Krein signature in the elliptic-hyperbolic transition of periodic orbits

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Abstract

In the linearization about periodic orbits, Krein signature is a symplectic invariant of Floquet multipliers on the unit circle. Of interest here is the role of Krein signature when two Floquet multipliers meet at $+1$ and undergo a elliptic to hyperbolic transition. It is shown, using the normal form at the transition, that the symplectic invariant at the transition point is determined by the Krein signature of the colliding elliptic Floquet multipliers, and vice versa.

1 Introduction

The lowest dimension in which the phenomena occurs is four dimensions and so restrict to this case. Consider a standard autonomous Hamiltonian system on $\mathbb{R}^4$,

$$\mathbf{J}\mathbf{u}_t = \nabla H(\mathbf{u})\,, \quad \mathbf{u}\in\mathbb{R}^4\,, \tag{1}$$

where $H(\mathbf{u})$ is a smooth Hamiltonian function, and $\mathbf{J}$ is the unit symplectic operator,

$$\mathbf{J} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2}$$

Suppose there exists a branch of periodic orbits $\widehat{\mathbf{u}}(t)$ of frequency $\omega > 0$. The periodic orbits can be parameterized by frequency, period, action or energy [1, 4]. For definiteness suppose they are parameterized by frequency.

Perturb about the family of periodic orbits $\mathbf{u} = \widehat{\mathbf{u}} + \mathbf{v}$, then $\mathbf{v}$ satisfies

$$\mathbf{J}\mathbf{v}_t = \nabla H(\widehat{\mathbf{u}} + \mathbf{v}) - \nabla H(\widehat{\mathbf{u}})\,. \tag{3}$$

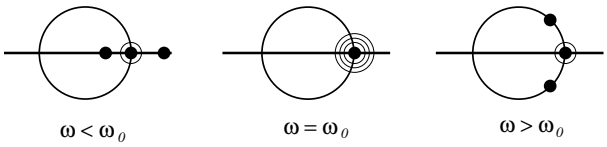

Figure 1: Schematic of the elliptic-hyperbolic transition of Floquet multipliers.

The linearized system

$$\mathbf{v}_t = \mathbf{A}(t)\mathbf{v}\,, \quad \mathbf{J}\mathbf{A}(t) = D^2H(\widehat{\mathbf{u}}(t))\,,$$

is in standard form for Floquet theory. There is one pair of Floquet multipliers at $+1$, and a second pair which may be elliptic or hyperbolic. Here the interest is in the case where there is a parameter value, say $\omega = \omega_0$, where there is a transition between the two cases, as shown in Figure 1.

Near the transition, the elliptic Floquet multipliers have a symplectic invariant, the Krein signature. Suppose μ is an elliptic Floquet multipier with complex eigenfunction $\xi(t)$. The Krein signature, $\mathcal{S}$, associated with the Floquet multiplier is defined by

$$\Omega(\overline{\xi},\xi) := \langle \mathbf{J}\overline{\xi},\xi\rangle = 2\mathrm{i}\mathcal{S}\,. \tag{4}$$

By scaling $\xi(t)$, the sign $\mathcal{S} = \pm 1$. MacKay [3] argues that there are exactly four cases in the transition depending on the Krein signature of the elliptic side of the transition. Here these four cases are identified and it is shown that the Krein signature in each case is determined by the symplectic invariant of the collision. This result is surprising since the symplectic invariant at the transition uses the fourth eigenvector in the Jordan chain which does not exist away from the transition. The strategy is to use the nonlinear normal form at the transition.

2 Normal form for elliptic-hyperbolic transition

There exists coordinates $(\phi, q, I, p) \in \mathbb{R}^4$ such that the leading-order normal form [2] for the nonlinear system (3) near the elliptic-hyperbolic transition is

$$\begin{aligned} -\dot{I} &= 0 \\ -\dot{p} &= I - \tfrac{1}{2}\kappa q^2 \\ \dot{\phi} &= q \\ \dot{q} &= s\,p\,. \end{aligned} \tag{1}$$

The parameter κ is proportional to the curvature of the energy-frequency curve, and $s = \pm 1$ is the symplectic sign of the transition, and a formula for it is given in [2] in terms of the generalized eigenvectors of the transition (see also §3 below). The main result here is that s is in fact equal to the Krein signature of the elliptic Floquet multipliers in the unfolding of the transition. The classification is shown in Figure 2.

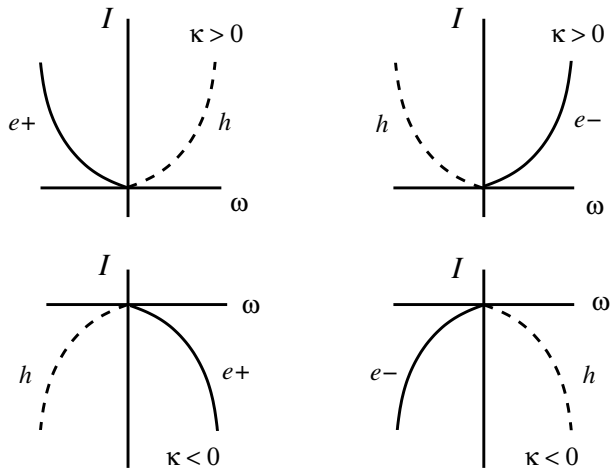

Figure 2: Four cases in the elliptic-hyperbolic transition. h denotes hyperbolic branch of periodic orbits, and $e\pm$ denoted elliptic branch with Krein signature $s = \pm 1$.

The curve of periodic orbits near the transition is a relative equilibrium of the normal form (1)

$$\begin{pmatrix} \phi(t) \\ q(t) \\ I(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \omega t + \phi_0 \\ q_0 \\ I_0 \\ 0 \end{pmatrix} .$$

Substitution gives $q_0 = \omega$[1] and

$$I_0 = \tfrac{1}{2}\kappa\omega^2 .$$

There are two cases $\kappa > 0$ and $\kappa < 0$ and they are shown in upper and lower diagram pairs, respectively, in Figure 2. On the other hand there are two possible elliptic configurations in each case. To see the distinction, compute the Krein signature of the elliptic side in each case.

The linearization of (1) about the basic state is

$$\mathbf{J}W_t = D^2H(Z_0)W ,$$

with $Z_0 = (\phi_0, \omega, I_0, 0)$, and

$$D^2H(Z_0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\kappa\omega & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} .$$

[1]The use of ω here is a slight abuse of notation since ω in the normal form should be interpreted as $\omega - \omega_0$ for the original system.

Take $W(t) = \mathrm{e}^{\lambda t}\widehat{W}$, then the characteristic equation for λ is

$$0 = \det \begin{bmatrix} 0 & 0 & \lambda & 0 \\ 0 & -\kappa\omega & 1 & \lambda \\ -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & s \end{bmatrix} = \lambda^2(\lambda^2 - s\kappa\omega)\,.$$

Hence the periodic orbit Z_0 is hyperbolic if $s\kappa\omega > 0$ and elliptic if $s\kappa\omega < 0$. Now suppose we are in the elliptic region, $-s\kappa\omega > 0$, and define

$$\alpha = \sqrt{-s\kappa\omega}\,,$$

and look at the elliptic eigenvalue $\lambda = \mathrm{i}\alpha$. The complex eigenvector ξ is

$$\xi = \mathbb{C} \begin{pmatrix} 1 \\ \mathrm{i}\alpha \\ 0 \\ -s\alpha^2 \end{pmatrix},$$

where $\mathbb{C}$ is an arbitrary complex scale factor. Compute the Krein signature (4)

$$2\mathrm{i}\mathcal{S} = \Omega(\overline{\xi}, \xi) = |\mathbb{C}|^2 2\mathrm{i}s\alpha^3\,.$$

Since $\alpha > 0$, take $\mathbb{C} = \alpha^{-3/2}$. Then the Krein sign $\mathcal{S}$ equals s.

3 A formula for s

Let $\theta = \omega t$ (back to ω for the original system) and define $\mathbf{L} := D^2H(\widehat{\mathbf{u}}) - \omega\mathbf{J}\frac{d}{d\theta}$. Then at the transition, zero is an eigenvalue of $\mathbf{L}$ of algebraic multiplicity four but geometric multiplicity one. The geometric eigenvector $\widehat{\xi}_1 = \widehat{\mathbf{u}}_\theta$, the tangent vector to the periodic orbit, and the first geometric eigenvector is $\widehat{\xi}_2 = \widehat{\mathbf{u}}_\omega$. The Jordan chain is

$$\mathbf{L}\widehat{\xi}_1 = 0\,, \quad \mathbf{L}\widehat{\xi}_2 = \mathbf{J}\widehat{\xi}_1\,, \quad \mathbf{L}\widehat{\xi}_3 = \mathbf{J}\widehat{\xi}_2\,, \quad \mathbf{L}\widehat{\xi}_4 = \mathbf{J}\widehat{\xi}_3\,.$$

The symplectic invariant s at the transition is defined by [2]

$$s = \mathrm{sign}\left(\int_0^{2\pi} \Omega(\widehat{\xi}_4, \widehat{\xi}_1)\,\mathrm{d}\theta \right).$$

References

[1] L. Bates & J. Śniatycki. *On the period-energy relation*, Proc. Amer. Math. Soc. **114** 877–878 (1992).

[2] T.J. Bridges & N.M. Donaldson. *Degenerate periodic orbits and homoclinic torus bifurcation*, Phys. Rev. Lett. **95**(10) 104301 (2005).

[3] R.S. MacKay. *Three topics in Hamiltonian dynamics*, in *Dynamical Systems and Chaos, Vol 2* (Hachioji), 34–43, World Scientific Publishers: Singapore (1994).

[4] J.-A. Sepulchre & R.S. Mackay. *Localized oscillations in conservative or dissipative networks of weakly coupled autonomous oscillators*, Nonlinearity **10** 679–713 (1997).