The Euler equations in fluid mechanics relative to a rotating-translating reference frame

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1   The spatial and body reference frames

In this technical report the details of the construction of the apparent accelerations, which appear in the Euler equations when viewed from a body fixed 3D moving frame, are presented. A moving frame has been widely used in the study of sloshing [4, 3]. However, there are small subtleties that have been overlooked in the derivation, and therefore a detailed derivation is presented here.

Let $\mathbf{X} = (X, Y, Z)$ be the *spatial frame*. This frame is fixed in space. Let $\mathbf{x} = (x, y, z)$ be the *body frame*. It is attached to the moving vessel. For Newton's law the acceleration in the spatial frame is required, but the analysis of the fluid motion is carried out in the body frame. The relationship between the two accelerations is developed using the kinetic theory of rigid bodies [6, 5].
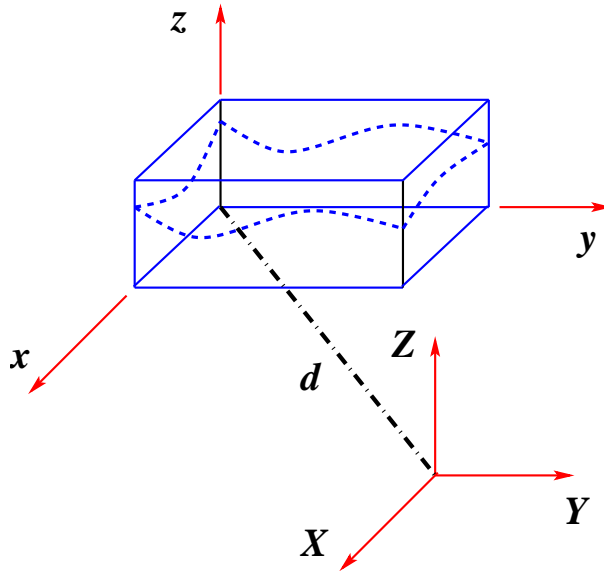

Figure 1: Schematic of spatial and body frames of reference.

Let $\mathbf{Q}(t)$ be a proper rotation in $\mathbb{R}^3$,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad \text{and} \quad \det[\mathbf{Q}] = 1.$$

Then the relation between the spatial displacement and the body displacement is

$$\mathbf{X} = \mathbf{Q}(\mathbf{x} + \mathbf{d}) + \mathbf{q}.$$

The term $\mathbf{Qd}$ could be incorporated into $\mathbf{q}$ but in cases where the linear displacements are zero it will be useful to maintain the distinction.

The relation between the body velocity and space velocity is

$$\dot{\mathbf{X}} = \dot{\mathbf{Q}}(\mathbf{x} + \mathbf{d}) + \mathbf{Q}\dot{\mathbf{x}} + \dot{\mathbf{q}}. \tag{1.1}$$

Define the *body angular velocity* by

$$\widehat{\boldsymbol{\Omega}} = \mathbf{Q}^T\dot{\mathbf{Q}} := \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}. \tag{1.2}$$

The convention for the entries of the skew-symmetric matrix $\widehat{\boldsymbol{\Omega}}$ is such that

$$\widehat{\boldsymbol{\Omega}}\mathbf{r} = \boldsymbol{\Omega} \times \mathbf{r}, \quad \text{for any} \quad \mathbf{r} \in \mathbb{R}^3, \quad \boldsymbol{\Omega} := \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}.$$

The body angular velocity is to be contrasted with the *spatial angular velocity* – the angular velocity viewed from the spatial frame – which is

$$\widehat{\boldsymbol{\Omega}}^{\text{spatial}} := \dot{\mathbf{Q}}\mathbf{Q}^T.$$

As vectors the spatial and body angular velocities are related by $\boldsymbol{\Omega}^{\text{spatial}} = \mathbf{Q}\boldsymbol{\Omega}$. The angular velocity without a superscript will represent the body angular velocity. It is the body angular velocity that will be used predominantly.

Express (1.1) in terms of the angular velocity

$$\dot{\mathbf{X}} = \mathbf{Q}(\dot{\mathbf{x}} + \boldsymbol{\Omega} \times (\mathbf{x} + \mathbf{d})) + \dot{\mathbf{q}}. \tag{1.3}$$

The spatial acceleration is then

$$\ddot{\mathbf{X}} = \dot{\mathbf{Q}}(\dot{\mathbf{x}} + \boldsymbol{\Omega} \times (\mathbf{x} + \mathbf{d})) + \mathbf{Q}(\ddot{\mathbf{x}} + \dot{\boldsymbol{\Omega}} \times (\mathbf{x} + \mathbf{d}) + \boldsymbol{\Omega} \times \dot{\mathbf{x}}) + \ddot{\mathbf{q}},$$

or

$$\ddot{\mathbf{X}} = \mathbf{Q}[\ddot{\mathbf{x}} + 2\boldsymbol{\Omega} \times \dot{\mathbf{x}} + \dot{\boldsymbol{\Omega}} \times (\mathbf{x} + \mathbf{d}) + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times (\mathbf{x} + \mathbf{d}) + \mathbf{Q}^T\ddot{\mathbf{q}}]. \tag{1.4}$$

The velocity $\dot{\mathbf{x}}$ and acceleration $\ddot{\mathbf{x}}$ represent the velocity and acceleration of a fluid particle relative to the body frame. They are Lagrangian variables. Replace them by their respective Eulerian quantities

$$\ddot{\mathbf{X}} = \mathbf{Q}\left[\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + \dot{\boldsymbol{\Omega}} \times (\mathbf{x} + \mathbf{d}) + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times (\mathbf{x} + \mathbf{d}) + \mathbf{Q}^T\ddot{\mathbf{q}}\right], \tag{1.5}$$

where $\frac{D\mathbf{u}}{Dt}$ is the material derivative and $\mathbf{u} = (u, v, w)$.

Newton's second law for the fluid of uniform density ρ is

$$\rho\ddot{\mathbf{X}} = \mathbf{F},$$

where $\mathbf{F}$ represents the forces present – *in the spatial frame*. The forces present are the pressure gradient and gravity, which relative to the spatial frame are,

$$\mathbf{F} := -\nabla_{\mathbf{X}} P - \rho\mathbf{g}, \quad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix},$$

where $P(\mathbf{X}, t)$ is the pressure field in the spatial frame, $\nabla_{\mathbf{X}}$ is the gradient in the spatial frame, and $g > 0$ is the gravitational constant. Define the pressure relative to the body frame by

$$p(\mathbf{x}, t) := P(\mathbf{Q}(\mathbf{x} + \mathbf{d}) + \mathbf{q}, t),$$

from which it follows

$$\nabla_{\mathbf{x}} p = \mathbf{Q}^T \nabla_{\mathbf{X}} P.$$

The force field is then

$$\mathbf{F} = \mathbf{Q}[-\nabla_{\mathbf{x}} p - \rho\mathbf{Q}^T\mathbf{g}].$$

Newton's second law then gives

$$\rho\ddot{\mathbf{X}} = -\mathbf{Q}\left[\nabla_{\mathbf{x}} p + \rho\mathbf{Q}^T\mathbf{g}\right].$$

Substituting in for the acceleration, dividing by ρ and multiplying by $\mathbf{Q}^T$, then gives the governing equations in the body frame,

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p = -2\mathbf{\Omega} \times \mathbf{u} - \dot{\mathbf{\Omega}} \times (\mathbf{x} + \mathbf{d}) - \mathbf{\Omega} \times (\mathbf{\Omega} \times (\mathbf{x} + \mathbf{d})) - \mathbf{Q}^T\ddot{\mathbf{q}} - \mathbf{Q}^T\mathbf{g}. \quad (1.6)$$

The gradient without a subscript will always denote a gradient in the body frame. These equations are the starting point for the derivation, analysis and numerics in [2].

1.1 Two-dimensional case

In the case of two-dimensional motions, the equations simplify dramatically. The simplification arises since proper rotation matrices in two-dimensions simplify to the form

$$\mathbf{Q}(t) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix},$$

for a scalar function $\theta(t)$. The body angular velocity and spatial angular velocity coincide since

$$\mathbf{Q}^T\dot{\mathbf{Q}} = \dot{\mathbf{Q}}\mathbf{Q}^T = \dot{\theta}\mathbf{J}, \quad \mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Taking spatial coordinates (X, Y) and body coordinates (x, y) with fluid velocities (u, v) the equations (1.6) simplify to

$$
\begin{aligned}
\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} &= -g\sin\theta + 2\dot{\theta}v + \ddot{\theta}(y + d_2) + \dot{\theta}^2(x + d_1) - \ddot{q}_1\cos\theta - \ddot{q}_2\sin\theta\,, \\
\frac{Dv}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial y} &= -g\cos\theta - 2\dot{\theta}u - \ddot{\theta}(x + d_1) + \dot{\theta}^2(y + d_2) + \ddot{q}_1\sin\theta - \ddot{q}_2\cos\theta\,,
\end{aligned}
\tag{1.7}
$$

which are the equations that are the starting point in [1].

References

[1] H. ALEMI ARDAKANI & T.J. BRIDGES. *Shallow-water sloshing in vessels undergoing prescribed rigid body motion in two dimensions*, Preprint (2010).

[2] H. ALEMI ARDAKANI & T.J. BRIDGES. *Shallow-water sloshing in vessels undergoing prescribed rigid body motion in three dimensions*, J. Fluid Mech. (in press, 2010).

[3] O.M. FALTINSEN & A.N. TIMOKHA. *Sloshing*, Cambridge University Press (2009).

[4] R.A. IBRAHIM. *Liquid Sloshing Dynamics*, Cambridge University Press (2005).

[5] R.M. MURRAY AND Z.X. LIN AND S.S. SASTRY. *A Mathematical Introduction to Robotic Manipulation*, CRC Press: Boca Raton (1994).

[6] O.M. O'REILLY. *Intermediate Dynamics for Engineers: a Unified Treatment of Newton-Euler and Lagrangian Mechanics*, Cambridge University Press (2008).