-\frac{1}{2}} & h_{i,j}^{n-\frac{1}{2}} & 0 \\ \widehat{\alpha_{i,j}^n} & U_{i,j}^{n-\frac{1}{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}\,, \quad {}_\star\mathbf{J}_{i,j} = \tfrac{\Delta t}{2\Delta y}h_{i,j}^\star \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\,. \tag{5.8}$$

A similar strategy is used to construct the discrete equations at $y=L_2$. The velocity boundary condition is $V=0$, with discretization given in (4.27). The discrete mass equation is

$$\begin{aligned} h_{i,JJ}^{n+1} - \tfrac{\Delta t}{2\Delta y}h_{i,JJ}^\star V_{i,JJ-1}^{n+1} &= h_{i,JJ}^{n+\frac{1}{2}} - \tfrac{\Delta t}{4\Delta x}h_{i,JJ}^{n+\frac{1}{2}}\left(U_{i+1,JJ}^{n+\frac{1}{2}} - U_{i-1,JJ}^{n+\frac{1}{2}}\right) \\ &\quad - \tfrac{\Delta t}{4\Delta x}U_{i,JJ}^{n+\frac{1}{2}}\left(h_{i+1,JJ}^{n+\frac{1}{2}} - h_{i-1,JJ}^{n+\frac{1}{2}}\right)\,. \end{aligned} \tag{5.9}$$

and the discrete $x-$momentum equation is

$$\begin{aligned} U_{i,JJ}^{n+1} + \left[2\Omega_2^{n+1} + \tfrac{1}{2}\Delta t\left(\dot{\Omega}_2^{n+1} + \Omega_1^{n+1}\Omega_3^{n+1}\right)\right]h_{i,JJ}^{n+1} &= U_{i,JJ}^{n+\frac{1}{2}} \\ &\quad - \tfrac{\Delta t}{4\Delta x}U_{i,JJ}^{n+\frac{1}{2}}\left(U_{i+1,JJ}^{n+\frac{1}{2}} - U_{i-1,JJ}^{n+\frac{1}{2}}\right) \\ &\quad - \tfrac{\Delta t}{4\Delta x}\widehat{\alpha_{i,JJ}^{n+1}}\left(h_{i+1,JJ}^{n+\frac{1}{2}} - h_{i-1,JJ}^{n+\frac{1}{2}}\right) \\ &\quad + 2\Omega_2^{n+1}h_{i,JJ}^{n+\frac{1}{2}} + \tfrac{1}{2}\Delta t\widehat{\beta}_{i,JJ}^{n+1}\,. \end{aligned} \tag{5.10}$$

Combining equations (4.27), (5.9) and (5.10) gives the equation for $j=JJ$

$$-{}_\star\mathbf{J}_{i,JJ}\mathbf{z}_{i,JJ-1}^{n+1} + \mathbf{P}^{n+1}\mathbf{z}_{i,JJ}^{n+1} = \mathbf{K}_{i,JJ}^{n+1}\mathbf{z}_{i-1,JJ}^{n+\frac{1}{2}} + \mathbf{L}^{n+1}\mathbf{z}_{i,JJ}^{n+\frac{1}{2}} - \mathbf{K}_{i,JJ}^{n+1}\mathbf{z}_{i+1,JJ}^{n+\frac{1}{2}} +_2 \boldsymbol{\beta}_{i,JJ}^{n+1}\,. \tag{5.11}$$

For fixed $h^\star$, $U^\star$ and $V^\star$ the following block linear system of equations is to be solved along vertical grid lines at interior lines $i=2,\ldots,II-1$,

$$\begin{aligned} \mathbf{P}^{n+1}\mathbf{z}_{i,1}^{n+1} + {}_\star\mathbf{J}_{i,1}\mathbf{z}_{i,2}^{n+1} &= \mathbf{K}_{i,1}^{n+1}\mathbf{z}_{i-1,1}^{n+\frac{1}{2}} + \mathbf{L}^{n+1}\mathbf{z}_{i,1}^{n+\frac{1}{2}} - \mathbf{K}_{i,1}^{n+1}\mathbf{z}_{i+1,1}^{n+\frac{1}{2}} +_2 \boldsymbol{\beta}_{i,1}^{n+1}\,, \\ -{}_\star\mathbf{C}_{i,2}^{n+1}\mathbf{z}_{i,1}^{n+1} + \mathbf{B}^{n+1}\mathbf{z}_{i,2}^{n+1} + {}_\star\mathbf{C}_{i,2}^{n+1}\mathbf{z}_{i,3}^{n+1} &= \mathbf{A}_{i,2}^{n+1}\mathbf{z}_{i-1,2}^{n+\frac{1}{2}} + \mathbf{D}^{n+1}\mathbf{z}_{i,2}^{n+\frac{1}{2}} - \mathbf{A}_{i,2}^{n+1}\mathbf{z}_{i+1,2}^{n+\frac{1}{2}} + \boldsymbol{\beta}_{i,2}^{n+1}\,, \\ -{}_\star\mathbf{C}_{i,3}^{n+1}\mathbf{z}_{i,2}^{n+1} + \mathbf{B}^{n+1}\mathbf{z}_{i,3}^{n+1} + {}_\star\mathbf{C}_{i,3}^{n+1}\mathbf{z}_{i,4}^{n+1} &= \mathbf{A}_{i,3}^{n+1}\mathbf{z}_{i-1,3}^{n+\frac{1}{2}} + \mathbf{D}^{n+1}\mathbf{z}_{i,3}^{n+\frac{1}{2}} - \mathbf{A}_{i,3}^{n+1}\mathbf{z}_{i+1,3}^{n+\frac{1}{2}} + \boldsymbol{\beta}_{i,3}^{n+1}\,, \\ \vdots \qquad & \vdots \\ -{}_\star\mathbf{J}_{i,JJ}\mathbf{z}_{i,JJ-1}^{n+1} + \mathbf{P}^{n+1}\mathbf{z}_{i,JJ}^{n+1} &= \mathbf{K}_{i,JJ}^{n+1}\mathbf{z}_{i-1,JJ}^{n+\frac{1}{2}} + \mathbf{L}^{n+1}\mathbf{z}_{i,JJ}^{n+\frac{1}{2}} - \mathbf{K}_{i,JJ}^{n+1}\mathbf{z}_{i+1,JJ}^{n+\frac{1}{2}} +_2 \boldsymbol{\beta}_{i,JJ}^{n+1}\,. \end{aligned} \tag{5.12}$$

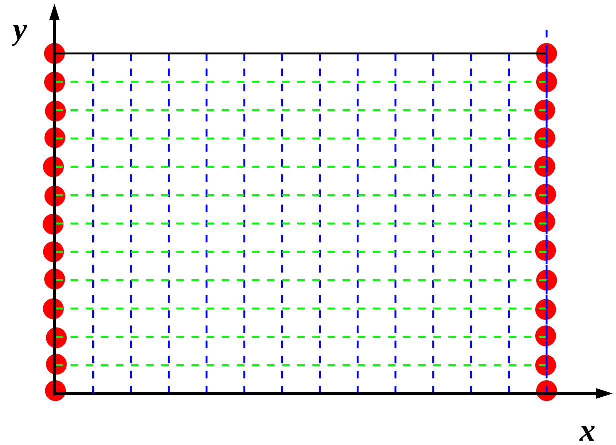

Figure 3: The grid layout with the vertical lines $i = 1$ and $i = II$ highlighted.

This system is iterated until $h^\star \to h^{n+1}$ and $V^\star \to V^{n+1}$. The equations along the grid lines $i = 1$ and $i = II$, highlighted in Figure 3, are solved separately.

5.2 Grid lines $(1, j)$ and (II, j) for $j = 1, \ldots, JJ$

The equations for grid lines $i = 1$ and $i = II$ are obtained by the boundary conditions and restriction of the governing equations to the boundary. The boundary conditions at $x = 0$ is $U(0, y, t) = 0$. The discrete version of this is equation (4.6). To obtain a boundary condition for h, use the mass equation at $x = 0$

$$h_t + hU_x + h^\star V_y + V^\star h_y = 0\,,$$

with discretization

$$h_{1,j}^{n+1} + \tfrac{\Delta t}{4\Delta y} h_{1,j}^\star \left(V_{1,j+1}^{n+1} - V_{1,j-1}^{n+1}\right) + \tfrac{\Delta t}{4\Delta y} V_{1,j}^\star \left(h_{1,j+1}^{n+1} - h_{1,j-1}^{n+1}\right) \;=\; \left(1 - \tfrac{\Delta t}{2\Delta x} U_{2,j}^{n+\frac{1}{2}}\right) h_{1,j}^{n+\frac{1}{2}}\,. \tag{5.13}$$

To obtain a boundary condition for V, use the $y-$momentum equation at $(1, j)$

$$V_t + V^\star V_y - 2\Omega_1 h_t + \left[\alpha(x,y,t) - \left(\Omega_1^2 + \Omega_2^2\right) h^\star\right] h_y = \left(\dot{\Omega}_1 - \Omega_2\Omega_3\right) h + \widetilde{\beta}(x,y,t)\,,$$

with discretization

$$\begin{aligned} &V_{1,j}^{n+1} + \tfrac{\Delta t}{4\Delta y} V_{1,j}^\star \left(V_{1,j+1}^{n+1} - V_{1,j-1}^{n+1}\right) - \left[2\Omega_1^{n+1} + \tfrac{1}{2}\Delta t \left(\dot{\Omega}_1^{n+1} - \Omega_2^{n+1}\Omega_3^{n+1}\right)\right] h_{1,j}^{n+1} \\ &+ \tfrac{\Delta t}{4\Delta y} \widehat{{}_\star\alpha_{1,j}^{n+1}} \left(h_{1,j+1}^{n+1} - h_{1,j-1}^{n+1}\right) = V_{1,j}^{n+\frac{1}{2}} - 2\Omega_1^{n+1} h_{1,j}^{n+\frac{1}{2}} + \tfrac{1}{2}\Delta t \widetilde{\beta}_{1,j}^{n+1}\,. \end{aligned} \tag{5.14}$$

Combining equations (4.6), (5.13) and (5.14) gives the equation for $(1, j)$

$$-{}_\star\overline{\mathbf{M}}_{1,j}^{n+1}\mathbf{z}_{1,j-1}^{n+1} + \overline{\mathbf{N}}^{n+1}\mathbf{z}_{1,j}^{n+1} + {}_\star\overline{\mathbf{M}}_{1,j}^{n+1}\mathbf{z}_{1,j+1}^{n+1} \;=\; \overline{\mathbf{O}}_{2,j}^{n+1}\mathbf{z}_{1,j}^{n+\frac{1}{2}} + {}_1\boldsymbol{\beta}_{1,j}^{n+1}\,, \tag{5.15}$$

with

$$
{}_{\star}\overline{\mathbf{M}}^{n}_{i,j} = \tfrac{\Delta t}{4\Delta y}\begin{bmatrix} V^{\star}_{i,j} & 0 & h^{\star}_{i,j} \\ 0 & 0 & 0 \\ {}_{\star}\widehat{\alpha^{n}_{i,j}} & 0 & V^{\star}_{i,j} \end{bmatrix}, \quad \overline{\mathbf{O}}^{n}_{i,j} = \begin{bmatrix} 1-\frac{\Delta t}{2\Delta x}U^{n-\frac{1}{2}}_{i,j} & 0 & 0 \\ 0 & 0 & 0 \\ -2\Omega^{n}_{1} & 0 & 1 \end{bmatrix},
$$

$$
\overline{\mathbf{N}}^{n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2\Omega^{n}_{1} - \frac{1}{2}\Delta t\left(\dot{\Omega}^{n}_{1} - \Omega^{n}_{2}\Omega^{n}_{3}\right) & 0 & 1 \end{bmatrix}. \tag{5.16}
$$

The mass equation at grid point $(1,1)$ is

$$
h_t + hU_x + h^{\star}V_y = 0\,,
$$

with discretization

$$
h^{n+1}_{1,1} + \tfrac{\Delta t}{2\Delta y}h^{\star}_{1,1}V^{n+1}_{1,2} \;=\; \left(1-\tfrac{\Delta t}{2\Delta x}U^{n+\frac{1}{2}}_{2,1}\right)h^{n+\frac{1}{2}}_{1,1}\,. \tag{5.17}
$$

Combining equation (5.17) with $V^{n+1}_{1,1}=0$ and $U^{n+1}_{1,1}=0$ gives

$$
\mathbf{I}\mathbf{z}^{n+1}_{1,1} + {}_{\star}\overline{\mathbf{Q}}_{1,1}\mathbf{z}^{n+1}_{1,2} \;=\; \overline{\mathbf{R}}^{n+\frac{1}{2}}_{2,1}\mathbf{z}^{n+\frac{1}{2}}_{1,1}\,, \tag{5.18}
$$

with

$$
{}_{\star}\overline{\mathbf{Q}}_{i,j} = \tfrac{\Delta t}{2\Delta y}h^{\star}_{i,j}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{\mathbf{R}}^{n}_{i,j} = \begin{bmatrix} 1-\frac{\Delta t}{2\Delta x}U^{n}_{i,j} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.19}
$$

Similarly the discrete mass equation at grid point $(1,JJ)$ is

$$
h^{n+1}_{1,JJ} - \tfrac{\Delta t}{2\Delta y}h^{\star}_{1,JJ}V^{n+1}_{1,JJ-1} \;=\; \left(1-\tfrac{\Delta t}{2\Delta x}U^{n+\frac{1}{2}}_{2,JJ}\right)h^{n+\frac{1}{2}}_{1,JJ}\,. \tag{5.20}
$$

Combining equation (5.20) with $V^{n+1}_{1,JJ}=0$ and $U^{n+1}_{1,JJ}=0$ gives

$$
-{}_{\star}\overline{\mathbf{Q}}_{1,JJ}\mathbf{z}^{n+1}_{1,JJ-1} + \mathbf{I}\mathbf{z}^{n+1}_{1,JJ} \;=\; \overline{\mathbf{R}}^{n+\frac{1}{2}}_{2,JJ}\mathbf{z}^{n+\frac{1}{2}}_{1,JJ}\,. \tag{5.21}
$$

For fixed $h^{\star}$ and $V^{\star}$ the following block linear system of equations is to be solved for the grid line $(1,j)$,

$$
\begin{aligned}
\mathbf{I}\mathbf{z}^{n+1}_{1,1} + {}_{\star}\overline{\mathbf{Q}}_{1,1}\mathbf{z}^{n+1}_{1,2} &= \overline{\mathbf{R}}^{n+\frac{1}{2}}_{2,1}\mathbf{z}^{n+\frac{1}{2}}_{1,1}\,,\\
-{}_{\star}\overline{\mathbf{M}}^{n+1}_{1,2}\mathbf{z}^{n+1}_{1,1} + \overline{\mathbf{N}}^{n+1}\mathbf{z}^{n+1}_{1,2} + {}_{\star}\overline{\mathbf{M}}^{n+1}_{1,2}\mathbf{z}^{n+1}_{1,3} &= \overline{\mathbf{O}}^{n+1}_{2,2}\mathbf{z}^{n+\frac{1}{2}}_{1,2} +_{1} \boldsymbol{\beta}^{n+1}_{1,2}\,,\\
-{}_{\star}\overline{\mathbf{M}}^{n+1}_{1,3}\mathbf{z}^{n+1}_{1,2} + \overline{\mathbf{N}}^{n+1}\mathbf{z}^{n+1}_{1,3} + {}_{\star}\overline{\mathbf{M}}^{n+1}_{1,3}\mathbf{z}^{n+1}_{1,4} &= \overline{\mathbf{O}}^{n+1}_{2,3}\mathbf{z}^{n+\frac{1}{2}}_{1,3} +_{1} \boldsymbol{\beta}^{n+1}_{1,3}\,,\\
&\vdots \quad \vdots\\
-{}_{\star}\overline{\mathbf{Q}}_{1,JJ}\mathbf{z}^{n+1}_{1,JJ-1} + \mathbf{I}\mathbf{z}^{n+1}_{1,JJ} &= \overline{\mathbf{R}}^{n+\frac{1}{2}}_{2,JJ}\mathbf{z}^{n+\frac{1}{2}}_{1,JJ}\,.
\end{aligned} \tag{5.22}
$$

A similar strategy is used along the right boundary $i = II$. The boundary condition is $U(L_1, y, t) = 0$, with discretization (4.11). The discrete mass equation at (II, j) is

$$\begin{aligned} h_{II,j}^{n+1} + \tfrac{\Delta t}{4\Delta y} h_{II,j}^{\star}\left(V_{II,j+1}^{n+1} - V_{II,j-1}^{n+1}\right) + \tfrac{\Delta t}{4\Delta y} V_{II,j}^{\star}\left(h_{II,j+1}^{n+1} - h_{II,j-1}^{n+1}\right) = \\ \left(1 + \tfrac{\Delta t}{2\Delta x} U_{II-1,j}^{n+\frac{1}{2}}\right) h_{II,j}^{n+\frac{1}{2}} , \end{aligned} \tag{5.23}$$

and the discrete $y-$momentum equation is

$$\begin{aligned} V_{II,j}^{n+1} + \tfrac{\Delta t}{4\Delta y} V_{II,j}^{\star}\left(V_{II,j+1}^{n+1} - V_{II,j-1}^{n+1}\right) - \left[2\Omega_1^{n+1} + \tfrac{1}{2}\Delta t\left(\dot{\Omega}_1^{n+1} - \Omega_2^{n+1}\Omega_3^{n+1}\right)\right] h_{II,j}^{n+1} \\ + \tfrac{\Delta t}{4\Delta y} \widehat{{}_{\star}\alpha_{II,j}^{n+1}}\left(h_{II,j+1}^{n+1} - h_{II,j-1}^{n+1}\right) = V_{II,j}^{n+\frac{1}{2}} - 2\Omega_1^{n+1} h_{II,j}^{n+\frac{1}{2}} + \tfrac{1}{2}\Delta t \widetilde{\beta}_{II,j}^{n+1} . \end{aligned} \tag{5.24}$$

Combining equations (4.11), (5.23) and (5.24) gives the equation for (II, j)

$$-{}_{\star}\overline{\mathbf{M}}_{II,j}^{n+1}\mathbf{z}_{II,j-1}^{n+1} + \overline{\mathbf{N}}^{n+1}\mathbf{z}_{II,j}^{n+1} + {}_{\star}\overline{\mathbf{M}}_{II,j}^{n+1}\mathbf{z}_{II,j+1}^{n+1} \;=\; \overline{\mathbf{S}}_{II-1,j}^{n+1}\mathbf{z}_{II,j}^{n+\frac{1}{2}} + {}_1\boldsymbol{\beta}_{II,j}^{n+1} , \tag{5.25}$$

with

$$\overline{\mathbf{S}}_{i,j}^{n} = \begin{bmatrix} 1 + \frac{\Delta t}{2\Delta x} U_{i,j}^{n-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 \\ -2\Omega_1^n & 0 & 1 \end{bmatrix} . \tag{5.26}$$

The discrete mass equation at lower right corner grid point $(II, 1)$ is

$$h_t + hU_x + h^{\star} V_y = 0 ,$$

with discretization

$$h_{II,1}^{n+1} + \tfrac{\Delta t}{2\Delta y} h_{II,1}^{\star} V_{II,2}^{n+1} \;=\; \left(1 + \tfrac{\Delta t}{2\Delta x} U_{II-1,1}^{n+\frac{1}{2}}\right) h_{II,1}^{n+\frac{1}{2}} . \tag{5.27}$$

Combining equation (5.27) with $V_{II,1}^{n+1} = 0$ and $U_{II,1}^{n+1} = 0$ gives

$$\mathbf{I}\mathbf{z}_{II,1}^{n+1} + {}_{\star}\overline{\mathbf{Q}}_{II,1}\mathbf{z}_{II,2}^{n+1} \;=\; \overline{\mathbf{T}}_{II-1,1}^{n+\frac{1}{2}}\mathbf{z}_{II,1}^{n+\frac{1}{2}} , \tag{5.28}$$

with

$$\overline{\mathbf{T}}_{i,j}^{n} \;=\; \begin{bmatrix} 1 + \frac{\Delta t}{2\Delta x} U_{i,j}^{n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{5.29}$$

Similarly the discrete mass equation at upper right cornder grid point (II, JJ) is

$$h_{II,JJ}^{n+1} - \tfrac{\Delta t}{2\Delta y} h_{II,JJ}^{\star} V_{II,JJ-1}^{n+1} \;=\; \left(1 + \tfrac{\Delta t}{2\Delta x} U_{II-1,JJ}^{n+\frac{1}{2}}\right) h_{II,JJ}^{n+\frac{1}{2}} . \tag{5.30}$$

Combining equation (5.30) with $V_{II,JJ}^{n+1} = 0$ and $U_{II,JJ}^{n+1} = 0$ gives

$$-{}_{\star}\overline{\mathbf{Q}}_{II,JJ}\mathbf{z}_{II,JJ-1}^{n+1} + \mathbf{I}\mathbf{z}_{II,JJ}^{n+1} \;=\; \overline{\mathbf{T}}_{II-1,JJ}^{n+\frac{1}{2}}\mathbf{z}_{II,JJ}^{n+\frac{1}{2}} . \tag{5.31}$$

For fixed $h^\star$ and $V^\star$ the following block linear system of equations is to be solved along the right boundary grid line $i = II$,

$$\begin{aligned}
\mathbf{I}\mathbf{z}_{II,1}^{n+1} + {}_\star\overline{\mathbf{Q}}_{II,1}\mathbf{z}_{II,2}^{n+1} &= \overline{\mathbf{T}}_{II-1,1}^{n+\frac{1}{2}}\mathbf{z}_{II,1}^{n+\frac{1}{2}}\,,\\
-{}_\star\overline{\mathbf{M}}_{II,2}^{n+1}\mathbf{z}_{II,1}^{n+1} + \overline{\mathbf{N}}^{n+1}\mathbf{z}_{II,2}^{n+1} + {}_\star\overline{\mathbf{M}}_{II,2}^{n+1}\mathbf{z}_{II,3}^{n+1} &= \overline{\mathbf{S}}_{II-1,2}^{n+1}\mathbf{z}_{II,2}^{n+\frac{1}{2}} +_1 \boldsymbol{\beta}_{II,2}^{n+1}\,,\\
-{}_\star\overline{\mathbf{M}}_{II,3}^{n+1}\mathbf{z}_{II,2}^{n+1} + \overline{\mathbf{N}}^{n+1}\mathbf{z}_{II,3}^{n+1} + {}_\star\overline{\mathbf{M}}_{II,3}^{n+1}\mathbf{z}_{II,4}^{n+1} &= \overline{\mathbf{S}}_{II-1,3}^{n+1}\mathbf{z}_{II,3}^{n+\frac{1}{2}} +_1 \boldsymbol{\beta}_{II,3}^{n+1}\,,\\
&\ \,\vdots \qquad \vdots\\
-{}_\star\overline{\mathbf{Q}}_{II,JJ}\mathbf{z}_{II,JJ-1}^{n+1} + \mathbf{I}\mathbf{z}_{II,JJ}^{n+1} &= \overline{\mathbf{T}}_{II-1,JJ}^{n+\frac{1}{2}}\mathbf{z}_{II,JJ}^{n+\frac{1}{2}}\,.
\end{aligned} \tag{5.32}$$

This completes the algorithm details for the second half step $n + \frac{1}{2} \mapsto n + 1$. For each fixed $h^\star$ and $V^\star$, it involves solving a linear block tridiagonal system for each $i = 1, \ldots, II$. Then the process is repeated with updates of $h^\star$ and $V^\star$ till convergence $h^\star \to h^{n+1}$ and $V^\star \to V^{n+1}$.

Numerical results using this scheme are reported in [4].

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