Review of the Dillingham, Falzarano & Pantazopoulos rotating three-dimensional shallow-water equations

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1 Introduction

Two derivations of the shallow water equations (SWEs) for fluid in a vessel that is undergoing a general rigid-body motion in three dimensions first appeared in the literature at about the same time, given independently by PANTAZOPOULOS [8, 9, 10] and DILLINGHAM & FALZARANO [3]. Both derivations follow the same strategy. Their respective derivations are an extension of the formulation for two-dimensional shallow water flow in a rotating frame in [2]. Their idea is to start with the classical SWEs

$$\begin{aligned} u_t + uu_x + vu_y + gh_x &= 0 \\ v_t + uv_x + vv_y + gh_y &= 0 \\ h_t + (hu)_x + (hv)_y &= 0\,, \end{aligned} \tag{1.1}$$

where $h(x, y, t)$ is the free surface elevation, u, v are representative horizontal velocities and $g > 0$ is the gravitational constant. They then deduce the acceleration

$$\mathbf{a}_{(\mathbf{x})} := (a_{(x)}, a_{(y)}, a_{(z)})\,,$$

of the body frame relative to an inertial frame. Then g is replaced by an average of $a_{(z)}$ and approximations for $a_{(x)}$ and $a_{(y)}$ are substituted into the right-hand side of the horizontal momentum equations. The resulting SWEs will be called the DFP SWEs. These equations were also later used in [4] to study the dynamical behaviour of an offshore supply vessel with water on deck.

The purpose of this report is to determine the precise approximations used in the derivation in order to compare with the new shallow-water equations found in [1].

2 Summary of DFP SWEs

Pantazopoulos [8, 9, 10] and Dillingham & Falzarano [3] propose the following form for the 3D rotating shallow-water equations

$$\begin{aligned} u_t + uu_x + vu_y + a_{(z)}h_x &= f_1 \\ v_t + uv_x + vv_y + a_{(z)}h_y &= f_2 \\ h_t + (hu)_x + (hv)_y &= 0\,, \end{aligned} \tag{2.2}$$

where f_1 and f_2 are representations of the horizontal accelerations due to the vessel motion. In the sequel we will use [10] as a guide to the derivation as it gives the most detail. The final equations in all the sources are the same modulo typographical errors. The expression given for f_1 is

$$\begin{aligned} f_1 = {} & -\ddot{n}_1\cos\theta - \ddot{n}_2\sin\phi\sin\theta + \ddot{n}_3\sin\theta\cos\phi - 2\omega_1\omega_2 x\sin\phi\sin\theta\cos\theta \\ & -\omega_1\omega_2 y\cos\phi\cos\theta - \omega_1\omega_2 z_d\sin\phi(\sin^2\theta - \cos^2\theta) + \omega_1^2 x\sin^2\theta \\ & -\omega_1^2 z_d\sin\theta\cos\theta + \omega_2^2 x(1-\sin^2\theta\sin^2\phi) - \omega_2^2 y\sin\phi\sin\theta\cos\phi \\ & +\omega_2^2 z_d\sin\theta\sin^2\phi\cos\theta + 2\omega_1 v\sin\theta - 2\omega_2 v\sin\phi\cos\theta \boxed{+\dot{\omega}_1 y\sin\theta} \\ & -\dot{\omega}_2 y\sin\phi\cos\theta - \dot{\omega}_2 z_d\cos\phi + g\sin\theta\cos\phi\,, \end{aligned} \tag{2.3}$$

after correcting typos and adding in a missing term (boxed in the above equation). This missing term also appears to be a typo since the boxed term is included in $a_{(x)}$ in [10] and $f_1 = -a_{(x)} + \dot{u}$. The expression for f_2 in [10] is

$$\begin{aligned} f_2 = {} & -\ddot{n}_2\cos\phi - \ddot{n}_3\sin\phi + \omega_1^2 y - \omega_2^2 x\sin\phi\sin\theta\cos\phi + \omega_2^2 y\sin^2\phi \\ & +\omega_2^2 z_d\sin\phi\cos\phi\cos\theta - \omega_1\omega_2 x\cos\phi\cos\theta - \omega_1\omega_2 z_d\sin\theta\cos\phi \\ & -2\omega_1 u\sin\theta + 2\omega_2 u\sin\phi\cos\theta - \dot{\omega}_1 x\sin\theta + \dot{\omega}_1 z_d\cos\theta \\ & +\dot{\omega}_2 x\sin\phi\cos\theta + \dot{\omega}_2 z_d\sin\theta\sin\phi - g\sin\phi\,, \end{aligned} \tag{2.4}$$

after correction of two typos. The vertical acceleration term is

$$\begin{aligned} a_{(z)} = {} & \ddot{n}_1\sin\theta - \ddot{n}_2\sin\phi\cos\theta + \ddot{n}_3\cos\phi\cos\theta + \omega_1\omega_2 x\sin\phi(\sin^2\theta - \cos^2\theta) \\ & +\omega_1\omega_2 y\sin\theta\cos\phi - 2\omega_1\omega_2 z_d\sin\phi\sin\theta\cos\theta \\ & +\omega_1^2 x\sin\theta\cos\theta - \omega_1^2 z_d\cos^2\theta - \omega_2^2 x\sin\theta\sin^2\phi\cos\theta \\ & -\omega_2^2 y\sin\phi\cos\phi\cos\theta - \omega_2^2\boxed{z_d}(1-\sin^2\phi\cos^2\theta) + 2\omega_1 v\cos\theta \\ & +2\omega_2 v\sin\phi\sin\theta - 2\omega_2 u\cos\phi + \dot{\omega}_1 y\cos\theta - \dot{\omega}_2 x\cos\phi \\ & +\dot{\omega}_2 y\sin\phi\sin\theta + g\cos\phi\cos\theta\,. \end{aligned} \tag{2.5}$$

In the term $\omega_2^2\boxed{z_d}(1-\sin^2\phi\cos^2\theta)$ in [8, 10], the boxed part, z_d, is missing and it has been reinstated here. The same equations are used in [3, 4, 9]. We have not seen reference [3] but [4] imply that the equations are the same as above.

3 Major assumption on the angular velocity

In these equations $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the *spatial angular velocity* and the reader will note that ω_3 does not appear anywhere in the above expressions. This missing term is because of the following assumption

$$\omega_3 = 0\,. \qquad \textbf{(DFP-1)}$$

This assumption is surprising. It is both inconsistent and unnecessary. It is inconsistent because the third component of the body angular velocity is not set to zero as well (see below). It is unnecessary because the authors develop these equations for numerical simulation and so inclusion of ω_3 is straightforward.

The use of spatial angular velocity – rather than body angular velocity – is also what causes the expressions for f_1, f_2 and the vertical acceleration to be excessively complicated.

4 Moving frame of reference

Let $\mathbf{X} = (X, Y, Z)$ be coordinates for a fixed spatial frame and let $\mathbf{x} = (x, y, z)$ be coordinates for the body frame. Then DFP use the following relationship between the fixed frame and the moving frame

$$\mathbf{X} = \mathbf{Q}(\mathbf{x} + \mathbf{d}) + \mathbf{n}\,,$$

with $\mathbf{n} = (n_1, n_2, n_3)$ the spatial translation of the moving frame. The following assumptions are imposed on $\mathbf{x}$ and $\mathbf{d}$

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}\,, \qquad \textbf{(DFP-2)}$$

and

$$\mathbf{d} = \begin{pmatrix} 0 \\ 0 \\ z_d \end{pmatrix}\,. \qquad \textbf{(DFP-3)}$$

The rotation matrix is restricted to

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ \sin\phi\sin\theta & \cos\phi & -\sin\phi\cos\theta \\ -\cos\phi\sin\theta & \sin\phi & \cos\phi\cos\theta \end{bmatrix}\,. \qquad \textbf{(DFP-4)}$$

This matrix is given explicitly in equation (6) on page 29 of [10]. The angles ϕ and θ are Euler angles associated with roll and pitch respectively. A derivation of (**DFP-4**) using rotation tensors is given in the next section.

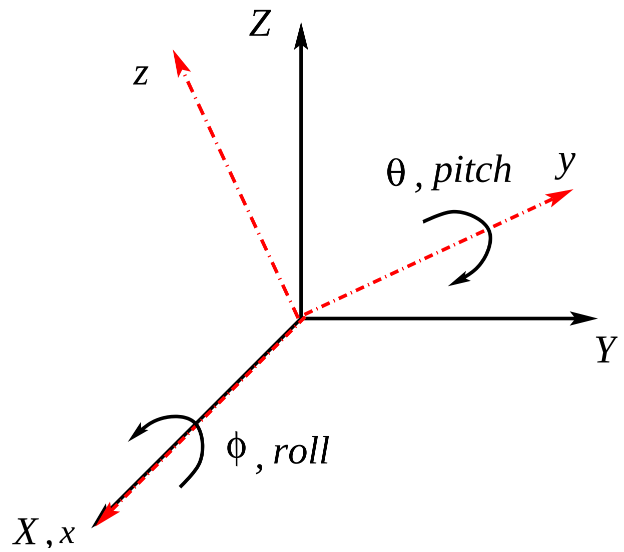

Figure 1: Schematic of the roll-pitch motion in terms of Euler angles ϕ and θ.

5 Prescribed roll-pitch motion of the vessel

A derivation of the rotation matrix (**DFP-4**) is given in terms of rotation tensors. Let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ be a basis for the spatial frame $\mathbf{X}$.

Using a coordinate-free rotation tensor formulation [7], the roll motion is

$$\mathbf{R} := \mathbf{L}(\phi, \mathbf{E}_1) = \cos\phi\,(\mathbf{I} - \mathbf{E}_1 \otimes \mathbf{E}_1) + \sin\phi\,\widehat{\mathbf{E}}_1 + \mathbf{E}_1 \otimes \mathbf{E}_1\,, \tag{5.1}$$

where

$$(\mathbf{a} \otimes \mathbf{b})\,\mathbf{c} := (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}\,, \quad \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3\,,$$

and

$$\widehat{\mathbf{a}} := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}\,. \tag{5.2}$$

The hat-matrix has the property that

$$\widehat{\mathbf{a}}\,\mathbf{b} = \mathbf{a} \times \mathbf{b}\,, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\,.$$

The rotation (5.1) is a counterclockwise rotation about the $X-$axis as shown schematically in Figure 1.

The matrix representation of the rotation (5.1) is

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\,.$$

However it is not a good idea to use a matrix representation until the full rotation matrix is constructed since the basis for the rotation is changing. The pitch rotation is a counterclockwise rotation about the $y-$axis. The axis of rotation is

$$\mathbf{a}_2 := \mathbf{R}\mathbf{E}_2 = \cos\phi\mathbf{E}_2 + \sin\phi\mathbf{E}_3\,.$$

The pitch rotation is therefore

$$\mathbf{P} := \mathbf{L}(\theta, \mathbf{a}_2) = \cos\theta\,(\mathbf{I} - \mathbf{a}_2 \otimes \mathbf{a}_2) + \sin\theta\,\widehat{\mathbf{a}}_2 + \mathbf{a}_2 \otimes \mathbf{a}_2\,.$$

The roll-pitch rotation is then

$$\mathbf{Q} = \mathbf{P}\mathbf{R}\,.$$

In order to construct a matrix representation of this composite rotation, use the property of rotation tensors

$$\mathbf{L}(\theta, \mathbf{R}\mathbf{b}) = \mathbf{R}\,\mathbf{L}(\theta, \mathbf{b})\,\mathbf{R}^T\,.$$

when $\mathbf{R}$ is any proper rotation (see [5, 6] for a proof of this identity). Applying this property to $\mathbf{Q}$ gives

$$\begin{aligned}
\mathbf{Q} &= \mathbf{P}\mathbf{R} \\
&= \mathbf{L}(\theta, \mathbf{R}\mathbf{E}_2)\mathbf{L}(\phi, \mathbf{E}_1) \\
&= \mathbf{R}\mathbf{L}(\theta, \mathbf{E}_2)\mathbf{R}^T\mathbf{R} \\
&= \mathbf{L}(\phi, \mathbf{E}_1)\mathbf{L}(\theta, \mathbf{E}_2)\,.
\end{aligned}$$

Since both matrices are now relative to the standard basis, a matrix representation can be constructed,

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

which when multiplied out gives (**DFP-4**).

The angular velocities are now easily computed. Considering ϕ and θ as functions of time

$$\dot{\mathbf{Q}}\mathbf{Q}^T = \begin{bmatrix} 0 & -\Omega_3^s & \Omega_2^s \\ \Omega_3^s & 0 & -\Omega_1^s \\ -\Omega_2^s & \Omega_1^s & 0 \end{bmatrix} = \widehat{\mathbf{\Omega}^s}\,,$$

where $\mathbf{\Omega}^s = (\Omega_1^s, \Omega_2^s, \Omega_3^s)$ is the *spatial* angular velocity vector. Now

$$\frac{d}{dt}\mathbf{L}(\gamma, \mathbf{E}_j)\mathbf{L}(\gamma, \mathbf{E}_j)^T = \dot{\gamma}\widehat{\mathbf{E}_j}\,,$$

and

$$\dot{\mathbf{Q}} = \dot{\mathbf{L}}(\phi, \mathbf{E}_1)\mathbf{L}(\theta, \mathbf{E}_2) + \mathbf{L}(\phi, \mathbf{E}_1)\dot{\mathbf{L}}(\theta, \mathbf{E}_2) = \dot{\phi}\widehat{\mathbf{E}_1}\mathbf{Q} + \dot{\theta}\mathbf{L}(\phi, \mathbf{E}_1)\widehat{\mathbf{E}_2}\mathbf{L}(\theta, \mathbf{E}_2)$$

and so

$$\widehat{\mathbf{\Omega}^s} = \dot{\mathbf{Q}}\mathbf{Q}^T = \dot{\phi}\widehat{\mathbf{E}_1} + \dot{\theta}\mathbf{L}(\phi, \mathbf{E}_1)\widehat{\mathbf{E}_2}\mathbf{L}(\phi, \mathbf{E}_1)^T \,.$$

Now use the identity

$$\widehat{\mathbf{Q}\mathbf{v}} = \mathbf{Q}\widehat{\mathbf{v}}\mathbf{Q}^T\,, \quad \text{for any } \mathbf{v} \in \mathbb{R}^3\,, \quad \text{and any } \mathbf{Q} \in \mathbf{SO}(3)\,,$$

to arrive at

$$\begin{aligned} \mathbf{\Omega}^s &= \dot{\phi}\mathbf{E}_1 + \dot{\theta}\mathbf{L}(\phi, \mathbf{E}_1)\mathbf{E}_2 \\ &= \dot{\phi}\mathbf{E}_1 + \dot{\theta}\mathbf{a}_2 \\ &= \dot{\phi}\mathbf{E}_1 + \dot{\theta}(\cos\phi\mathbf{E}_2 + \sin\phi\mathbf{E}_3)\,, \end{aligned}$$

or in components

$$\mathbf{\Omega}^s = \begin{pmatrix} \dot{\phi} \\ \dot{\theta}\cos\phi \\ \dot{\theta}\sin\phi \end{pmatrix} .$$

The body angular velocity and spatial angular velocity are related by

$$\mathbf{\Omega}^b = \mathbf{Q}^T\mathbf{\Omega}^s = \begin{pmatrix} \dot{\phi}\cos\theta \\ \dot{\theta} \\ \dot{\phi}\sin\theta \end{pmatrix} .$$

If we assume that θ is small then

$$\Omega^s \approx \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \phi\dot{\theta} \end{pmatrix} ,$$

and

$$\Omega^b \approx \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \theta\dot{\phi} \end{pmatrix} .$$

If we further neglect the quadratic terms $\phi\dot{\theta}$ and $\theta\dot{\phi}$ then the angular velocities reduce to

$$\Omega^s = \Omega^b = \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ 0 \end{pmatrix} .$$

This is the approximation used explicitly by [4] and appears to be implicitly used in [8, 10]. Formalizing this assumption

$$\phi \approx 0 \quad \text{and} \quad \theta \approx 0\,. \qquad (\mathbf{DFP\text{-}5})$$

The assumption $\phi\dot{\theta} \approx 0$ implies assumption (**DFP-1**). However there is an inconsistency in this assumption in that $\dot{\phi}\theta$ is *not* assumed to be small! Hence Ω_3^s is assumed to be small but Ω_3^b is not assumed to be small.

On the other hand since the aim is to numerically simulate, neither assumption (**DFP-1**) or assumption (**DFP-5**) is necessary.

6 Simplifying f_1 and f_2

The authors [4, 8, 9, 10] all use the spatial angular velocity in their derivation. But the equations are relative to the body frame. It is certainly correct, but it leads to cumbersome equations. In this section it is shown that the terms f_1 and f_2 simplify considerably when the body angular velocity is used. To simplify notation use

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)\,,$$

for the spatial angular velocity as in [8, 10] and use

$$\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)\,,$$

for the body angular velocity as in [1].

With the assumption (**DFP-1**), the body angular velocity can be expressed in terms of the spatial angular velocity using $\boldsymbol{\omega} = \mathbf{Q}\boldsymbol{\Omega}$,

$$\begin{aligned} \Omega_1 &= \omega_1 \cos\theta + \omega_2 \sin\phi \sin\theta \\ \Omega_2 &= \omega_2 \cos\phi \\ \Omega_3 &= \omega_1 \sin\theta - \omega_2 \sin\phi \cos\theta\,. \end{aligned} \tag{6.3}$$

The translation and gravity terms can be expressed using $\mathbf{Q}$,

$$\ddot{\mathbf{n}} \cdot \mathbf{Q}\mathbf{e}_1 = \ddot{n}_1 \cos\theta + \ddot{n}_2 \sin\phi \sin\theta - \ddot{n}_3 \cos\phi \sin\theta\,,$$

and

$$g\mathbf{e}_3 \cdot \mathbf{Q}\mathbf{e}_1 = -g\cos\phi\sin\theta\,.$$

Using these two expressions, (6.3), and the formulae in Appendix A, f_1 simplifies to

$$\begin{aligned} f_1 \;=\; & -\ddot{\mathbf{n}} \cdot \mathbf{Q}\mathbf{e}_1 + (\Omega_2^2 + \Omega_3^2)x - \Omega_1\Omega_2 y - \Omega_1\Omega_3 z_d \\ & +2\Omega_3 v - \dot{\Omega}_2 z_d + \dot{\Omega}_3 y - g\mathbf{e}_3 \cdot \mathbf{Q}\mathbf{e}_1\,. \end{aligned} \tag{6.4}$$

A similar construction shows that

$$\begin{aligned} f_2 \;=\; & -\ddot{\mathbf{n}} \cdot \mathbf{Q}\mathbf{e}_2 + (\Omega_1^2 + \Omega_3^2)y - \Omega_1\Omega_2 x - \Omega_2\Omega_3 z_d \\ & -2\Omega_3 u + \dot{\Omega}_1 z_d - \dot{\Omega}_3 x - g\mathbf{e}_3 \cdot \mathbf{Q}\mathbf{e}_2\,. \end{aligned} \tag{6.5}$$

The simplification in the expressions (6.4) and (6.5) over the original expressions (2.3) and (2.4) is remarkable. The simplification is due first to the use of the body angular velocity and secondly to the explicit use of the rotation operator.

To see that these are equivalent to (2.3) and (2.4) substitute (6.3) into (6.4) to recover (2.3) and substitute (6.3) into (6.5) to recover (2.4).

The vertical acceleration term can also be simplified to

$$\begin{aligned} a_{(z)} \;=\; & \ddot{\mathbf{n}} \cdot \mathbf{Q}\mathbf{e}_3 - (\Omega_1^2 + \Omega_2^2)z_d + \Omega_1\Omega_3 x + \Omega_2\Omega_3 y \\ & +2\Omega_1 v - 2\Omega_2 u + \dot{\Omega}_1 y - \dot{\Omega}_2 x + g\mathbf{e}_3 \cdot \mathbf{Q}\mathbf{e}_3\,. \end{aligned} \tag{6.6}$$

That this expression is equivalent to (2.5) is verified by substitution of (6.3) into (6.6).

7 Comparison of the surface equations [1] with the DFP SWEs

With the coefficients now expressed in terms of the body angular velocity we can compare the DFP SWEs with the new surface equations derived in [1]. The surface momentum equations derived in [1], neglecting surface tension are

$$\begin{aligned} U_t + UU_x + VU_y + a_{11}\,h_x + a_{12}\,h_y &= b_1\,, \\ V_t + UV_x + VV_y + a_{21}\,h_x + a_{22}\,h_y &= b_2\,, \end{aligned} \tag{7.1}$$

with

$$\begin{aligned} a_{11} &= 2\Omega_1 V + \mathbf{Q}\mathbf{e}_3\cdot\ddot{\mathbf{q}} + g\mathbf{Q}\mathbf{e}_3\cdot\mathbf{e}_3 - (\Omega_1^2+\Omega_2^2)(h+d_3) \\ &\quad -(\dot{\Omega}_2 - \Omega_1\Omega_3)(x+d_1) + (\dot{\Omega}_1 + \Omega_3\Omega_2)(y+d_2) \\ a_{22} &= -2\Omega_2 U - (\Omega_1^2+\Omega_2^2)(h+d_3) + \mathbf{Q}\mathbf{e}_3\cdot\ddot{\mathbf{q}} + g\mathbf{Q}\mathbf{e}_3\cdot\mathbf{e}_3 \\ &\quad +(\dot{\Omega}_1 + \Omega_2\Omega_3)(y+d_2) - (\dot{\Omega}_2 - \Omega_1\Omega_3)(x+d_1)\,. \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} a_{12} &= 2\Omega_2 V\,, \\ a_{21} &= -2\Omega_1 U \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} b_1 &= -2\Omega_2 h_t + 2\Omega_3 V - \mathbf{Q}\mathbf{e}_1\cdot\ddot{\mathbf{q}} - g\mathbf{Q}\mathbf{e}_1\cdot\mathbf{e}_3 + (\Omega_2^2+\Omega_3^2)(x+d_1) \\ &\quad +(\dot{\Omega}_3 - \Omega_1\Omega_2)(y+d_2) - (\dot{\Omega}_2 + \Omega_1\Omega_3)(h+d_3)\,, \\ b_2 &= 2\Omega_1 h_t - 2\Omega_3 U - \mathbf{Q}\mathbf{e}_2\cdot\ddot{\mathbf{q}} - g\mathbf{Q}\mathbf{e}_2\cdot\mathbf{e}_3 + (\Omega_1^2+\Omega_3^2)(y+d_2) \\ &\quad -(\dot{\Omega}_3 + \Omega_1\Omega_2)(x+d_1) + (\dot{\Omega}_1 - \Omega_2\Omega_3)(h+d_3)\,. \end{aligned} \tag{7.4}$$

The advantage of these equations is that the vessel motion is exact with the only assumption being neglect of the vertical acceleration at the free surface [1].

The velocity field (U,V) is not the same as the velocity field in the DFP SWEs. Assume for purposes of comparison that $(U,V)\approx(u,v)$, and the the coefficients in the two systems can be compared.

First note that $\mathbf{q}=\mathbf{n}$, $\omega_3=0$ in the DFP equations and $d_3=z_d$. Comparing,

$$\begin{aligned} a_{11} &= a_{(z)} + 2\Omega_2 U - (\Omega_1^2+\Omega_2^2)h - (\dot{\Omega}_2 - \Omega_1\Omega_3)d_1 + (\dot{\Omega}_1 + \Omega_2\Omega_3)d_2 \\ a_{12} &= a_{12}^{\mathrm{DFP}} + 2\Omega_2 V \\ a_{21} &= a_{21}^{\mathrm{DFP}} - 2\Omega_1 U \\ a_{22} &= a_{(z)} - 2\Omega_1 V - (\Omega_1^2+\Omega_2^2)h - (\dot{\Omega}_2 - \Omega_1\Omega_3)d_1 + (\dot{\Omega}_1 + \Omega_2\Omega_3)d_2\,. \end{aligned}$$

In the DFP formulation, the coefficients a_{12} and a_{21} are identically zero. The right-hand side coefficient comparison is

$$\begin{aligned} b_1 &= f_1 - 2\Omega_2 h_t + (\Omega_2^2+\Omega_3^2)d_1 + (\dot{\Omega}_3 - \Omega_1\Omega_2)d_2 - (\dot{\Omega}_2 + \Omega_1\Omega_3)h \\ b_2 &= f_2 + 2\Omega_1 h_t + (\Omega_1^2+\Omega_3^2)d_2 - (\dot{\Omega}_3 + \Omega_1\Omega_2)d_1 + (\dot{\Omega}_1 - \Omega_2\Omega_3)h \end{aligned}$$

If we invoke the assumption (**DFP-3**) then these relations simplify to

$$
\begin{aligned}
a_{11} &= a_{(z)} + 2\Omega_2 U - (\Omega_1^2 + \Omega_2^2)h \\
a_{12} &= a_{12}^{\mathrm{DFP}} + 2\Omega_2 V \\
a_{21} &= a_{21}^{\mathrm{DFP}} - 2\Omega_1 U \\
a_{22} &= a_{(z)} - 2\Omega_1 V - (\Omega_1^2 + \Omega_2^2)h \\
b_1 &= f_1 - 2\Omega_2 h_t - (\dot{\Omega}_2 + \Omega_1\Omega_3)h \\
b_2 &= f_2 + 2\Omega_1 h_t + (\dot{\Omega}_1 - \Omega_2\Omega_3)h\,.
\end{aligned}
$$

Therefore in order to reduce the surface equations to the DFP equations we need the further assumptions

$$
\begin{aligned}
|2\Omega_2 U - (\Omega_1^2 + \Omega_2^2)h| &\approx 0 \\
|2\Omega_1 V + (\Omega_1^2 + \Omega_2^2)h| &\approx 0 \\
|2\Omega_2 h_t + (\dot{\Omega}_2 + \Omega_1\Omega_3)h| &\approx 0 \\
|2\Omega_1 h_t + (\dot{\Omega}_1 - \Omega_2\Omega_3)h| &\approx 0 \\
|\Omega_2 V h_y| &\approx 0 \\
|\Omega_1 U h_x| &\approx 0\,.
\end{aligned}
\qquad (\textbf{DFP-6})
$$

With these additional assumptions, and equivalence of the velocity fields, the equations reduce to the same form. These assumptions are however quite severe and are not justified in general.

— Appendix —

A Vector identities used in transforming spatial coordinates to body coordinates

Using the transformation (6.3)

$$
2\boldsymbol{\Omega} \times \mathbf{u} = \begin{pmatrix} -2\Omega_3 v \\ 2\Omega_3 u \\ 2\Omega_1 v - 2\Omega_2 u \end{pmatrix} = \begin{pmatrix} -2\omega_1 v \sin\theta + 2\omega_2 v \sin\phi\cos\theta \\ 2\omega_1 u \sin\theta - 2\omega_2 u \sin\phi\cos\theta \\ 2\omega_1 v \cos\theta + 2\omega_2 v \sin\phi\sin\theta - 2\omega_2 u \cos\phi \end{pmatrix}.
$$

Now let $\mathbf{x} = (x, y, z_d)$ where z_d is the constant vertical distance from the centre of gravity to the deck. Then

$$
\begin{aligned}
\dot{\boldsymbol{\Omega}} \times \mathbf{x} &= \begin{pmatrix} \dot{\Omega}_2 z_d - \dot{\Omega}_3 y \\ -\dot{\Omega}_1 z_d + \dot{\Omega}_3 x \\ \dot{\Omega}_1 y - \dot{\Omega}_2 x \end{pmatrix} \\
&= \begin{pmatrix} \dot{\omega}_2 z_d \cos\phi - y(\dot{\omega}_1 \sin\theta - \dot{\omega}_2 \sin\phi \cos\theta) \\ -z_d(\dot{\omega}_1 \cos\theta + \dot{\omega}_2 \sin\phi \sin\theta) + x(\dot{\omega}_1 \sin\theta - \dot{\omega}_2 \sin\phi \cos\theta) \\ y(\dot{\omega}_1 \cos\theta + \dot{\omega}_2 \sin\phi \sin\theta) - \dot{\omega}_2 x \cos\phi \end{pmatrix} .
\end{aligned}
$$

Similarly

$$
\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}) = \begin{pmatrix} -(\Omega_2^2 + \Omega_3^2)x + \Omega_1\Omega_2 y + \Omega_1\Omega_3 z_d \\ \Omega_1\Omega_2 x - (\Omega_1^2 + \Omega_3^2)y + \Omega_2\Omega_3 z_d \\ \Omega_1\Omega_3 x + \Omega_2\Omega_3 y - (\Omega_1^2 + \Omega_2^2) z_d \end{pmatrix} .
$$

Translating into spatial angular velocities

$$
\begin{aligned}
\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})_1 &= -x\omega_1^2 \sin^2\theta - x\omega_2^2(1 - \sin^2\phi \sin^2\theta) + 2x\omega_1\omega_2 \sin\phi \sin\theta \cos\theta \\
&\quad + y\omega_1\omega_2 \cos\phi \cos\theta + y\omega_2^2 \sin\phi \cos\phi \sin\theta + z_d\omega_1^2 \sin\theta \cos\theta \\
&\quad + z_d\omega_1\omega_2 \sin\phi(\sin^2\theta - \cos^2\theta) - z_d\omega_2^2 \sin^2\phi \sin\theta \cos\theta ,
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})_2 &= x\omega_1\omega_2 \cos\theta \cos\phi + x\omega_2^2 \sin\phi \cos\phi \sin\theta - y\omega_1^2 \\
&\quad - y\omega_2^2 \sin^2\phi + \omega_1\omega_2 z_d \cos\phi \sin\theta - \omega_2^2 z_d \sin\phi \cos\phi \cos\theta ,
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})_3 &= x\omega_1^2 \sin\theta \cos\theta - x\omega_1\omega_2 \sin\phi(\cos^2\theta - \sin^2\theta) - x\omega_2^2 \sin^2\phi \sin\theta \cos\theta \\
&\quad + y\omega_1\omega_2 \cos\phi \sin\theta - y\omega_2^2 \sin\phi \cos\phi \cos\theta - 2z_d\omega_1\omega_2 \sin\phi \sin\theta \cos\theta \\
&\quad - z_d\omega_1^2 \cos^2\theta - z_d\omega_2^2 \sin^2\phi \sin^2\theta - z_d\omega_2^2 \cos^2\phi ,
\end{aligned}
$$

B Sketch of the derivation of Pantazopoulos [10]

In his thesis, Pantazopoulos gives a derivation of the acceleration terms relative to the moving frame. Here a sketch of that derivation is given which shows how the use of the spatial angular velocity complicates the equations. The general relation between $\mathbf{X}$ and $\mathbf{x}$ is

$$\mathbf{X} = \mathbf{Q}(\mathbf{x} + \mathbf{d}) + \mathbf{n} , \tag{B-1}$$

where $\mathbf{Q}$ is a rotation matrix, $\mathbf{d}$ is the distance to the centre of rotation and $\mathbf{n}$ is the displacement of the body frame. In [10] it is assumed that $\mathbf{x} = (x, y, 0)$, $\mathbf{d} = (0, 0, z_d)$ and $\mathbf{Q}$ is restricted to the form (**DFP-4**). To simplify notation let

$$\mathbf{x} = (x, y, z_d) , \tag{B-2}$$

Hence (B-1) simplifies to

$$\mathbf{X} = \mathbf{Q}\mathbf{x} + \mathbf{n}\,, \tag{B-3}$$

With derivative

$$\dot{\mathbf{X}} = \dot{\mathbf{Q}}\mathbf{x} + \mathbf{Q}\dot{\mathbf{x}} + \dot{\mathbf{n}}\,. \tag{B-4}$$

Use the spatial angular velocity $\boldsymbol{\omega}$ defined by

$$\dot{\mathbf{Q}}\mathbf{Q}^T = \widehat{\boldsymbol{\omega}}\,,$$

where the hat-map is defined in (5.2). Hence (B-4) has the equivalent representation

$$\dot{\mathbf{X}} = \boldsymbol{\omega} \times (\mathbf{X} - \mathbf{n}) + \mathbf{Q}\dot{\mathbf{x}} + \dot{\mathbf{n}}\,. \tag{B-5}$$

Note also that

$$\dot{\mathbf{X}} - \dot{\mathbf{n}} = \boldsymbol{\omega} \times (\mathbf{Q}\mathbf{x}) + \mathbf{Q}\dot{\mathbf{x}}\,. \tag{B-6}$$

Differentiate (B-5) again to obtain the absolute acceleration

$$\ddot{\mathbf{X}} = \dot{\boldsymbol{\omega}} \times (\mathbf{X} - \mathbf{n}) + \boldsymbol{\omega} \times (\dot{\mathbf{X}} - \dot{\mathbf{n}}) + \dot{\mathbf{Q}}\dot{\mathbf{x}} + \mathbf{Q}\ddot{\mathbf{x}} + \ddot{\mathbf{n}}\,, \tag{B-7}$$

or after substitution,

$$\ddot{\mathbf{X}} = \ddot{\mathbf{n}} + \mathbf{Q}\ddot{\mathbf{x}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{Q}\mathbf{x})) + 2\boldsymbol{\omega} \times (\mathbf{Q}\dot{\mathbf{x}}) + \dot{\boldsymbol{\omega}} \times (\mathbf{Q}\mathbf{x})\,. \tag{B-8}$$

This latter equation is in fact equation (5) in [10] with the identifications

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{\mathbf{n}} \\ \ddot{\boldsymbol{\rho}} &= \mathbf{Q}\ddot{\mathbf{x}} \\ \boldsymbol{\rho} &= \mathbf{Q}\mathbf{x} \\ \dot{\boldsymbol{\rho}} &= \mathbf{Q}\dot{\mathbf{x}}\,. \end{aligned}$$

[10] makes the assumption $\boldsymbol{\omega} = (\omega_1, \omega_2, 0)$ as noted in (**DFP-1**). A problem is that the right-hand side of (B-8) has a mixture of spatial vectors and body vectors: $\mathbf{x}$ is the position in the body coordinates, but $\boldsymbol{\omega}$ is the spatial angular velocity.

[10] proceeds to derive the acceleration relative to the body. Let

$$\mathbf{A}^s := \ddot{\mathbf{X}}\,,$$

be the acceleration relative to the fixed frame and let

$$\mathbf{A}^b = \mathbf{Q}^T\mathbf{A}^s\,, \tag{B-9}$$

be the acceleration viewed from the body frame. In [10] this latter vector is denoted by

$$\mathbf{A}^b := \begin{pmatrix} a_{(x)} \\ a_{(y)} \\ a_{(z)} \end{pmatrix}\,.$$

[10] proceeds to derive explicit expressions for the components of $\mathbf{A}^b$ (in equation (10a) (10b) and (10c) on page 33 of [10]). However a general expression can be found which highlights why the formulae in [10] are so complicated. Combining (B-8) with (B-9) gives

$$\mathbf{A}^b = \mathbf{Q}^T\mathbf{A}^s = \mathbf{Q}^T\left(\ddot{\mathbf{n}} + \mathbf{Q}\ddot{\mathbf{x}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times(\mathbf{Q}\mathbf{x})) + 2\boldsymbol{\omega}\times(\mathbf{Q}\dot{\mathbf{x}}) + \dot{\boldsymbol{\omega}}\times(\mathbf{Q}\mathbf{x})\right).$$

Before analyzing this expression, note that $\dot{\mathbf{x}}$ is the Lagrangian velocity and $\ddot{\mathbf{x}}$ the Lagrangian acceleration of a fluid particle. Replacing these terms gives

$$\mathbf{A}^b = \mathbf{Q}^T\mathbf{A}^s = \mathbf{Q}^T\left(\ddot{\mathbf{n}} + \mathbf{Q}\frac{D\mathbf{u}}{Dt} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times(\mathbf{Q}\mathbf{x})) + 2\boldsymbol{\omega}\times(\mathbf{Q}\mathbf{u}) + \dot{\boldsymbol{\omega}}\times(\mathbf{Q}\mathbf{x})\right). \tag{B-10}$$

To simplify further use the following fundamental identity [7]: for any vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$ and any invertible 3×3 matrix $\mathbf{M}$,

$$\mathbf{M}\mathbf{a}\times\mathbf{M}\mathbf{b} = \det(\mathbf{M})\left(\mathbf{M}^{-1}\right)^T\,\mathbf{a}\times\mathbf{b}\,.$$

If $\mathbf{M}$=$\mathbf{Q}$ and $\mathbf{Q}$ is a proper rotation ($\det(\mathbf{Q})=1$ and $\mathbf{Q}^{-1}=\mathbf{Q}^T$) then this formula simplifes to

$$\mathbf{Q}\mathbf{a}\times\mathbf{Q}\mathbf{b} = \mathbf{Q}(\mathbf{a}\times\mathbf{b})\,.$$

Apply this formula to (B-10)

$$\mathbf{A}^b = \mathbf{Q}^T\ddot{\mathbf{n}} + \frac{D\mathbf{u}}{Dt} + \mathbf{Q}^T\boldsymbol{\omega}\times(\mathbf{Q}^T\boldsymbol{\omega}\times\mathbf{x}) + 2\mathbf{Q}^T\boldsymbol{\omega}\times\mathbf{u} + \mathbf{Q}^T\dot{\boldsymbol{\omega}}\times\mathbf{x}\,. \tag{B-11}$$

Hence it is clear that if we insist on using the spatial angular velocity then the equation will be very complicated. [10] treats gravity as an acceleration (it is in fact a body force) and so $\mathbf{A}^b$ becomes

$$\mathbf{A}^b = \mathbf{Q}^T\ddot{\mathbf{n}} + \frac{D\mathbf{u}}{Dt} + \mathbf{Q}^T\boldsymbol{\omega}\times(\mathbf{Q}^T\boldsymbol{\omega}\times\mathbf{x}) + 2\mathbf{Q}^T\boldsymbol{\omega}\times\mathbf{u} + \mathbf{Q}^T\dot{\boldsymbol{\omega}}\times\mathbf{x} + g\mathbf{Q}^T\mathbf{e}_3\,, \tag{B-12}$$

where $g>0$ is the gravitational constant. It is precisely the components of $\mathbf{A}^b$ in (B-12) which appear in equation (10) on page 33 of [10].

If instead we replace the spatial angular velocity with the body angular velocity

$$\boldsymbol{\Omega} := \mathbf{Q}^T\boldsymbol{\omega}\,,$$

then the equation simplifies dramatically

$$\mathbf{A}^b = \mathbf{Q}^T\ddot{\mathbf{n}} + \frac{D\mathbf{u}}{Dt} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) + 2\boldsymbol{\Omega}\times\mathbf{u} + \dot{\boldsymbol{\Omega}}\times\mathbf{x} + g\mathbf{Q}^T\mathbf{e}_3\,. \tag{B-13}$$

In general, one can use either (B-12) or (B-13) to write out the components of $\mathbf{A}^b$, but the latter expression is less complicated.

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